Differential Forms

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Preface

Introduction

For most math undergraduates one's first encounter with differential forms is the change of variables formula in multivariable calculus, i.e., the formula

$$\int_U f^*\phi|\det J_f|dx = \int_V \phi dy. \tag{1}$$

In this formula, U and V are bounded open subsets of $\mathbf{R}^n$, $\phi\colon V \to \mathbf{R}$ is a bounded continuous function, $f: U \to V$ is a bijective differentiable map, $f^*\phi\colon U \to \mathbf{R}$ is the function $\phi \circ f$, and $\det J_f(x)$ is the determinant of the Jacobian matrix:

$$J_f(x) := \left(\frac{\partial f_i}{\partial x_j}(x)\right).$$

As for the "dx" and "dy", their presence in (1) can be accounted for by the fact that in single-variable calculus, with $U = (a,b)$, $V = (c,d)$, $f\colon (a,b) \to (c,d)$, and $y = f(x)$ a C^1 function with positive first derivative, the tautological equation $\frac{dy}{dx} = \frac{df}{dx}$ can be rewritten in the form

$$d(f^*y) = f^*dy$$

and (1) can be written more suggestively as

$$\int_U f^*(\phi dy) = \int_V \phi dy. \tag{2}$$

One of the goals of this text on *differential forms* is to legitimize this interpretation of equation (1) in n dimensions and in fact, more generally, show that an analogue of this formula is true when U and V are n-dimensional manifolds.

Another related goal is to prove an important topological generalization of the change of variables formula (1). This formula asserts that if we drop the assumption that f be a bijection and just require f to be proper (i.e., that preimages of compact subsets of V to be compact subsets of U) then the formula (1) can be replaced by

$$\int_U f^*(\phi dy) = \deg(f)\int_V \phi dy, \tag{3}$$

where $\deg(f)$ is a topological invariant of f that roughly speaking counts, with plus and minus signs, the number of preimage points of a generically chosen point of V.[1]

This degree formula is just one of a host of results which connect the theory of differential forms with topology, and one of the main goals of this book will explore some of the other examples. For instance, for U an open subset of $\mathbf{R}^2$, we define $\Omega^0(U)$ to be the vector space of C^∞ functions on U. We define the vector space $\Omega^1(U)$ to be the space of formal sums

$$f_1\,dx_1 + f_2\,dx_2, \tag{4}$$

where $f_1, f_2 \in C^\infty(U)$. We define the vector space $\Omega^2(U)$ to be the space of expressions of the form

$$f\,dx_1 \wedge dx_2, \tag{5}$$

where $f \in C^\infty(U)$, and for $k > 2$ define $\Omega^k(U)$ to the zero vector space.

On these vector spaces one can define operators

$$d\colon \Omega^i(U) \to \Omega^{i+1}(U) \tag{6}$$

by the recipes

$$df := \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2 \tag{7}$$

for $i = 0$,

$$d(f_1\,dx_1 + f_2 dx_2) = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx \tag{8}$$

for $i = 1$, and $d = 0$ for $i > 1$. It is easy to see that the operator

$$d^2\colon \Omega^i(U) \to \Omega^{i+2}(U) \tag{9}$$

is zero. Hence,

$$\operatorname{im}(d\colon \Omega^{i-1}(U) \to \Omega^i(U)) \subset \ker(d\colon \Omega^i(U) \to \Omega^{i+1}(U)),$$

and this enables one to define the *de Rham cohomology groups of U* as the quotient vector space

$$H^i(U) := \frac{\ker(d\colon \Omega^i(U) \to \Omega^{i+1}(U))}{\operatorname{im}(d\colon \Omega^{i-1}(U) \to \Omega^i(U))}. \tag{10}$$

It turns out that these cohomology groups are topological invariants of U and are, in fact, isomorphic to the cohomology groups of U defined by the algebraic topologists. Moreover, by slightly generalizing the definitions in equations (4), (5) and (7)–(10) one can define these groups for open subsets of $\mathbf{R}^n$ and, with a bit

[1]It is our feeling that this formula should, like formula (1), be part of the standard calculus curriculum, particularly in view of the fact that there now exists a beautiful elementary proof of it by Peter Lax (see [6, 8, 9]).

more effort, for arbitrary C^{∞} manifolds (as we will do in Chapter 5); and their existence will enable us to describe interesting connections between problems in multivariable calculus and differential geometry on the one hand and problems in topology on the other.

To make the context of this book easier for our readers to access we will devote the rest of this introduction to the following annotated table of contents, chapter-by-chapter descriptions of the topics that we will be covering.

Organization

Chapter 1: Multilinear algebra

As we mentioned above one of our objectives is to legitimatize the presence of the dx and dy in formula (1), and translate this formula into a theorem about differential forms. However a rigorous exposition of the theory of differential forms requires a lot of algebraic preliminaries, and these will be the focus of Chapter 1. We'll begin, in §§1.1 and 1.2, by reviewing material that we hope most of our readers are already familiar with: the definition of vector space, the notions of *basis*, of *dimension*, of *linear mapping*, of *bilinear form*, and of *dual space* and *quotient space*. Then in §1.3 we will turn to the main topics of this chapter, the concept of k-tensor and (the future key ingredient in our exposition of the theory of differential forms in Chapter 2) the concept of alternating k-tensor. Those k-tensors come up in fact in two contexts: as *alternating k-tensors*, and as *exterior forms*, i.e., in the first context as a subspace of the space of k-tensors and in the second as a quotient space of the space of k-tensors. Both descriptions of k-tensors will be needed in our later applications. For this reason the second half of Chapter 1 is mostly concerned with exploring the relationships between these two descriptions and making use of these relationships to define a number of basic operations on exterior forms such as the wedge product operation (see §1.6), the interior product operation (see §1.7) and the pullback operation (see §1.8). We will also make use of these results in §1.9 to define the notion of an *orientation* for an n-dimensional vector space, a notion that will, among other things, enable us to simplify the change of variables formula (1) by getting rid of the absolute value sign in the term $|\det J_f|$.

Chapter 2: The concept of a differential form

The expressions in equations (4), (5), (7) and (8) are typical examples of differential forms, and if this were intended to be a text for undergraduate physics majors we would define differential forms by simply commenting that they're expressions of this type. We'll begin this chapter, however, with the following more precise definition: Let U be an open subset of $\mathbf{R}^n$. Then a *k-form* ω on U is a "function" which to each $p \in U$ assigns an element of $\Lambda^k(T_p^\star U)$, T_pU being the tangent space to U at p, $T_p^\star U$ its vector space dual, and $\Lambda^k(T_p^\star)$ the kth-order exterior power of $T_p^\star U$. (It turns out, fortunately, not to be too hard to reconcile this definition with the physics definition above.) Differential 1-forms are perhaps best understood as the dual objects to vector fields, and in §§2.1 and 2.2 we elaborate on this observation, and recall for future use some standard facts about vector fields and their integral curves. Then in § 2.3 we will turn to the topic of k-forms and in the exercises at the end of § 2.3 discuss a lot of explicit examples (that we strongly urge readers of this text to stare at). Then in §§ 2.4–2.7 we will discuss in detail three fundamental operations on differential forms of which we've already gotten preliminary glimpses in equation (3) and equations (5)–(9), namely the wedge product operation, the exterior differential operation, and the pullback operation. Also, to add to this list in § 2.5 we will discuss the interior product operation of vector fields on differential forms, a generalization of the duality pairing of vector fields with 1-forms that we alluded to earlier. (In order to get a better feeling for this material we strongly recommend that one take a look at § 2.7 where these operations are related to the div, curl, and grad operations in freshman calculus.) In addition this section contains some interesting applications of the material above to physics, in § 2.8 to electrodynamics and Maxwell's equation, as well as to classical mechanisms and the Hamilton–Jacobi equations.

Chapter 3: Integration of forms

As we mentioned above, the change of variables formula in integral calculus is a special case of a more general result: the degree formula; and we also cited a paper of Peter Lax which contains an elementary proof of this formula which will hopefully induce future authors of elementary text books in multivariate calculus to include it in their treatment of the Riemann integral. In this chapter we will also give a proof of this result but not, regrettably, Lax's proof. The reason why not is that we want to relate this result to another result which will have some important de Rham theoretic applications when we get to Chapter 5. To describe this result let U be a connected open set in $\mathbf{R}^n$ and $\omega = f dx_1 \wedge \cdots \wedge dx_n$ a compactly supported n-form on U. We will prove the following theorem.

Theorem 11. *The integral $\int_U \omega = \int_U f dx_1 \cdots dx_n$ of ω over U is zero if and only if $\omega = dv$ where v is a compactly supported $(n-1)$-form on U.*

An easy corollary of this result is the following weak version of the degree theorem.

Corollary 12. *Let U and V be connected open subsets of $\mathbf{R}^n$ and $f : U \to V$ a proper mapping. Then there exists a constant $\delta(f)$ with the property that for every compactly supported n-form ω on V*

$$\int_U f^* \omega = \delta(f) \int_V \omega.$$

Thus to prove the degree theorem it suffices to show that this $\delta(f)$ is the $\deg(f)$ in formula (3) and this turns out not to be terribly hard.

The degree formula has a lot of interesting applications and at the end of Chapter 3 we will describe two of them. The first, the Brouwer fixed point theorem, asserts that if B^n is the closed unit ball in $\mathbf{R}^n$ and $f : B^n \to B^n$ is a continuous map, then f has a fixed point, i.e., there is a point $x_0 \in B^n$ that gets mapped onto itself by f. (We will show that if this were not true the degree theorem would lead to an egregious contradiction.)

The second is the fundamental theorem of algebra. If

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$$

is a polynomial function of the complex variable z, then it has to have a complex root z_0 satisfying $p(z_0) = 0$. (We'll show that both these results are more-or-less one line corollaries of the degree theorem.)

Chapter 4: Forms on manifolds

In the previous two chapters the differential forms that we have been considering have been forms defined on open subsets of $\mathbf{R}^n$. In this chapter we'll make the transition to *forms defined on manifolds*; and to prepare our readers for this transition, include at the beginning of this chapter a brief introduction to the theory of manifolds. (It is a tradition in undergraduate courses to define n-dimensional manifolds as n-dimensional submanifolds of $\mathbf{R}^N$, i.e., as n-dimensional generalizations of curves and surfaces in $\mathbf{R}^3$, whereas the tradition in graduate level courses is to define them as abstract entities. Since this is intended to be a text book for undergraduates, we'll adopt the first of these approaches, our reluctance to doing so being somewhat tempered by the fact that, thanks to the Whitney embedding theorem, these two approaches are the same.) In §4.1 we will review a few general facts about manifolds, in §4.2, define the crucial notion of the *tangent space* T_pX *to a manifold* X *at a point* p and in §§4.3 and 4.4 show how to define differential forms on X by defining a *k-form* to be, as in §2.4, a function ω which assigns to each $p \in X$ an element ω_p of $\Lambda^k(T_p^\star X)$. Then in §4.4 we will define what it means for a manifold to be *oriented* and in §4.5 show that if X is an oriented n-dimensional manifold and ω a compactly supported n-form, the integral of ω is well-defined. Moreover, we'll prove in this section a manifold version of the change of variables formula and in §4.6 prove manifold versions of two standard results in integral calculus: the divergence theorem and Stokes theorem, setting the stage for the main results of Chapter 4: the manifold version of the degree theorem (see §4.7) and a number of applications of this theorem (among them the Jordan–Brouwer separation theorem, the Gauss–Bonnet theorem and the Index theorem for vector fields (see §§4.8 and 4.9).

Chapter 5: Cohomology via forms

Given an n-dimensional manifold let $\Omega^k(X)$ be the space of differential forms on X of degree k, let $d \colon \Omega^k(X) \to \Omega^{k+1}(X)$ be exterior differentiation and let $H^k(X)$ be the *kth de Rham cohomology group of* X: the quotient of the vector space

$$Z^k(X) := \ker(d \colon \Omega^k(X) \to \Omega^{k+1}(X))$$

by the vector space

$$B^k(X) := \operatorname{im}(d \colon \Omega^{k-1}(X) \to \Omega^k(X)).$$

In Chapter 5 we will make a systematic study of these groups and also show that some of the results about differential forms that we proved in earlier chapters can be interpreted as cohomological properties of these groups. For instance, in § 5.1 we will show that the wedge product and pullback operations on forms that we discussed in Chapter 2 gives rise to analogous operations in cohomology and that for a compact oriented n-dimensional manifold the integration operator on forms that we discussed in Chapter 4 gives rise to a pairing

$$H^k(X) \times H^{n-k}(X) \to \mathbf{R},$$

and in fact more generally if X is non-compact, a pairing

$$H^k_c(X) \times H^{n-k}(X) \to \mathbf{R}, \tag{13}$$

where the groups $H^i_c(X)$ are the de Rham cohomology groups that one gets by replacing the spaces of differential forms $\Omega^i(X)$ by the corresponding spaces $\Omega^i_c(X)$ of compactly supported differential forms.

One problem one runs into in the definition of these cohomology groups is that since the spaces $\Omega^k(X)$ and $\Omega^k_c(X)$ are infinite dimensional in general, there's no guarantee that the same won't be the case for the spaces $H^k(X)$ and $H^k_c(X)$, and we'll address this problem in § 5.3. More explicitly, we will say that X has *finite topology* if it admits a finite covering by open sets $U_1, \dots, U_N$ such that for every multi-index $I = (i_1, \dots, i_k)$, where $1 \le i_r \le N$, the intersection

$$U_I := U_{i_1} \cap \cdots \cap U_{i_k} \tag{14}$$

is either empty or is diffeomorphic to a convex open subset of $\mathbf{R}^n$. We will show that if X has finite topology, then its cohomology groups are finite dimensional and, secondly, we will show that many manifolds have finite topology. (For instance, if X is compact then X has finite topology; for details see §§ 5.2 and 5.3.)

We mentioned above that if X is not compact it has two types of cohomology groups: the groups $H^k(X)$ and the groups $H^{n-k}_c(X)$. In § 5.5 we will show that if X is oriented then for $\nu \in \Omega^k(X)$ and $\omega \in \Omega^{n-k}_c(X)$ the integration operation

$$(\nu, \omega) \mapsto \int_X \nu \wedge \omega$$

gives rise to a pairing

$$H^k(X) \times H^{n-k}_c(X) \to \mathbf{R},$$

and that if X is connected this pairing is non-degenerate, i.e., defines a bijective linear transformation

$$H^{n-k}_c(X) \xrightarrow{\sim} H^k(X)^\star.$$

This results in the *Poincaré duality theorem* and it has some interesting implications which we will explore in §§ 5.5–5.7. For instance in § 5.5 we will show that if Y and Z are closed oriented submanifolds of X and Z is compact and of codimension equal to the dimension of Y, then the *intersection number* $I(Y, Z)$ of Y and Z in X is well-defined (no matter how badly Y and Z intersect). In § 5.6 we will show that if X is a compact oriented manifold and $f : X \to X$ a C^∞ map one can define the *Lefschetz number* of f as the intersection number $I(\Delta_X, \mathrm{graph}(f))$ where Δ_X is the diagonal in $X \times X$, and view this as a "topological count" of the number of fixed points of the map f.

Finally in §5.8 we will show that if X has finite topology (in the sense we described above), then one can define an alternative set of cohomology groups for X, the *Čech cohomology groups* of a cover $\mathcal{U}$, and we will sketch a proof of the assertion that this Čech cohomology and de Rham cohomology coincide, leaving a lot of details as exercises (however, with some hints supplied).

Appendices

In Appendix A we review some techniques in multivariable calculus that enable one to reduce global problems to local problems and illustrate these techniques by describing some typical applications. In particular we show how these techniques can be used to make sense of "improper integrals" and to extend differentiable maps $f : X \to \mathbf{R}$ on arbitrary subsets $X \subset \mathbf{R}^m$ to open neighborhoods of these sets.

In Appendix B we discuss another calculus issue: how to solve systems of equations

$$\begin{cases} f_1(x) = y_1 \\ \quad\vdots \\ f_N(x) = y_N \end{cases}$$

for x in terms of y (here $f_1, \dots, f_N$ are differentiable functions on an open subset of $\mathbf{R}^n$). To answer this question we prove a somewhat refined version of what in elementary calculus texts is referred to as the *implicit function theorem* and derive as corollaries of this theorem two results that we make extensive use of in Chapter 4: the canonical submersion theorem and the canonical immersion theorem.

Finally, in Appendix C we discuss a concept which plays an important role in the formulation of the "finite topology" results that we discussed in Chapter 5, namely the concept of a "good cover". In more detail let X be an n-dimensional manifold an let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of open subsets of X with the property that $\bigcup_{\alpha \in I} U_\alpha = X$.

Then $\mathcal{U}$ is a *good cover* of X if for every finite subset

$$\{\alpha_1, \dots, \alpha_r\} \subset I$$

the intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_r}$ is either empty or is diffeomorphic to a convex open subset of $\mathbf{R}^n$. In Appendix C we will prove that good covers always exist.

Notational Conventions

Below we provide a list of a few of our common notational conventions.

- $:=$ used to define a term; the term being defined appears to the left of the colon
- $\subset$ subset
- $\to$ denotes an map (of sets, vector spaces, etc.)

- $\hookrightarrow$ inclusion map
- $\twoheadrightarrow$ surjective map
- $\xrightarrow{\sim}$ a map that is an isomorphism
- $\smallsetminus$ difference of sets: if $A \subset X$, then $X \smallsetminus A$ is the complement of A in X

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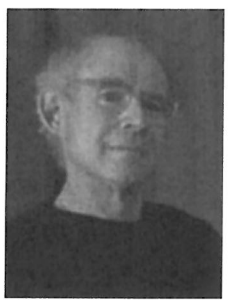

Victor Guillemin received the B.A. in mathematics from Harvard in 1959, the M.A. from the University of Chicago in 1960, and the Ph.D. from Harvard in 1962, under the direction of Shlomo Sternberg. Following an instructorship at Columbia University, Professor Guillemin joined the mathematics faculty at the Massachusetts Institute of Technology in 1966 (professor in 1973). He was appointed the Norbert Wiener Professor of Mathematics, 1994–1999. Professor Guillemin's research interests are in differential geometry and symplectic geometry. He was elected Fellow of the American Academy of Arts & Sciences (1984) and Member of the National Academy of Sciences (1985). He subsequently received the Guggenheim and Humboldt fellowships. In 2003, Professor Guillemin received the Leroy P. Steele Prize for Lifetime Achievement of the American Mathematical Society.

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Contents

Chapter 1

Multilinear Algebra

1.1. Background

We will list below some definitions and theorems that are part of the curriculum of a standard theory-based course in linear algebra.[1]

Definition 1.1.1. A *(real) vector space* is a set V the elements of which we refer to as ***vectors***. The set V is equipped with two vector space operations:

(1) *Vector space addition*: Given vectors $v_1, v_2 \in V$ one can add them to get a third vector, $v_1 + v_2$.
(2) *Scalar multiplication*: Given a vector $v \in V$ and a real number λ, one can multiply v by λ to get a vector λv.

These operations satisfy a number of standard rules: associativity, commutativity, distributive laws, etc. which we assume you're familiar with. (See Exercise 1.1.i.) In addition we assume that the reader is familiar with the following definitions and theorems.

(1) The ***zero vector*** in V is the unique vector $0 \in V$ with the property that for every vector $v \in V$, we have $v + 0 = 0 + v = v$ and $\lambda v = 0$ if $\lambda \in \mathbf{R}$ is zero.
(2) *Linear independence*: A (finite) set of vectors, $v_1, \dots, v_k \in V$ is ***linearly independent*** if the map

$$\mathbf{R}^k \to V, \quad (c_1, \dots, c_k) \mapsto c_1 v_1 + \cdots + c_k v_k \tag{1.1.2}$$

is injective.
(3) A (finite) set of vectors $v_1, \dots, v_k \in V$ ***spans*** V if the map (1.1.2) is surjective.
(4) Vectors $v_1, \dots, v_k \in V$ are a ***basis*** of V if they span V and are linearly independent; in other words, if the map (1.1.2) is bijective. This means that every vector v can be written *uniquely* as a sum

$$v = \sum_{i=1}^{n} c_i v_i. \tag{1.1.3}$$

[1] Such a course is a prerequisite for reading this text.

(5) If V is a vector space with a basis $v_1, \dots, v_k$, then V is said to be ***finite dimensional***, and k is the ***dimension*** of V. (It is a theorem that this definition is legitimate: every basis has to have the same number of vectors.) In this chapter all the vector spaces we'll encounter will be finite dimensional. We write $\dim(V)$ for the dimension of V.

(6) A subset $U \subset V$ is a ***subspace*** if it is vector space in its own right, i.e., for all $v, v_1, v_2 \in U$ and $\lambda \in \mathbf{R}$, both λv and $v_1 + v_2$ are in U (where the addition and scalar multiplication is given as vectors of V).

(7) Let V and W be vector spaces. A map $A : V \to W$ is ***linear*** if for $v, v_1, v_2 \in U$ and $\lambda \in \mathbf{R}$ we have

$$A(\lambda v) = \lambda A v \tag{1.1.4}$$

and

$$A(v_1 + v_2) = Av_1 + Av_2.$$

(8) Let $A : V \to W$ be a linear map of vector spaces. The ***kernel*** of A is the subset of V defined by

$$\ker(A) := \{ v \in V \mid A(v) = 0 \},$$

i.e., is the set of vectors in V which A sends to the zero vector in W. By equation (1.1.4) and item (7), the subset $\ker(A)$ is a subspace of V.

(9) By (1.1.4) and (7) the set-theoretic image $\mathrm{im}(A)$ of A is a subspace of W. We call $\mathrm{im}(A)$ the ***image*** A of A. The following is an important rule for keeping track of the dimensions of $\ker A$ and $\mathrm{im}\, A$:

$$\dim(V) = \dim(\ker(A)) + \dim(\mathrm{im}(A)). \tag{1.1.5}$$

(10) *Linear mappings and matrices*: Let $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_m$ a basis of W. Then by (1.1.3) Av_j can be written uniquely as a sum,

$$Av_j = \sum_{i=1}^{m} c_{i,j} w_i, \quad c_{i,j} \in \mathbf{R}. \tag{1.1.6}$$

The $m \times n$ matrix of real numbers $(c_{i,j})$ is the ***matrix*** associated with A. Conversely, given such an $m \times n$ matrix, there is a unique linear map A with the property (1.1.6).

(11) An ***inner product*** on a vector space is a map

$$B : V \times V \to \mathbf{R}$$

with the following three properties:

(a) *Bilinearity*: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbf{R}$ we have

$$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$

and

$$B(\lambda v, w) = \lambda B(v, w).$$

(b) *Symmetry*: For vectors $v, w \in V$ we have $B(v, w) = B(w, v)$.

(c) *Positivity*: For every vector $v \in V$ we have $B(v, v) \geq 0$. Moreover, if $v \neq 0$ then $B(v, v) > 0$.

Remark 1.1.7. Notice that by property (11b), property (11a) is equivalent to

$$B(w, \lambda v) = \lambda B(w, v)$$

and

$$B(w, v_1 + v_2) = B(w, v_1) + B(w, v_2).$$

Example 1.1.8. The map (1.1.2) is a linear map. The vectors $v_1, \dots, v_k$ span V if its image of this map is V, and the $v_1, \dots, v_k$ are linearly independent if the kernel of this map is the zero vector in $\mathbf{R}^k$.

The items on the list above are just a few of the topics in linear algebra that we're assuming our readers are familiar with. We have highlighted them because they're easy to state. However, understanding them requires a heavy dollop of that indefinable quality "mathematical sophistication", a quality which will be in heavy demand in the next few sections of this chapter. We will also assume that our readers are familiar with a number of more low-brow linear algebra notions: matrix multiplication, row and column operations on matrices, transposes of matrices, determinants of $n \times n$ matrices, inverses of matrices, Cramer's rule, recipes for solving systems of linear equations, etc. (See [10, §§ 1.1 and 1.2] for a quick review of this material.)

Exercises for § 1.1

Exercise 1.1.i. Our basic example of a vector space in this course is $\mathbf{R}^n$ equipped with the vector addition operation

$$(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$$

and the scalar multiplication operation

$$\lambda(a_1, \dots, a_n) = (\lambda a_1, \dots, \lambda a_n).$$

Check that these operations satisfy the axioms below.

(1) *Commutativity*: $v + w = w + v$.
(2) *Associativity*: $u + (v + w) = (u + v) + w$.
(3) For the zero vector $0 := (0, \dots, 0)$ we have $v + 0 = 0 + v$.
(4) $v + (-1)v = 0$.
(5) $1v = v$.
(6) *Associative law for scalar multiplication*: $(ab)v = a(bv)$.
(7) *Distributive law for scalar addition*: $(a + b)v = av + bv$.
(8) *Distributive law for vector addition*: $a(v + w) = av + aw$.

Exercise 1.1.ii. Check that the standard basis vectors of $\mathbf{R}^n$: $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, etc. *are* a basis.

Exercise 1.1.iii. Check that the standard inner product on $\mathbf{R}^n$

$$B((a_1, \dots, a_n), (b_1, \dots, b_n)) = \sum_{i=1}^{n} a_i b_i$$

is an inner product.

1.2. Quotient and dual spaces

In this section, we will discuss a couple of items which are frequently, but not always, covered in linear algebra courses, but which we'll need for our treatment of multilinear algebra in §§1.3–1.8.

The quotient spaces of a vector space

Definition 1.2.1. Let V be a vector space and W a vector subspace of V. A ***W-coset*** is a set of the form

$$v + W := \{v + w \mid w \in W\}.$$

It is easy to check that if $v_1 - v_2 \in W$, the cosets $v_1 + W$ and $v_2 + W$ coincide while if $v_1 - v_2 \notin W$, the cosets $v_1 + W$ and $v_2 + W$ are disjoint. Thus the distinct W-cosets decompose V into a *disjoint* collection of subsets of V.

Notation 1.2.2. Let V be a vector space and $W \subset V$ a subspace. We write V/W for the set of *distinct* W-cosets in V.

Definition 1.2.3. Let V be a vector space and $W \subset V$ a subspace. Define a vector addition operation on V/W by setting

$$(v_1 + W) + (v_2 + W) := (v_1 + v_2) + W \tag{1.2.4}$$

and define a scalar multiplication operation on V/W by setting

$$\lambda(v + W) := (\lambda v) + W. \tag{1.2.5}$$

These operations make V/W into a vector space, called the ***quotient space*** of V by W.

It is easy to see that the operations on V/W from Definition 1.2.3 are well-defined. For instance, suppose $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$. Then $v_1 - v_1'$ and $v_2 - v_2'$ are in W, so

$$(v_1 + v_2) - (v_1' + v_2') \in W,$$

and hence $(v_1 + v_2) + W = (v_1' + v_2') + W$.

Definition 1.2.6. Let V be a vector space and $W \subset V$ a subspace. Define a ***quotient map***

$$\pi : V \to V/W \tag{1.2.7}$$

by setting $\pi(v) := v + W$. It is clear from equations (1.2.4) and (1.2.5) that π is linear, and that it maps V surjectively onto V/W.

Observation 1.2.8. Note that the zero vector in the vector space V/W is the zero coset $0 + W = W$. Hence $v \in \ker(\pi)$ if and only if $v + W = W$, i.e., $v \in W$. In other words, $\ker(\pi) = W$.

In Definitions 1.2.3 and 1.2.6, V and W do not have to be finite dimensional, but if they are then by equation (1.1.5) we have

$$\dim(V/W) = \dim(V) - \dim(W).$$

We leave the following easy proposition as an exercise.

Proposition 1.2.9. *Let $A\colon V \to U$ be a linear map of vector spaces. If $W \subset \ker(A)$ there exists a unique linear map $A^\sharp\colon V/W \to U$ with the property that $A = A^\sharp \circ \pi$, where $\pi\colon V \to V/W$ is the quotient map.*

The dual space of a vector space

Definition 1.2.10. Let V be a vector space. Write $V^\star$ for the set of all linear functions $\ell\colon V \to \mathbf{R}$. Define a vector space structure on $V^\star$ as follows: if $\ell_1, \ell_2 \in V^\star$, then define the sum $\ell_1 + \ell_2$ to be the map $V \to \mathbf{R}$ given by

$$(\ell_1 + \ell_2)(v) := \ell_1(v) + \ell_2(v),$$

which is clearly linear. If $\ell \in V^\star$ is a linear function and $\lambda \in \mathbf{R}$, define $\lambda\ell$ by

$$(\lambda\ell)(v) := \lambda \cdot \ell(v),$$

then $\lambda\ell$ is clearly linear.

The vector space $V^\star$ is called the ***dual space*** of V.

Suppose V is n-dimensional, and let $e_1, \dots, e_n$ be a basis of V. Then every vector $v \in V$ can be written uniquely as a sum

$$v = c_1e_1 + \cdots + c_ne_n, \quad c_i \in \mathbf{R}.$$

Let

$$e_i^\star(v) = c_i. \tag{1.2.11}$$

If $v = c_1e_1 + \cdots + c_ne_n$ and $v' = c_1'e_1 + \cdots + c_n'e_n$ then $v+v' = (c_1+c_1')e_1 + \cdots + (c_n+c_n')e_n$, so

$$e_i^\star(v+v') = c_i + c_i' = e_i^\star(v) + e_i^\star(v').$$

This shows that $e_i^\star(v)$ is a linear function of V and hence $e_i^\star \in V^\star$.

Claim 1.2.12. *If V is an n-dimensional vector space with basis $e_1, \dots, e_n$, then $e_1^\star, \dots, e_n^\star$ is a basis of $V^\star$.*

Proof. First of all note that by (1.2.11)

$$e_i^\star(e_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{1.2.13}$$

If $\ell \in V^\star$ let $\lambda_i = \ell(e_i)$ and let $\ell' = \sum_{i=1}^n \lambda_i e_i^\star$. Then by (1.2.13)

$$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^\star(e_j) = \lambda_j = \ell(e_j),$$

i.e., ℓ and ℓ' take identical values on the basis vectors, e_j. Hence $\ell = \ell'$.

Suppose next that $\sum_{i=1}^n \lambda_i e_i^\star = 0$. Then by (1.2.13), $\lambda_j = (\sum_{i=1}^n \lambda_i e_i^\star)(e_j) = 0$ for $j = 1, \dots, n$. Hence the vectors $e_1^\star, \dots, e_n^\star$ are linearly independent. ☐

Let V and W be vector spaces and $A\colon V \to W$ a linear map. Given $\ell \in W^\star$ the composition, $\ell \circ A$, of A with the linear map, $\ell\colon W \to \mathbf{R}$, is linear, and hence is an element of $V^\star$. We will denote this element by $A^\star \ell$, and we will denote by

$$A^\star : W^\star \to V^\star$$

the map $\ell \mapsto A^\star \ell$. It is clear from the definition that

$$A^\star(\ell_1 + \ell_2) = A^\star \ell_1 + A^\star \ell_2$$

and that

$$A^\star(\lambda \ell) = \lambda A^\star \ell,$$

i.e., $A^\star$ is linear.

Definition 1.2.14. Let V and W be vector spaces and $A\colon V \to W$ a linear map. We call the map $A^\star\colon W^\star \to V^\star$ defined above the ***transpose*** of the map A.

We conclude this section by giving a matrix description of $A^\star$. Let $e_1, \dots, e_n$ be a basis of V and $f_1, \dots, f_m$ a basis of W; let $e_1^\star, \dots, e_n^\star$ and $f_1^\star, \dots, f_m^\star$ be the dual bases of $V^\star$ and $W^\star$. Suppose A is defined in terms of $e_1, \dots, e_n$ and $f_1, \dots, f_m$ by the $m \times n$ matrix, $(a_{i,j})$, i.e., suppose

$$Ae_j = \sum_{i=1}^{n} a_{i,j} f_i.$$

Claim 1.2.15. *The linear map $A^\star$ is defined, in terms of $f_1^\star, \dots, f_m^\star$ and $e_1^\star, \dots, e_n^\star$, by the transpose matrix $(a_{j,i})$.*

Proof. Let

$$A^\star f_i^\star = \sum_{j=1}^{m} c_{j,i} e_j^\star.$$

Then

$$A^\star f_i^\star(e_j) = \sum_{k=1}^{n} c_{k,i} e_k^\star(e_j) = c_{j,i}$$

by (1.2.13). On the other hand

$$A^\star f_i^\star(e_j) = f_i^\star(Ae_j) = f_i^\star\left(\sum_{k=1}^{m} a_{k,j} f_k\right) = \sum_{k=1}^{m} a_{k,j} f_i^\star(f_k) = a_{i,j}$$

so $a_{i,j} = c_{j,i}$. □

Exercises for §1.2

Exercise 1.2.i. Let V be an n-dimensional vector space and W a k-dimensional subspace. Show that there exists a basis $e_1, \dots, e_n$ of V with the property that $e_1, \dots, e_k$ is a basis of W.

Hint: Induction on $n - k$. To start the induction suppose that $n - k = 1$. Let $e_1, \dots, e_{n-1}$ be a basis of W and e_n any vector in $V \smallsetminus W$.

Exercise 1.2.ii. In Exercise 1.2.i show that the vectors $f_i := \pi(e_{k+i})$, $i = 1, \dots, n-k$ are a basis of V/W, where $\pi\colon V \to V/W$ is the quotient map.

Exercise 1.2.iii. In Exercise 1.2.i let U be the linear span of the vectors e_{k+i} for $i = 1, \dots, n-k$. Show that the map

$$U \to V/W, \quad u \mapsto \pi(u),$$

is a vector space isomorphism, i.e., show that it maps U bijectively onto V/W.

Exercise 1.2.iv. Let U, V and W be vector spaces and let $A\colon V \to W$ and $B\colon U \to V$ be linear mappings. Show that $(AB)^\star = B^\star A^\star$.

Exercise 1.2.v. Let $V = \mathbf{R}^2$ and let W be the x_1-axis, i.e., the one-dimensional subspace

$$\{(x_1, 0) \mid x_1 \in \mathbf{R}\}$$

of $\mathbf{R}^2$.

(1) Show that the W-cosets are the lines, $x_2 = a$, parallel to the x_1-axis.
(2) Show that the sum of the cosets "$x_2 = a$" and "$x_2 = b$" is the coset "$x_2 = a + b$".
(3) Show that the scalar multiple of the coset, "$x_2 = c$" by the number, λ, is the coset, "$x_2 = \lambda c$".

Exercise 1.2.vi.

(1) Let $(V^\star)^\star$ be the dual of the vector space, $V^\star$. For every $v \in V$, let $\mathrm{ev}_v\colon V^\star \to \mathbf{R}$ be the ***evaluation function*** $\mathrm{ev}_v(\ell) = \ell(v)$. Show that the ev_v is a linear function on $V^\star$, i.e., an element of $(V^\star)^\star$, and show that the map

$$\mathrm{ev} = \mathrm{ev}_{(-)}\colon V \to (V^\star)^\star, \quad v \mapsto \mathrm{ev}_v \tag{1.2.16}$$

is a linear map of V into $(V^\star)^\star$.

(2) If V is *finite dimensional*, show that the map (1.2.16) is bijective. Conclude that there is a *natural* identification of V with $(V^\star)^\star$, i.e., that V and $(V^\star)^\star$ are two descriptions of the same object.

Hint: $\dim(V^\star)^\star = \dim V^\star = \dim V$, so by equation (1.1.5) it suffices to show that (1.2.16) is injective.

Exercise 1.2.vii. Let W be a vector subspace of a finite-dimensional vector space V and let

$$W^\perp = \{\ell \in V^\star \mid \ell(w) = 0 \text{ for all } w \in W\}.$$

$W^\perp$ is called the ***annihilator*** of W in $V^\star$. Show that $W^\perp$ is a subspace of $V^\star$ of dimension $\dim V - \dim W$.

Hint: By Exercise 1.2.i we can choose a basis, $e_1, \dots, e_n$ of V such that $e_1, \dots, e_k$ is a basis of W. Show that $e^\star_{k+1}, \dots, e^\star_n$ is a basis of $W^\perp$.

Exercise 1.2.viii. Let V and V' be vector spaces and $A\colon V \to V'$ a linear map. Show that if $W \subset \ker(A)$, then there exists a linear map $B\colon V/W \to V'$ with the property that $A = B \circ \pi$ (where π is the quotient map (1.2.7)). In addition show that this linear map is injective if and only if $\ker(A) = W$.

Exercise 1.2.ix. Let W be a subspace of a finite-dimensional vector space V. From the inclusion map, $\iota\colon W^{\perp} \to V^{\star}$, one gets a transpose map,

$$\iota^{\star}\colon (V^{\star})^{\star} \to (W^{\perp})^{\star}$$

and, by composing this with (1.2.16), a map

$$\iota^{\star} \circ \mathrm{ev}\colon V \to (W^{\perp})^{\star}.$$

Show that this map is onto and that its kernel is W. Conclude from Exercise 1.2.viii that there is a *natural* bijective linear map

$$\nu\colon V/W \to (W^{\perp})^{\star}$$

with the property $\nu \circ \pi = \iota^{\star} \circ \mathrm{ev}$. In other words, V/W and $(W^{\perp})^{\star}$ are two descriptions of the same object. (This shows that the "quotient space" operation and the "dual space" operation are closely related.)

Exercise 1.2.x. Let V_1 and V_2 be vector spaces and $A\colon V_1 \to V_2$ a linear map. Verify that for the transpose map $A^{\star}\colon V_2^{\star} \to V_1^{\star}$ we have:

$$\ker(A^{\star}) = \mathrm{im}(A)^{\perp}$$

and

$$\mathrm{im}(A^{\star}) = \ker(A)^{\perp}.$$

Exercise 1.2.xi. Let V be a vector space.

(1) Let $B\colon V \times V \to \mathbf{R}$ be an inner product on V. For $v \in V$ let

$$\ell_v\colon V \to \mathbf{R}$$

be the function: $\ell_v(w) = B(v, w)$. Show that ℓ_v is linear and show that the map

$$L\colon V \to V^{\star}, \quad v \mapsto \ell_v \tag{1.2.17}$$

is a linear mapping.

(2) If V is finite dimensional, prove that L bijective. Conclude that if V has an inner product one gets from it a *natural* identification of V with $V^{\star}$.

Hint: Since $\dim V = \dim V^{\star}$ it suffices by equation (1.1.5) to show that $\ker(L) = 0$. Now note that if $v \neq 0$, $\ell_v(v) = B(v, v)$ is a positive number.

Exercise 1.2.xii. Let V be an n-dimensional vector space and $B\colon V \times V \to \mathbf{R}$ an inner product on V. A basis, $e_1, \ldots, e_n$ of V is ***orthonormal*** if

$$B(e_i, e_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{1.2.18}$$

(1) Show that an orthonormal basis exists.

Hint: By induction let $e_1, \ldots, e_k$ be vectors with the property (1.2.18) and let v be a vector which is not a linear combination of these vectors. Show that the vector

$$w = v - \sum_{i=1}^{k} B(e_i, v) e_i$$

is non-zero and is orthogonal to the e_i's. Now let $e_{k+1} = \lambda w$, where $\lambda = B(w,w)^{-\frac{1}{2}}$.

(2) Let $e_1, \dots e_n$ and $e_1', \dots e_n'$ be two orthonormal bases of V and let

$$e_j' = \sum_{i=1}^{n} a_{i,j} e_i.$$

Show that

$$\sum_{i=1}^{n} a_{i,j} a_{i,k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases} \tag{1.2.19}$$

(3) Let A be the matrix $(a_{i,j})$. Show that equation (1.2.19) can be written more compactly as the matrix identity

$$AA^{\intercal} = \mathrm{id}_n, \tag{1.2.20}$$

where id_n is the $n \times n$ identity matrix and $A^{\intercal}$ is the transpose of the matrix A.

(4) Let $e_1, \dots, e_n$ be an orthonormal basis of V and $e_1^\star, \dots, e_n^\star$ the dual basis of $V^\star$. Show that the mapping (1.2.17) is the mapping, $Le_i = e_i^\star$, $i = 1, \dots n$.

1.3. Tensors

Definition 1.3.1. Let V be an n-dimensional vector space and let V^k be the set of all k-tuples $(v_1, \dots, v_k)$, where $v_1, \dots, v_k \in V$, that is, the k-fold direct sum of V with itself. A function

$$T : V^k \to \mathbf{R}$$

is said to be ***linear in its*** i**th *variable*** if, when we fix vectors, $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$, the map $V \to \mathbf{R}$ defined by

$$v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k) \tag{1.3.2}$$

is linear.

If T is linear in its ith variable for $i = 1, \dots, k$ it is said to be ***k-linear***, or alternatively is said to be a ***k-tensor***. We write $\mathcal{L}^k(V)$ for the set of all k-tensors in V. We will agree that 0-tensors are just the real numbers, that is $\mathcal{L}^0(V) := \mathbf{R}$.

Let $T_1, T_2 \colon V^k \to \mathbf{R}$ be linear functions. It is clear from (1.3.2) that if T_1 and T_2 are k-linear, so is $T_1 + T_2$. Similarly if T is k-linear and λ is a real number, λT is k-linear. Hence $\mathcal{L}^k(V)$ is a vector space. Note that for $k = 1$, "k-linear" just means 'linear", so $\mathcal{L}^1(V) = V^\star$.

Definition 1.3.3. Let n and k be positive integers. A ***multi-index of n of length k*** is a k-tuple $I = (i_1, \dots, i_k)$ of integers with $1 \le i_r \le n$ for $r = 1, \dots, k$.

Example 1.3.4. Let n be a positive integer. The multi-indices of n of length 2 are in bijection the square of pairs of integers

$$(i,j)\,, \quad 1 \le i,j \le n,$$

and there are exactly n^2 of them.

We leave the following generalization as an easy exercise.

Lemma 1.3.5. *Let n and k be positive integers. There are exactly n^k multi-indices of n of length k.*

Now fix a basis $e_1, \dots, e_n$ of V. For $T \in \mathcal{L}^k(V)$ write

$$T_I := T(e_{i_1}, \dots, e_{i_k}) \tag{1.3.6}$$

for every multi-index I of length k.

Proposition 1.3.7. *The real numbers T_I determine T, i.e., if T and T' are k-tensors and $T_I = T'_I$ for all I, then $T = T'$.*

Proof. By induction on n. For $n = 1$ we proved this result in §1.1. Let's prove that if this assertion is true for $n-1$, it is true for n. For each e_i let T_i be the $(k-1)$-tensor

$$(v_1, \dots, v_{n-1}) \mapsto T(v_1, \dots, v_{n-1}, e_i).$$

Then for $v = c_1e_1 + \cdots + c_ne_n$

$$T(v_1, \dots, v_{n-1}, v) = \sum_{i=1}^{n} c_i T_i(v_1, \dots, v_{n-1}),$$

so the T_i's determine T. Now apply the inductive hypothesis. □

The tensor product operation

Definition 1.3.8. If T_1 is a k-tensor and T_2 is an ℓ-tensor, one can define a $k+\ell$-tensor, $T_1 \otimes T_2$, by setting

$$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k)T_2(v_{k+1}, \dots, v_{k+\ell}).$$

This tensor is called ***the tensor product*** of T_1 and T_2.

We note that if T_1 is a 0-tensor, i.e., scalar, then tensor product with T_1 is just scalar multiplication by T_1, and similarly if T_2 is a 0-tensor. That is $a \otimes T = T \otimes a = aT$ for $a \in \mathbf{R}$ and $T \in \mathcal{L}^k(V)$.

Properties 1.3.9. Suppose that we are given a k_1-tensor T_1, a k_2-tensor T_2, and a k_3-tensor T_3 on a vector space V.

(1) *Associativity*: One can define a $(k_1 + k_2 + k_3)$-tensor $T_1 \otimes T_2 \otimes T_3$ by setting

$$\begin{aligned}(T_1 \otimes T_2 \otimes T_3)&(v_1, \dots, v_{k+\ell+m}) \\ &:= T_1(v_1, \dots, v_k)T_2(v_{k+1}, \dots, v_{k+\ell})T_3(v_{k+\ell+1}, \dots, v_{k+\ell+m}).\end{aligned}$$

Alternatively, one can define $T_1 \otimes T_2 \otimes T_3$ by defining it to be the tensor product of $(T_1 \otimes T_2) \otimes T_3$ or the tensor product of $T_1 \otimes (T_2 \otimes T_3)$. It is easy to see that both these tensor products are identical with $T_1 \otimes T_2 \otimes T_3$:

$$(T_1 \otimes T_2) \otimes T_3 = T_1 \otimes T_2 \otimes T_3 = T_1 \otimes (T_2 \otimes T_3).$$

(2) *Distributivity of scalar multiplication:* We leave it as an easy exercise to check that if λ is a real number then

$$\lambda(T_1 \otimes T_2) = (\lambda T_1) \otimes T_2 = T_1 \otimes (\lambda T_2).$$

(3) *Left and right distributive laws:* If $k_1 = k_2$, then

$$(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$$

and if $k_2 = k_3$, then

$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3.$$

A particularly interesting tensor product is the following. For $i = 1, \dots, k$ let $\ell_i \in V^\star$ and let

$$T = \ell_1 \otimes \cdots \otimes \ell_k. \tag{1.3.10}$$

Thus, by definition,

$$T(v_1, \dots, v_k) = \ell_1(v_1) \cdots \ell_k(v_k). \tag{1.3.11}$$

A tensor of the form (1.3.11) is called a ***decomposable*** k-tensor. These tensors, as we will see, play an important role in what follows. In particular, let $e_1, \dots, e_n$ be a basis of V and $e_1^\star, \dots, e_n^\star$ the dual basis of $V^\star$. For every multi-index I of length k let

$$e_I^\star = e_{i_1}^\star \otimes \cdots \otimes e_{i_k}^\star.$$

Then if J is another multi-index of length k,

$$e_I^\star(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1, & I = J, \\ 0, & I \neq J \end{cases} \tag{1.3.12}$$

by (1.2.13), (1.3.10) and (1.3.11). From (1.3.12) it is easy to conclude the following.

Theorem 1.3.13. *Let V be a vector space with basis $e_1, \dots, e_n$ and let $0 \le k \le n$ be an integer. The k-tensors $e_I^\star$ of (1.3.12) are a basis of $\mathcal{L}^k(V)$.*

Proof. Given $T \in \mathcal{L}^k(V)$, let

$$T' = \sum_I T_I e_I^\star,$$

where the T_I's are defined by (1.3.6). Then

$$T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_I e_I^\star(e_{j_1}, \dots, e_{j_k}) = T_J \tag{1.3.14}$$

by (1.3.12); however, by Proposition 1.3.7 the T_J's determine T, so $T' = T$. This proves that the $e_I^\star$'s are a spanning set of vectors for $\mathcal{L}^k(V)$. To prove they're a basis, suppose

$$\sum_I C_I e_I^\star = 0$$

for constants, $C_I \in \mathbf{R}$. Then by (1.3.14) with $T' = 0$, $C_J = 0$, so the $e_I^\star$'s are linearly independent. □

As we noted in Lemma 1.3.5, there are exactly n^k multi-indices of length k and hence n^k basis vectors in the set $\{e_I^\star\}_I$, so we have proved.

Corollary 1.3.15. *Let V be an n-dimensional vector space. Then* $\dim(\mathcal{L}^k(V)) = n^k$.

The pullback operation

Definition 1.3.16. Let V and W be finite-dimensional vector spaces and let $A: V \to W$ be a linear mapping. If $T \in \mathcal{L}^k(W)$, we define

$$A^\star T: V^k \to \mathbf{R}$$

to be the function

$$(A^\star T)(v_1, \ldots, v_k) := T(Av_1, \ldots, Av_k). \tag{1.3.17}$$

It is clear from the linearity of A that this function is linear in its ith variable for all i, and hence is a k-tensor. We call $A^\star T$ the ***pullback*** of T by the map A.

Proposition 1.3.18. *The map*

$$A^\star: \mathcal{L}^k(W) \to \mathcal{L}^k(V), \quad T \mapsto A^\star T,$$

is a linear mapping.

We leave this as an exercise. We also leave as an exercise the identity

$$A^\star(T_1 \otimes T_2) = A^\star(T_1) \otimes A^\star(T_2) \tag{1.3.19}$$

for $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^m(W)$. Also, if U is a vector space and $B: U \to V$ a linear mapping, we leave for you to check that

$$(AB)^\star T = B^\star(A^\star T) \tag{1.3.20}$$

for all $T \in \mathcal{L}^k(W)$.

Exercises for § 1.3

Exercise 1.3.i. Verify that there are exactly n^k multi-indices of length k.

Exercise 1.3.ii. Prove Proposition 1.3.18.

Exercise 1.3.iii. Verify equation (1.3.19).

Exercise 1.3.iv. Verify equation (1.3.20).

Exercise 1.3.v. Let $A: V \to W$ be a linear map. Show that if ℓ_i, $i = 1, \ldots, k$ are elements of $W^\star$

$$A^\star(\ell_1 \otimes \cdots \otimes \ell_k) = A^\star(\ell_1) \otimes \cdots \otimes A^\star(\ell_k).$$

Conclude that $A^\star$ maps decomposable k-tensors to decomposable k-tensors.

Exercise 1.3.vi. Let V be an n-dimensional vector space and ℓ_i, $i = 1, 2$, elements of $V^\star$. Show that $\ell_1 \otimes \ell_2 = \ell_2 \otimes \ell_1$ if and only if ℓ_1 and ℓ_2 are linearly dependent.

Hint: Show that if ℓ_1 and ℓ_2 are linearly independent there exist vectors, v_i, $i =, 1, 2$ in V with property

$$\ell_i(v_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Now compare $(\ell_1 \otimes \ell_2)(v_1, v_2)$ and $(\ell_2 \otimes \ell_1)(v_1, v_2)$. Conclude that if $\dim V \geq 2$ the tensor product operation is not commutative, i.e., it is usually not true that $\ell_1 \otimes \ell_2 = \ell_2 \otimes \ell_1$.

Exercise 1.3.vii. Let T be a k-tensor and v a vector. Define $T_v : V^{k-1} \to \mathbf{R}$ to be the map

$$T_v(v_1, \dots, v_{k-1}) := T(v, v_1, \dots, v_{k-1}). \tag{1.3.21}$$

Show that T_v is a $(k-1)$-tensor.

Exercise 1.3.viii. Show that if T_1 is an r-tensor and T_2 is an s-tensor, then if $r > 0$,

$$(T_1 \otimes T_2)_v = (T_1)_v \otimes T_2.$$

Exercise 1.3.ix. Let $A : V \to W$ be a linear map mapping, and $v \in V$. Write $w := Av$. Show that for $T \in \mathcal{L}^k(W)$, $A^*(T_w) = (A^*T)_v$.

1.4. Alternating k-tensors

We will discuss in this section a class of k-tensors which play an important role in multivariable calculus. In this discussion we will need some standard facts about the "permutation group". For those of you who are already familiar with this object (and I suspect most of you are) you can regard the paragraph below as a chance to re-familiarize yourselves with these facts.

Permutations

Definition 1.4.1. Let Σ_k be the k-element set $\Sigma_k := \{1, 2, \dots, k\}$. A ***permutation of order*** k is a bijecton $\sigma : \Sigma_k \xrightarrow{\sim} \Sigma_k$. Given two permutations σ_1 and σ_2, their ***product*** $\sigma_1\sigma_2$ is the composition of $\sigma_1 \circ \sigma_2$, i.e., the map,

$$i \mapsto \sigma_1(\sigma_2(i)).$$

For every permutation σ, one denotes by σ^{-1} the inverse permutation given by the inverse bijection of σ, i.e., defined by

$$\sigma(i) := j \iff \sigma^{-1}(j) = i.$$

Let S_k be the set of all permutations of order k. One calls S_k the ***permutation group*** of Σ_k or, alternatively, the ***symmetric group on k letters***.

It is easy to check the following.

Lemma 1.4.2. *The group S_k has $k!$ elements.*

Definition 1.4.3. Let k be a positive integer. For every $1 \le i < j \le k$, let $\tau_{i,j}$ be the permutation

$$\tau_{i,j}(\ell) := \begin{cases} j, & \ell = i, \\ i, & \ell = j, \\ \ell, & \ell \neq i, j. \end{cases}$$

The permutation $\tau_{i,j}$ is called a ***transposition***, and if $j = i + 1$, then $\tau_{i,j}$ is called an ***elementary transposition***.

Theorem 1.4.4. *Every permutation in S_k can be written as a product of (a finite number of) transpositions.*

Proof. We prove this by induction on k. The base case when $k = 2$ is obvious.

For the induction step, suppose that we know the claim for S_{k-1}. Given $\sigma \in S_k$, we have $\sigma(k) = i$ if and only if $\tau_{i,k}\sigma(k) = k$. Thus $\tau_{i,k}\sigma$ is, in effect, a permutation of Σ_{k-1}. By induction, $\tau_{i,k}\sigma$ can be written as a product of transpositions, so

$$\sigma = \tau_{i,k}(\tau_{i,k}\sigma)$$

can be written as a product of transpositions. □

Theorem 1.4.5. *Every transposition can be written as a product of elementary transpositions.*

Proof. Let $\tau = \tau_{ij}$, $i < j$. With i fixed, argue by induction on j. Note that for $j > i + 1$

$$\tau_{ij} = \tau_{j-1,j}\tau_{i,j-1}\tau_{j-1,j}.$$

Now apply the inductive hypothesis to $\tau_{i,j-1}$. □

Corollary 1.4.6. *Every permutation can be written as a product of elementary transpositions.*

The sign of a permutation

Definition 1.4.7. Let $x_1, \dots, x_k$ be the coordinate functions on $\mathbf{R}^k$. For $\sigma \in S_k$ we define

$$(-1)^\sigma := \prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}. \tag{1.4.8}$$

Notice that the numerator and denominator in (1.4.8) are identical up to sign. Indeed, if $p = \sigma(i) < \sigma(j) = q$, the term, $x_p - x_q$, occurs once and just once in the numerator and once and just once in the denominator; and if $q = \sigma(i) > \sigma(j) = p$, the term, $x_p - x_q$, occurs once and just once in the numerator and its negative, $x_q - x_p$, once and just once in the numerator. Thus

$$(-1)^\sigma = \pm 1.$$

In light of this, we call $(-1)^\sigma$ the ***sign*** of σ.

Claim 1.4.9. *For $\sigma, \tau \in S_k$ we have*

$$(-1)^{\sigma\tau} = (-1)^\sigma(-1)^\tau.$$

That is, the sign defines a group homomorphism $S_k \to \{\pm 1\}$.

Proof. By definition,

$$(-1)^{\sigma\tau} = \prod_{i<j} \frac{x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{x_i - x_j}.$$

We write the right-hand side as a product of

$$\prod_{i<j} \frac{x_{\tau(i)} - x_{\tau(j)}}{x_i - x_j} = (-1)^\tau$$

and

$$\prod_{i<j} \frac{x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{x_{\tau(i)} - x_{\tau(j)}}. \tag{1.4.10}$$

For $i < j$, let $p = \tau(i)$ and $q = \tau(j)$ when $\tau(i) < \tau(j)$ and let $p = \tau(j)$ and $q = \tau(i)$ when $\tau(j) < \tau(i)$. Then

$$\frac{x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{x_{\tau(i)} - x_{\tau(j)}} = \frac{x_{\sigma(p)} - x_{\sigma(q)}}{x_p - x_q}$$

(i.e., if $\tau(i) < \tau(j)$, the numerator and denominator on the right *equal* the numerator and denominator on the left and, if $\tau(j) < \tau(i)$ are *negatives* of the numerator and denominator on the left). Thus equation (1.4.10) becomes

$$\prod_{p<q} \frac{x_{\sigma(p)} - x_{\sigma(q)}}{x_p - x_q} = (-1)^\sigma.$$ □

We'll leave for you to check that if τ is a transposition, $(-1)^\tau = -1$ and to deduce the following proposition.

Proposition 1.4.11. *If σ is the product of an odd number of transpositions, $(-1)^\sigma = -1$ and if σ is the product of an even number of transpositions, $(-1)^\sigma = +1$.*

Alternation

Definition 1.4.12. Let V be an n-dimensional vector space and $T \in \mathcal{L}^k(v)$ a k-tensor. For $\sigma \in S_k$, define $T^\sigma \in \mathcal{L}^k(V)$ to be the k-tensor

$$T^\sigma(v_1, \dots, v_k) := T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}). \tag{1.4.13}$$

Proposition 1.4.14.

(1) *If $T = \ell_1 \otimes \cdots \otimes \ell_k$, $\ell_i \in V^\star$, then $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$.*
(2) *The assignment $T \mapsto T^\sigma$ is a linear map $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$.*
(3) *If $\sigma, \tau \in S_k$, we have $T^{\sigma\tau} = (T^\sigma)^\tau$.*

Proof. To (1), we note that by equation (1.4.13)

$$(\ell_1 \otimes \cdots \otimes \ell_k)^\sigma(v_1, \dots, v_k) = \ell_1(v_{\sigma^{-1}(1)}) \cdots \ell_k(v_{\sigma^{-1}(k)}).$$

Setting $\sigma^{-1}(i) = q$, the ith term in this product is $\ell_{\sigma(q)}(v_q)$; so the product can be rewritten as

$$\ell_{\sigma(1)}(v_1) \dots \ell_{\sigma(k)}(v_k)$$

or

$$(\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)})(v_1, \dots, v_k).$$

We leave the proof of (2) as an exercise.

Now we prove (3). Let $T = \ell_1 \otimes \cdots \otimes \ell_k$. Then

$$T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)} = \ell'_1 \otimes \cdots \otimes \ell'_k,$$

where $\ell_j' = \ell_{\sigma(j)}$. Thus

$$(T^\sigma)^\tau = \ell'_{\tau(1)} \otimes \cdots \otimes \ell'_{\tau(k)}.$$

But if $\tau(i) = j$, $\ell'_{\tau(j)} = \ell_{\sigma(\tau(j))}$. Hence

$$(T^\sigma)^\tau \ell_{\sigma\tau(1)} \otimes \cdots \otimes \ell_{\sigma\tau(k)} = T^{\sigma\tau}. \qquad \square$$

Definition 1.4.15. Let V be a vector space and $k \geq 0$ an integer. A k-tensor $T \in \mathcal{L}^k(V)$ is ***alternating*** if $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.

We denote by $\mathcal{A}^k(V)$ the set of all alternating k-tensors in $\mathcal{L}^k(V)$. By (2) this set is a vector subspace of $\mathcal{L}^k(V)$.

It is not easy to write down simple examples of alternating k-tensors. However, there is a method, called the *alternation operation* Alt for constructing such tensors.

Definition 1.4.16. Let V be a vector space and k a non-negative integer. The ***alternation operation*** on $\mathcal{L}^k(V)$ is defined as follows: given $T \in \mathcal{L}^k(V)$ let

$$\mathrm{Alt}(T) := \sum_{\tau \in S_k} (-1)^\tau T^\tau.$$

The alternation operation enjoys the following properties.

Proposition 1.4.17. *For $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$,*

(1) $\mathrm{Alt}(T)^\sigma = (-1)^\sigma \,\mathrm{Alt}\, T$;
(2) *if $T \in \mathcal{A}^k(V)$, then* $\mathrm{Alt}\, T = k!\, T$;
(3) $\mathrm{Alt}(T^\sigma) = \mathrm{Alt}(T)^\sigma$;
(4) *the map*

$$\mathrm{Alt}\colon \mathcal{L}^k(V) \to \mathcal{L}^k(V), \quad T \mapsto \mathrm{Alt}(T)$$

is linear.

Proof. To prove (1) we note that by Proposition 1.4.14:

$$\mathrm{Alt}(T)^\sigma = \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} = (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma}.$$

But as τ runs over S_k, $\tau\sigma$ runs over S_k, and hence the right-hand side is $(-1)^\sigma \,\mathrm{Alt}(T)$.

To prove (2), note that if $T \in \mathcal{A}^k(V)$

$$\mathrm{Alt}\, T = \sum_{\tau \in S_k} (-1)^\tau T^\tau = \sum_{\tau \in S_k} (-1)^\tau (-1)^\tau T = k!\, T.$$

To prove (3), we compute:

$$\mathrm{Alt}(T^\sigma) = \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} = (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma}$$
$$= (-1)^\sigma \,\mathrm{Alt}(T) = \mathrm{Alt}(T)^\sigma.$$

Finally, (4) is an easy corollary of (2) of Proposition 1.4.14. $\square$

We will use this alternation operation to construct a basis for $\mathcal{A}^k(V)$. First, however, we require some notation.

Definition 1.4.18. Let $I = (i_1, \dots, i_k)$ be a multi-index of length k.

(1) I is ***repeating*** if $i_r = i_s$ for some $r \neq s$.
(2) I is ***strictly increasing*** if $i_1 < i_2 < \cdots < i_r$.
(3) For $\sigma \in S_k$, write $I^\sigma := (i_{\sigma(1)}, \dots, i_{\sigma(k)})$.

Remark 1.4.19. If I is non-repeating there is a unique $\sigma \in S_k$ so that I^σ is strictly increasing.

Let $e_1, \dots, e_n$ be a basis of V and let

$$e_I^\star = e_{i_1}^\star \otimes \cdots \otimes e_{i_k}^\star$$

and

$$\psi_I = \operatorname{Alt}(e_I^\star).$$

Proposition 1.4.20.

(1) $\psi_{I^\sigma} = (-1)^\sigma \psi_I$.
(2) *If I is repeating, $\psi_I = 0$.*
(3) *If I and J are strictly increasing,*

$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1, & I = J, \\ 0, & I \neq J. \end{cases}$$

Proof. To prove (1) we note that $(e_I^\star)^\sigma = e_{I^\sigma}^\star$; so

$$\operatorname{Alt}(e_{I^\sigma}^\star) = \operatorname{Alt}(e_I^\star)^\sigma = (-1)^\sigma \operatorname{Alt}(e_I^\star).$$

To prove (2), suppose $I = (i_1, \dots, i_k)$ with $i_r = i_s$ for $r \neq s$. Then if $\tau = \tau_{i_r, i_s}$, $e_I^\star = e_{I^\tau}^\star$ so

$$\psi_I = \psi_{I^\tau} = (-1)^\tau \psi_I = -\psi_I.$$

To prove (3), note that by definition

$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \sum_\tau (-1)^\tau e_{I^\tau}^\star(e_{j_1}, \dots, e_{j_k}).$$

But by (1.3.12)

$$e_{I^\tau}^\star(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1, & I^\tau = J, \\ 0, & I^\tau \neq J. \end{cases} \tag{1.4.21}$$

Thus if I and J are strictly increasing, I^τ is strictly increasing if and only if $I^\tau = I$, and (1.4.21) is non-zero if and only if $I = J$. □

Now let T be in $\mathcal{A}^k(V)$. By Theorem 1.3.13,

$$T = \sum_J a_J e_J^\star, \quad a_J \in \mathbf{R}.$$

Since $k!\,T = \operatorname{Alt}(T)$,

$$T = \frac{1}{k!} \sum a_J \operatorname{Alt}(e_J^\star) = \sum b_J \psi_J.$$

We can discard all repeating terms in this sum since they are zero; and for every non-repeating term, J, we can write $J = I^\sigma$, where I is strictly increasing, and hence $\psi_J = (-1)^\sigma \psi_I$.

Conclusion 1.4.22. *We can write T as a sum*

$$T = \sum_I c_I \psi_I \tag{1.4.23}$$

with I's strictly increasing.

Claim 1.4.24. *The c_I's are unique.*

Proof. For J strictly increasing

$$T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I \psi_I(e_{j_1}, \dots, e_{j_k}) = c_J. \tag{1.4.25}$$

By (1.4.23) the ψ_I's, I strictly increasing, are a spanning set of vectors for $\mathcal{A}^k(V)$, and by (1.4.25) they are linearly independent, so we have proved.

Proposition 1.4.26. *The alternating tensors, ψ_I, I strictly increasing, are a basis for $\mathcal{A}^k(V)$.*

Thus $\dim \mathcal{A}^k(V)$ is equal to the number of strictly increasing multi-indices I of length k. We leave for you as an exercise to show that this number is equal to the binomial coefficient

$$\binom{n}{k} := \frac{n!}{(n-k)!\,k!} \tag{1.4.27}$$

if $1 \le k \le n$. □

Hint: Show that every strictly increasing multi-index of length k determines a k element subset of $\{1, \dots, n\}$ and vice versa.

Note also that if $k > n$ every multi-index

$$I = (i_1, \dots, i_k)$$

of length k has to be repeating: $i_r = i_s$ for some $r \neq s$ since the i_p's lie on the interval $1 \le i \le n$. Thus by Proposition 1.4.17

$$\psi_I = 0$$

for all multi-indices of length $k > 0$ and

$$\mathcal{A}^k(V) = 0.$$

Exercises for § 1.4

Exercise 1.4.i. Show that there are exactly $k!$ permutations of order k.

Hint: Induction on k: Let $\sigma \in S_k$, and let $\sigma(k) = i$, $1 \le i \le k$. Show that $\tau_{i,k}\sigma$ leaves k fixed and hence is, in effect, a permutation of Σ_{k-1}.

Exercise 1.4.ii. Prove that if $\tau \in S_k$ is a transposition, $(-1)^\tau = -1$ and deduce from this Proposition 1.4.11.

Exercise 1.4.iii. Prove assertion (2) in Proposition 1.4.14.

Exercise 1.4.iv. Prove that $\dim \mathcal{A}^k(V)$ is given by (1.4.27).

Exercise 1.4.v. Verify that for $i < j - 1$

$$\tau_{i,j} = \tau_{j-1,j}\tau_{i,j-1}, \tau_{j-1,j}.$$

Exercise 1.4.vi. For $k = 3$ show that every one of the six elements of S_3 is either a transposition or can be written as a product of two transpositions.

Exercise 1.4.vii. Let $\sigma \in S_k$ be the "cyclic" permutation

$$\sigma(i) := i + 1, \quad i = 1, \dots, k - 1$$

and $\sigma(k) := 1$. Show explicitly how to write σ as a product of transpositions and compute $(-1)^\sigma$.

Hint: Same hint as in Exercise 1.4.i.

Exercise 1.4.viii. In Exercise 1.3.vii show that if T is in $\mathcal{A}^k(V)$, T_v is in $\mathcal{A}^{k-1}(V)$. Show in addition that for $v, w \in V$ and $T \in \mathcal{A}^k(V)$ we have $(T_v)_w = -(T_w)_v$.

Exercise 1.4.ix. Let $A: V \to W$ be a linear mapping. Show that if T is in $\mathcal{A}^k(W)$, $A^\star T$ is in $\mathcal{A}^k(V)$.

Exercise 1.4.x. In Exercise 1.4.ix show that if T is in $\mathcal{L}^k(W)$ then $\text{Alt}(A^\star T) = A^\star(\text{Alt}(T))$, i.e., show that the Alt operation commutes with the pullback operation.

1.5. The space $\Lambda^k(V^\star)$

In § 1.4 we showed that the image of the alternation operation, $\text{Alt}: \mathcal{L}^k(V) \to \mathcal{L}^k(V)$, is $\mathcal{A}^k(V)$. In this section we will compute the kernel of Alt.

Definition 1.5.1. A decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$, with $\ell_1, \dots, \ell_k \in V^\star$, is ***redundant*** if for some index i we have $\ell_i = \ell_{i+1}$.

Let $\mathcal{I}^k(V) \subset \mathcal{L}^k(V)$ be the linear span of the set of redundant k-tensors.

Note that for $k = 1$ the notion of redundant does not really make sense; a single vector $\ell \in \mathcal{L}^1(V^\star)$ cannot be "redundant" so we decree

$$\mathcal{I}^1(V) := 0.$$

Proposition 1.5.2. *If* $T \in \mathcal{I}^k(V)$ *then* $\text{Alt}(T) = 0$.

Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ with $\ell_i = \ell_{i+1}$. Then if $\tau = \tau_{i,i+1}$, $T^\tau = T$ and $(-1)^\tau = -1$. Hence $\text{Alt}(T) = \text{Alt}(T^\tau) = \text{Alt}(T)^\tau = -\text{Alt}(T)$; so $\text{Alt}(T) = 0$. □

Proposition 1.5.3. *If* $T \in \mathcal{I}^r(V)$ *and* $T' \in \mathcal{L}^s(V)$, *then* $T \otimes T'$ *and* $T' \otimes T$ *are in* $\mathcal{I}^{r+s}(V)$.

Proof. We can assume that T and T' are decomposable, i.e., $T = \ell_1 \otimes \cdots \otimes \ell_r$ and $T' = \ell_1' \otimes \cdots \otimes \ell_s'$ and that T is redundant: $\ell_i = \ell_{i+1}$. Then

$$T \otimes T' = \ell_1 \otimes \cdots \ell_{i-1} \otimes \ell_i \otimes \ell_i \otimes \cdots \ell_r \otimes \ell_1' \otimes \cdots \otimes \ell_s'$$

is redundant and hence in $\mathcal{I}^{r+s}$. The argument for $T' \otimes T$ is similar. □

Proposition 1.5.4. *If $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, then*

$$T^\sigma = (-1)^\sigma T + S, \tag{1.5.5}$$

where S is in $\mathcal{I}^k(V)$.

Proof. We can assume T is decomposable, i.e., $T = \ell_1 \otimes \cdots \otimes \ell_k$. Let's first look at the simplest possible case: $k = 2$ and $\sigma = \tau_{1,2}$. Then

$$\begin{aligned} T^\sigma - (-1)^\sigma T &= \ell_1 \otimes \ell_2 + \ell_2 \otimes \ell_1 \\ &= \frac{1}{2}((\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2), \end{aligned}$$

and the terms on the right are redundant, and hence in $\mathcal{I}^2(V)$. Next let k be arbitrary and $\sigma = \tau_{i,i+1}$. If $T_1 = \ell_1 \otimes \cdots \otimes \ell_{i-2}$ and $T_2 = \ell_{i+2} \otimes \cdots \otimes \ell_k$, then

$$T - (-1)^\sigma T = T_1 \otimes (\ell_i \otimes \ell_{i+1} + \ell_{i+1} \otimes \ell_i) \otimes T_2$$

is in $\mathcal{I}^k(V)$ by Proposition 1.5.3 and the computation above.

The general case: By Theorem 1.4.5, σ can be written as a product of m elementary transpositions, and we'll prove (1.5.5) by induction on m.

We have just dealt with the case $m = 1$.

The induction step: the "$m - 1$" case implies the "m" case. Let $\sigma = \tau\beta$ where β is a product of $m - 1$ elementary transpositions and τ is an elementary transposition. Then

$$\begin{aligned} T^\sigma &= (T^\beta)^\tau = (-1)^\tau T^\beta + \cdots \\ &= (-1)^\tau (-1)^\beta T + \cdots \\ &= (-1)^\sigma T + \cdots, \end{aligned}$$

where the "dots" are elements of $\mathcal{I}^k(V)$, and the induction hypothesis was used in the second line. □

Corollary 1.5.6. *If $T \in \mathcal{L}^k(V)$, then*

$$\mathrm{Alt}(T) = k!\,T + W, \tag{1.5.7}$$

where W is in $\mathcal{I}^k(V)$.

Proof. By definition $\mathrm{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$, and by Proposition 1.5.4, $T^\sigma = (-1)^\sigma T + W_\sigma$, with $W_\sigma \in \mathcal{I}^k(V)$. Thus

$$\mathrm{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma (-1)^\sigma T + \sum_{\sigma \in S_k} (-1)^\sigma W_\sigma = k!\,T + W,$$

where $W = \sum_{\sigma \in S_k} (-1)^\sigma W_\sigma$. □

Corollary 1.5.8. *Let V be a vector space and $k \geq 1$. Then*

$$\mathcal{I}^k(V) = \ker(\mathrm{Alt}\colon \mathcal{L}^k(V) \to \mathcal{A}^k(V)).$$

Proof. We have already proved that if $T \in \mathcal{I}^k(V)$, then $\mathrm{Alt}(T) = 0$. To prove the converse assertion we note that if $\mathrm{Alt}(T) = 0$, then by (1.5.7)

$$T = -\frac{1}{k!}W$$

with $W \in \mathcal{I}^k(V)$. □

Putting these results together we conclude the following.

Theorem 1.5.9. *Every element $T \in \mathcal{L}^k(V)$ can be written uniquely as a sum $T = T_1 + T_2$, where $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{I}^k(V)$.*

Proof. By (1.5.7), $T = T_1 + T_2$ with

$$T_1 = \frac{1}{k!}\,\mathrm{Alt}(T)$$

and

$$T_2 = -\frac{1}{k!}W.$$

To prove that this decomposition is unique, suppose $T_1 + T_2 = 0$, with $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{I}^k(V)$. Then

$$0 = \mathrm{Alt}(T_1 + T_2) = k!\,T_1$$

so $T_1 = 0$, and hence $T_2 = 0$. □

Definition 1.5.10. Let V be a finite-dimensional vector space and $k \geq 0$. Define

$$\Lambda^k(V^\star) := \mathcal{L}^k(V)/\mathcal{I}^k(V), \tag{1.5.11}$$

i.e., let $\Lambda^k(V^\star)$ be the quotient of the vector space $\mathcal{L}^k(V)$ by the subspace $\mathcal{I}^k(V)$. By (1.2.7) one has a linear map:

$$\pi\colon \mathcal{L}^k(V) \to \Lambda^k(V^\star), \quad T \mapsto T + \mathcal{I}^k(V), \tag{1.5.12}$$

which is onto and has $\mathcal{I}^k(V)$ as kernel.

Theorem 1.5.13. *The map $\pi\colon \mathcal{L}^k(V) \to \Lambda^k(V^\star)$ maps $\mathcal{A}^k(V)$ bijectively onto $\Lambda^k(V^\star)$.*

Proof. By Theorem 1.5.9 every $\mathcal{I}^k(V)$ coset $T + \mathcal{I}^k(V)$ contains a unique element T_1 of $\mathcal{A}^k(V)$. Hence for every element of $\Lambda^k(V^\star)$ there is a unique element of $\mathcal{A}^k(V)$ which gets mapped onto it by π. □

Remark 1.5.14. Since $\Lambda^k(V^\star)$ and $\mathcal{A}^k(V)$ are isomorphic as vector spaces many treatments of multilinear algebra avoid mentioning $\Lambda^k(V^\star)$, reasoning that $\mathcal{A}^k(V)$ is a perfectly good substitute for it and that one should, if possible, not make two different definitions for what is essentially the same object. This is a justifiable point of view (and is the point of view taken by in [**4**, **10**, **12**]). There are, however, some advantages to distinguishing between $\mathcal{A}^k(V)$ and $\Lambda^k(V^\star)$, as we shall see in § 1.6.

Exercises for § 1.5

Exercise 1.5.i. A k-tensor $T \in \mathcal{L}^k(V)$ is ***symmetric*** if $T^\sigma = T$ for all $\sigma \in S_k$. Show that the set $\mathcal{S}^k(V)$ of symmetric k tensors is a vector subspace of $\mathcal{L}^k(V)$.

Exercise 1.5.ii. Let $e_1, \dots, e_n$ be a basis of V. Show that every symmetric 2-tensor is of the form

$$\sum_{1 \le i,j \le n} a_{i,j} e_i^\star \otimes e_j^\star ,$$

where $a_{i,j} = a_{j,i}$ and $e_1^\star, \dots, e_n^\star$ are the dual basis vectors of $V^\star$.

Exercise 1.5.iii. Show that if T is a symmetric k-tensor, then, for $k \ge 2$, T is in $\mathcal{I}^k(V)$.

Hint: Let σ be a transposition and deduce from the identity, $T^\sigma = T$, that T has to be in the kernel of Alt.

Exercise 1.5.iv (a warning). In general $\mathcal{S}^k(V) \neq \mathcal{I}^k(V)$. Show, however, that if $k = 2$ these two spaces are equal.

Exercise 1.5.v. Show that if $\ell \in V^\star$ and $T \in \mathcal{L}^{k-2}(V)$, then $\ell \otimes T \otimes \ell$ is in $\mathcal{I}^k(V)$.

Exercise 1.5.vi. Show that if ℓ_1 and ℓ_2 are in $V^\star$ and $T \in \mathcal{L}^{k-2}(V)$, then

$$\ell_1 \otimes T \otimes \ell_2 + \ell_2 \otimes T \otimes \ell_1$$

is in $\mathcal{I}^k(V)$.

Exercise 1.5.vii. Given a permutation $\sigma \in S_k$ and $T \in \mathcal{I}^k(V)$, show that $T^\sigma \in \mathcal{I}^k(V)$.

Exercise 1.5.viii. Let $\mathcal{W}(V)$ be a subspace of $\mathcal{L}^k(V)$ having the following two properties.

(1) For $S \in \mathcal{S}^2(V)$ and $T \in \mathcal{L}^{k-2}(V)$, $S \otimes T$ is in $\mathcal{W}(V)$.
(2) For T in $\mathcal{W}(V)$ and $\sigma \in S_k$, T^σ is in $\mathcal{W}(V)$.

Show that $\mathcal{W}(V)$ has to contain $\mathcal{I}^k(V)$ and conclude that $\mathcal{I}^k(V)$ is the smallest subspace of $\mathcal{L}^k(V)$ having properties (1) and (2).

Exercise 1.5.ix. Show that there is a bijective linear map

$$\alpha \colon \Lambda^k(V^\star) \xrightarrow{\sim} \mathcal{A}^k(V)$$

with the property

$$\alpha\pi(T) = \frac{1}{k!}\,\mathrm{Alt}(T) \tag{1.5.15}$$

for all $T \in \mathcal{L}^k(V)$, and show that α is the inverse of the map of $\mathcal{A}^k(V)$ onto $\Lambda^k(V^\star)$ described in Theorem 1.5.13.

Hint: Exercise 1.2.viii.

Exercise 1.5.x. Let V be an n-dimensional vector space. Compute the dimension of $\mathcal{S}^k(V)$.

Hints:

(1) Introduce the following symmetrization operation on tensors $T \in \mathcal{L}^k(V)$:

$$\mathrm{Sym}(T) = \sum_{\tau \in S_k} T^\tau.$$

Prove that this operation has properties (2)–(4) of Proposition 1.4.17 and, as a substitute for (1), has the property: $\mathrm{Sym}(T)^\sigma = \mathrm{Sym}(T)$.

(2) Let $\phi_I = \mathrm{Sym}(e_I^\star)$, $e_I^\star = e_{i_1}^\star \otimes \cdots \otimes e_{i_n}^\star$. Prove that $\{\phi_I \mid I \text{ is non-decreasing}\}$ form a basis of $S^k(V)$.

(3) Conclude that $\dim(\mathcal{S}^k(V))$ is equal to the number of non-decreasing multi-indices of length k: $1 \le i_1 \le i_2 \le \cdots \le \ell_k \le n$.

(4) Compute this number by noticing that the assignment

$$(i_1, \ldots, i_k) \mapsto (i_1 + 0, i_2 + 1, \ldots, i_k + k - 1)$$

is a bijection between the set of these non-decreasing multi-indices and the set of increasing multi-indices $1 \le j_1 < \cdots < j_k \le n + k - 1$.

1.6. The wedge product

The tensor algebra operations on the spaces $\mathcal{L}^k(V)$ which we discussed in §§ 1.2 and 1.3, i.e., the "tensor product operation" and the "pullback" operation, give rise to similar operations on the spaces, $\Lambda^k(V^\star)$. We will discuss in this section the analogue of the tensor product operation.

Definition 1.6.1. Given $\omega_i \in \Lambda^{k_i}(V^\star)$, $i = 1, 2$ we can, by equation (1.5.12), find a $T_i \in \mathcal{L}^{k_i}(V)$ with $\omega_i = \pi(T_i)$. Then $T_1 \otimes T_2 \in \mathcal{L}^{k_1+k_2}(V)$. The ***wedge product*** $\omega_1 \wedge \omega_2$ is defined by

$$\omega_1 \wedge \omega_2 \coloneqq \pi(T_1 \otimes T_2) \in \Lambda^{k_1+k_2}(V^\star). \tag{1.6.2}$$

Claim 1.6.3. *This wedge product is well defined, i.e., does not depend on our choices of T_1 and T_2.*

Proof. Let $\pi(T_1) = \pi(T_1') = \omega_1$. Then $T_1' = T_1 + W_1$ for some $W_1 \in \mathcal{I}^{k_1}(V)$, so

$$T_1' \otimes T_2 = T_1 \otimes T_2 + W_1 \otimes T_2.$$

But $W_1 \in \mathcal{I}^{k_1}(V)$ implies $W_1 \otimes T_2 \in \mathcal{I}^{k_1+k_2}(V)$ and this implies:

$$\pi(T_1' \otimes T_2) = \pi(T_1 \otimes T_2).$$

A symmetric argument shows that $\omega_1 \wedge \omega_2$ is well-defined, independent of the choice of T_2. □

More generally let $\omega_i \in \Lambda^{k_i}(V^\star)$, for $i = 1, 2, 3$, and let $\omega_i = \pi(T_i)$, $T_i \in \mathcal{L}^{k_i}(V)$. Define

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \in \Lambda^{k_1+k_2+k_3}(V^\star)$$

by setting

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = \pi(T_1 \otimes T_2 \otimes T_3).$$

As above it is easy to see that this is well-defined independent of the choice of T_1, T_2 and T_3. It is also easy to see that this triple wedge product is just the wedge product of $\omega_1 \wedge \omega_2$ with ω_3 or, alternatively, the wedge product of ω_1 with $\omega_2 \wedge \omega_3$, i.e.,

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3).$$

We leave for you to check: for $\lambda \in \mathbf{R}$

(1.6.4) $$\lambda(\omega_1 \wedge \omega_2) = (\lambda\omega_1) \wedge \omega_2 = \omega_1 \wedge (\lambda\omega_2)$$

and verify the two distributive laws:

(1.6.5) $$(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3$$

and

(1.6.6) $$\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3.$$

As we noted in § 1.4, $\mathcal{I}^k(V) = 0$ for $k = 1$, i.e., there are no non-zero "redundant" k tensors in degree $k = 1$. Thus

$$\Lambda^1(V^\star) = V^\star = \mathcal{L}^1(V).$$

A particularly interesting example of a wedge product is the following. Let $\ell_1, \dots, \ell_k \in V^\star = \Lambda^1(V^\star)$. If $T = \ell_1 \otimes \dots \otimes \ell_k$, then

(1.6.7) $$\ell_1 \wedge \dots \wedge \ell_k = \pi(T) \in \Lambda^k(V^\star).$$

We will call (1.6.7) a ***decomposable element*** of $\Lambda^k(V^\star)$.

We will prove that these elements satisfy the following wedge product identity. For $\sigma \in S_k$:

(1.6.8) $$\ell_{\sigma(1)} \wedge \dots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \dots \wedge \ell_k.$$

Proof. For every $T \in \mathcal{L}^k(V)$, $T = (-1)^\sigma T + W$ for some $W \in \mathcal{I}^k(V)$ by Proposition 1.5.4. Therefore since $\pi(W) = 0$

$$\pi(T^\sigma) = (-1)^\sigma \pi(T).$$

In particular, if $T = \ell_1 \otimes \dots \otimes \ell_k$, $T^\sigma = \ell_{\sigma(1)} \otimes \dots \otimes \ell_{\sigma(k)}$, so

$$\begin{aligned}\pi(T^\sigma) &= \ell_{\sigma(1)} \wedge \dots \wedge \ell_{\sigma(k)} = (-1)^\sigma \pi(T) \\ &= (-1)^\sigma \ell_1 \wedge \dots \wedge \ell_k.\end{aligned}$$ □

In particular, for ℓ_1 and $\ell_2 \in V^\star$

(1.6.9) $$\ell_1 \wedge \ell_2 = -\ell_2 \wedge \ell_1$$

and for ℓ_1, ℓ_2 and $\ell_3 \in V^\star$

$$\ell_1 \wedge \ell_2 \wedge \ell_3 = -\ell_2 \wedge \ell_1 \wedge \ell_3 = \ell_2 \wedge \ell_3 \wedge \ell_1.$$

More generally, it is easy to deduce from equation (1.6.8) the following result (which we'll leave as an exercise).

Theorem 1.6.10. *If* $\omega_1 \in \Lambda^r(V^\star)$ *and* $\omega_2 \in \Lambda^s(V^\star)$, *then*

(1.6.11) $$\omega_1 \wedge \omega_2 = (-1)^{rs} \omega_2 \wedge \omega_1.$$

Hint: It suffices to prove this for decomposable elements i.e., for $\omega_1 = \ell_1 \wedge \cdots \wedge \ell_r$ and $\omega_2 = \ell'_1 \wedge \cdots \wedge \ell'_s$. Now make rs applications of (1.6.9).

Let $e_1, \ldots, e_n$ be a basis of V and let $e_1^\star, \ldots, e_n^\star$ be the dual basis of $V^\star$. For every multi-index I of length k,

$$e_{i_1}^\star \wedge \cdots \wedge e_{i_k}^\star = \pi(e_I^\star) = \pi(e_{i_1}^\star \otimes \cdots \otimes e_{i_k}^\star). \tag{1.6.12}$$

Theorem 1.6.13. *The elements* (1.6.12), *with I strictly increasing, are basis vectors of $\Lambda^k(V^\star)$.*

Proof. The elements $\psi_I = \operatorname{Alt}(e_I^\star)$, for I strictly increasing, are basis vectors of $\mathcal{A}^k(V)$ by Proposition 1.4.26; so their images, $\pi(\psi_I)$, are a basis of $\Lambda^k(V^\star)$. But

$$\begin{aligned}\pi(\psi_I) = \pi\left(\sum_{\sigma\in S_k}(-1)^\sigma (e_I^\star)^\sigma\right) &= \sum_{\sigma\in S_k}(-1)^\sigma \pi(e_I^\star)^\sigma \\ &= \sum_{\sigma\in S_k}(-1)^\sigma(-1)^\sigma \pi(e_I^\star) = k!\,\pi(e_I^\star).\end{aligned}$$

□

Exercises for § 1.6

Exercise 1.6.i. Prove the assertions (1.6.4), (1.6.5), and (1.6.6).

Exercise 1.6.ii. Verify the multiplication law in equation (1.6.11) for the wedge product.

Exercise 1.6.iii. Given $\omega \in \Lambda^r(V^\star)$ let ω^k be the k-fold wedge product of ω with itself, i.e., let $\omega^2 = \omega \wedge \omega$, $\omega^3 = \omega \wedge \omega \wedge \omega$, etc.

(1) Show that if r is odd then, for $k > 1$, $\omega^k = 0$.
(2) Show that if ω is decomposable, then, for $k > 1$, $\omega^k = 0$.

Exercise 1.6.iv. If ω and μ are in $\Lambda^r(V^\star)$ prove:

$$(\omega + \mu)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^\ell \wedge \mu^{k-\ell}.$$

Hint: As in freshman calculus, prove this binomial theorem by induction using the identity: $\binom{k}{\ell} = \binom{k-1}{\ell-1} + \binom{k-1}{\ell}$.

Exercise 1.6.v. Let ω be an element of $\Lambda^2(V^\star)$. By definition the ***rank*** of ω is k if $\omega^k \neq 0$ and $\omega^{k+1} = 0$. Show that if

$$\omega = e_1 \wedge f_1 + \cdots + e_k \wedge f_k$$

with $e_i, f_i \in V^\star$, then ω is of rank $\leq k$.

Hint: Show that

$$\omega^k = k!\, e_1 \wedge f_1 \wedge \cdots \wedge e_k \wedge f_k.$$

Exercise 1.6.vi. Given $e_i \in V^\star$, $i = 1, \ldots, k$ show that $e_1 \wedge \cdots \wedge e_k \neq 0$ if and only if the e_i's are linearly independent.

Hint: Induction on k.

1.7. The interior product

We'll describe in this section another basic product operation on the spaces $\Lambda^k(V^*)$. As above we'll begin by defining this operator on the $\mathcal{L}^k(V)$'s.

Definition 1.7.1. Let V be a vector space and k a non-negative integer. Given $T \in \mathcal{L}^k(V)$ and $v \in V$, let $\iota_v T$ be the $(k-1)$-tensor which takes the value

$$(\iota_v T)(v_1, \dots, v_{k-1}) := \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1})$$

on the $k-1$-tuple of vectors, $v_1, \dots, v_{k-1}$, i.e., in the rth summand on the right, v gets inserted between v_{r-1} and v_r. (In particular, the first summand is $T(v, v_1, \dots, v_{k-1})$ and the last summand is $(-1)^{k-1} T(v_1, \dots, v_{k-1}, v)$.)

It is clear from the definition that if $v = v_1 + v_2$

$$\iota_v T = \iota_{v_1} T + \iota_{v_2} T, \tag{1.7.2}$$

and if $T = T_1 + T_2$

$$\iota_v T = \iota_v T_1 + \iota_v T_2, \tag{1.7.3}$$

and we will leave for you to verify by inspection the following two lemmas.

Lemma 1.7.4. *If T is the decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$, then*

$$\iota_v T = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k, \tag{1.7.5}$$

where the "hat" over ℓ_r means that ℓ_r is deleted from the tensor product.

Lemma 1.7.6. *If $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$, then*

$$\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2. \tag{1.7.7}$$

We will next prove the important identity.

Lemma 1.7.8. *Let V be a vector space and $T \in \mathcal{L}^k(V)$. Then for all $v \in V$ we have*

$$\iota_v(\iota_v T) = 0. \tag{1.7.9}$$

Proof. It suffices by linearity to prove this for decomposable tensors and since (1.7.9) is trivially true for $T \in \mathcal{L}^1(V)$, we can by induction assume (1.7.9) is true for decomposable tensors of degree $k-1$. Let $\ell_1 \otimes \cdots \otimes \ell_k$ be a decomposable tensor of degree k. Setting $T := \ell_1 \otimes \cdots \otimes \ell_{k-1}$ and $\ell = \ell_k$ we have

$$\iota_v(\ell_1 \otimes \cdots \otimes \ell_k) = \iota_v(T \otimes \ell) = \iota_v T \otimes \ell + (-1)^{k-1} \ell(v) T$$

by (1.7.7). Hence

$$\iota_v(\iota_v(T \otimes \ell)) = \iota_v(\iota_v T) \otimes \ell + (-1)^{k-2} \ell(v) \iota_v T + (-1)^{k-1} \ell(v) \iota_v T.$$

But by induction the first summand on the right is zero and the two remaining summands cancel each other out. ☐

From (1.7.9) we can deduce a slightly stronger result: For $v_1, v_2 \in V$

$$\iota_{v_1}\iota_{v_2} = -\iota_{v_2}\iota_{v_1}. \tag{1.7.10}$$

Proof. Let $v = v_1 + v_2$. Then $\iota_v = \iota_{v_1} + \iota_{v_2}$ so

$$\begin{aligned} 0 = \iota_v\iota_v &= (\iota_{v_1} + \iota_{v_2})(\iota_{v_1} + \iota_{v_2}) \\ &= \iota_{v_1}\iota_{v_1} + \iota_{v_1}\iota_{v_2} + \iota_{v_2}\iota_{v_1} + \iota_{v_2}\iota_{v_2} \\ &= \iota_{v_1}\iota_{v_2} + \iota_{v_2}\iota_{v_1} \end{aligned}$$

since the first and last summands are zero by (1.7.9). □

We'll now show how to define the operation ι_v on $\Lambda^k(V^\star)$. We'll first prove the following lemma.

Lemma 1.7.11. *If $T \in \mathcal{L}^k(V)$ is redundant, then so is $\iota_v T$.*

Proof. Let $T = T_1 \otimes \ell \otimes \ell \otimes T_2$ where ℓ is in $V^\star$, T_1 is in $\mathcal{L}^p(V)$ and T_2 is in $\mathcal{L}^q(V)$. Then by equation (1.7.7)

$$\iota_v T = \iota_v T_1 \otimes \ell \otimes \ell \otimes T_2 + (-1)^p T_1 \otimes \iota_v(\ell \otimes \ell) \otimes T_2 + (-1)^{p+2} T_1 \otimes \ell \otimes \ell \otimes \iota_v T_2.$$

However, the first and the third terms on the right are redundant and

$$\iota_v(\ell \otimes \ell) = \ell(v)\ell - \ell(v)\ell$$

by equation (1.7.5). □

Now let π be the projection (1.5.12) of $\mathcal{L}^k(V)$ onto $\Lambda^k(V^\star)$ and for $\omega = \pi(T) \in \Lambda^k(V^\star)$ define

$$\iota_v\omega = \pi(\iota_v T). \tag{1.7.12}$$

To show that this definition is legitimate we note that if $\omega = \pi(T_1) = \pi(T_2)$, then $T_1 - T_2 \in \mathcal{I}^k(V)$, so by Lemma 1.7.11 $\iota_v T_1 - \iota_v T_2 \in \mathcal{I}^{k-1}$ and hence

$$\pi(\iota_v T_1) = \pi(\iota_v T_2).$$

Therefore, (1.7.12) does not depend on the choice of T.

By definition, ι_v is a linear map $\Lambda^k(V^\star) \to \Lambda^{k-1}(V^\star)$. We call this the ***interior product operation***. From the identities (1.7.2)–(1.7.12) one gets, for $v, v_1, v_2 \in V$, $\omega \in \Lambda^k(V^\star)$, $\omega_1 \in \Lambda^p(V^\star)$, and $\omega_2 \in \Lambda^2(V^\star)$

$$\iota_{(v_1+v_2)}\omega = \iota_{v_1}\omega + \iota_{v_2}\omega, \tag{1.7.13}$$

$$\iota_v(\omega_1 \wedge \omega_2) = \iota_v\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \iota_v\omega_2, \tag{1.7.14}$$

$$\iota_v(\iota_v\omega) = 0 \tag{1.7.15}$$

and

$$\iota_{v_1}\iota_{v_2}\omega = -\iota_{v_2}\iota_{v_1}\omega.$$

Moreover if $\omega = \ell_1 \wedge \cdots \wedge \ell_k$ is a decomposable element of $\Lambda^k(V^\star)$ one gets from (1.7.5)

$$\iota_v \omega = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \wedge \cdots \wedge \hat{\ell}_r \wedge \cdots \wedge \ell_k.$$

In particular if $e_1, \dots, e_n$ is a basis of V, $e_1^\star, \dots, e_n^\star$ the dual basis of $V^\star$ and $\omega_I = e_{i_1}^\star \wedge \cdots \wedge e_{i_k}^\star$, $1 \le i_1 < \cdots < i_k \le n$, then $\iota_{e_j} \omega_I = 0$ if $j \notin I$ and if $j = i_r$

$$\iota_{e_j} \omega_I = (-1)^{r-1} \omega_{I_r}, \tag{1.7.16}$$

where $I_r = (i_1, \dots, \hat{\imath}_r, \dots, i_k)$ (i.e., I_r is obtained from the multi-index I by deleting i_r).

Exercises for § 1.7

Exercise 1.7.i. Prove Lemma 1.7.4.

Exercise 1.7.ii. Prove Lemma 1.7.6.

Exercise 1.7.iii. Show that if $T \in \mathcal{A}^k$, $\iota_v T = k T_v$ where T_v is the tensor (1.3.21). In particular conclude that $\iota_v T \in \mathcal{A}^{k-1}(V)$. (See Exercise 1.4.viii.)

Exercise 1.7.iv. Assume the dimension of V is n and let Ω be a non-zero element of the one-dimensional vector space $\Lambda^n(V^\star)$. Show that the map

$$\rho \colon V \to \Lambda^{n-1}(V^\star), \quad v \mapsto \iota_v \Omega, \tag{1.7.17}$$

is a bijective linear map.

Hint: One can assume $\Omega = e_1^\star \wedge \cdots \wedge e_n^\star$ where $e_1, \dots, e_n$ is a basis of V. Now use equation (1.7.16) to compute this map on basis elements.

Exercise 1.7.v (cross-product). Let V be a three-dimensional vector space, B an inner product on V and Ω a non-zero element of $\Lambda^3(V^\star)$. Define a map

$$- \times - \colon V \times V \to V$$

by setting

$$v_1 \times v_2 := \rho^{-1}(L v_1 \wedge L v_2)$$

where ρ is the map (1.7.17) and $L \colon V \to V^\star$ the map (1.2.17). Show that this map is linear in v_1, with v_2 fixed and linear in v_2 with v_1 fixed, and show that $v_1 \times v_2 = -v_2 \times v_1$.

Exercise 1.7.vi. For $V = \mathbf{R}^3$ let e_1, e_2 and e_3 be the standard basis vectors and B the standard inner product. (See § 1.1.) Show that if $\Omega = e_1^\star \wedge e_2^\star \wedge e_3^\star$ the cross-product above is the standard cross-product:

$$e_1 \times e_2 = e_3,$$

$$e_2 \times e_3 = e_1,$$

$$e_3 \times e_1 = e_2.$$

Hint: If B is the standard inner product, then $L e_i = e_i^\star$.

Remark 1.7.18. One can make this standard cross-product look even more standard by using the calculus notation: $e_1 = \hat{\imath}$, $e_2 = \hat{\jmath}$, and $e_3 = \hat{k}$.

1.8. The pullback operation on $\Lambda^k(V^\star)$

Let V and W be vector spaces and let A be a linear map of V into W. Given a k-tensor $T \in \mathcal{L}^k(W)$, recall that the *pullback* $A^\star T$ is the k-tensor

$$(A^\star T)(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$

in $\mathcal{L}^k(V)$. (See §1.3 and equation (1.3.17).) In this section we'll show how to define a similar pullback operation on $\Lambda^k(V^\star)$.

Lemma 1.8.1. *If* $T \in \mathcal{I}^k(W)$, *then* $A^\star T \in \mathcal{I}^k(V)$.

Proof. It suffices to verify this when T is a redundant k-tensor, i.e., a tensor of the form

$$T = \ell_1 \otimes \cdots \otimes \ell_k,$$

where $\ell_r \in W^\star$ and $\ell_i = \ell_{i+1}$ for some index, i. But by equation (1.3.19),

$$A^\star T = A^\star \ell_1 \otimes \cdots \otimes A^\star \ell_k$$

and the tensor on the right is redundant since $A^\star \ell_i = A^\star \ell_{i+1}$. □

Now let ω be an element of $\Lambda^k(W^\star)$ and let $\omega = \pi(T)$ where T is in $\mathcal{L}^k(W)$. We define

$$A^\star \omega = \pi(A^\star T). \tag{1.8.2}$$

Claim 1.8.3. *The left-hand side of equation* (1.8.2) *is well-defined.*

Proof. If $\omega = \pi(T) = \pi(T')$, then $T = T' + S$ for some $S \in \mathcal{I}^k(W)$, and $A^\star T = A^\star T' + A^\star S$. But $A^\star S \in \mathcal{I}^k(V)$, so

$$\pi(A^\star T') = \pi(A^\star T).$$ □

Proposition 1.8.4. *The map* $A^\star : \Lambda^k(W^\star) \to \Lambda^k(V^\star)$ *sending* $\omega \mapsto A^\star\omega$ *is linear. Moreover,*

(1) *if* $\omega_i \in \Lambda^{k_i}(W^\star)$, $i = 1, 2$, *then*

$$A^\star(\omega_1 \wedge \omega_2) = A^\star(\omega_1) \wedge A^\star(\omega_2);$$

(2) *if* U *is a vector space and* $B : U \to V$ *a linear map, then for* $\omega \in \Lambda^k(W^\star)$,

$$B^\star A^\star \omega = (AB)^\star \omega.$$

We'll leave the proof of these three assertions as exercises. As a hint, they follow immediately from the analogous assertions for the pullback operation on tensors. (See equations (1.3.19) and (1.3.20).)

As an application of the pullback operation we'll show how to use it to define the notion of *determinant* for a linear mapping.

Definition 1.8.5. Let V be an n-dimensional vector space. Then $\dim \Lambda^n(V^\star) = \binom{n}{n} = 1$; i.e., $\Lambda^n(V^\star)$ is a *one-dimensional* vector space. Thus if $A\colon V \to V$ is a linear mapping, the induced pullback mapping:

$$A^\star\colon \Lambda^n(V^\star) \to \Lambda^n(V^\star),$$

is just "multiplication by a constant". We denote this constant by $\det(A)$ and call it the ***determinant*** of A, Hence, by definition,

$$A^\star\omega = \det(A)\omega \tag{1.8.6}$$

for all $\omega \in \Lambda^n(V^\star)$.

From equation (1.8.6) it is easy to derive a number of basic facts about determinants.

Proposition 1.8.7. *If A and B are linear mappings of V into V, then*

$$\det(AB) = \det(A)\det(B).$$

Proof. By (2) and

$$\begin{aligned}(AB)^\star\omega = \det(AB)\omega &= B^\star(A^\star\omega)\\ &= \det(B)A^\star\omega = \det(B)\det(A)\omega,\end{aligned}$$

so $\det(AB) = \det(A)\det(B)$. ☐

Proposition 1.8.8. *Write* $\mathrm{id}_V\colon V \to V$ *for the identity map. Then* $\det(\mathrm{id}_V) = 1$.

We'll leave the proof as an exercise. As a hint, note that $\mathrm{id}_V^\star$ is the identity map on $\Lambda^n(V^\star)$.

Proposition 1.8.9. *If $A\colon V \to V$ is not surjective, then* $\det(A) = 0$.

Proof. Let W be the image of A. Then if A is not onto, the dimension of W is less than n, so $\Lambda^n(W^\star) = 0$. Now let $A = i_W B$ where i_W is the inclusion map of W into V and B is the mapping, A, regarded as a mapping from V to W. Thus if ω is in $\Lambda^n(V^\star)$, then by (2)

$$A^\star\omega = B^\star i_W^\star\omega$$

and since $i_W^\star\omega$ is in $\Lambda^n(W^\star)$ it is zero. ☐

We will derive by wedge product arguments the familiar "matrix formula" for the determinant. Let V and W be n-dimensional vector spaces and let $e_1, \dots, e_n$ be a basis for V and $f_1, \dots, f_n$ a basis for W. From these bases we get dual bases, $e_1^\star, \dots, e_n^\star$ and $f_1^\star, \dots, f_n^\star$, for $V^\star$ and $W^\star$. Moreover, if A is a linear map of V into W and $(a_{i,j})$ the $n \times n$ matrix describing A in terms of these bases, then the transpose map, $A^\star\colon W^\star \to V^\star$, is described in terms of these dual bases by the $n \times n$ transpose matrix, i.e., if

$$Ae_j = \sum_{i=1}^n a_{i,j} f_i,$$

then

$$A^\star f_j^\star = \sum_{i=1}^{n} a_{j,i} e_i^\star .$$

(See § 1.2.) Consider now $A^\star(f_1^\star \wedge \cdots \wedge f_n^\star)$. By (1),

$$\begin{aligned} A^\star(f_1^\star \wedge \cdots \wedge f_n^\star) &= A^\star f_1^\star \wedge \cdots \wedge A^\star f_n^\star \\ &= \sum_{1 \le k_1, \ldots, k_n \le n} (a_{1,k_1} e_{k_1}^\star) \wedge \cdots \wedge (a_{n,k_n} e_{k_n}^\star). \end{aligned}$$

Thus,

$$A^\star(f_1^\star \wedge \cdots \wedge f_n^\star) = \sum a_{1,k_1} \ldots a_{n,k_n} \, e_{k_1}^\star \wedge \cdots \wedge e_{k_n}^\star .$$

If the multi-index, $k_1, \ldots, k_n$, is repeating, then $e_{k_1}^\star \wedge \cdots \wedge e_{k_n}^\star$ is zero, and if it is not repeating then we can write

$$k_i = \sigma(i) \quad i = 1, \ldots, n$$

for some permutation, σ, and hence we can rewrite $A^\star(f_1^\star \wedge \cdots \wedge f_n^\star)$ as

$$A^\star(f_1^\star \wedge \cdots \wedge f_n^\star) = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \quad (e_1^\star \wedge \cdots \wedge e_n^\star)^\sigma .$$

But

$$(e_1^\star \wedge \cdots \wedge e_n^\star)^\sigma = (-1)^\sigma e_1^\star \wedge \cdots \wedge e_n^\star$$

so we get finally the formula

$$A^\star(f_1^\star \wedge \cdots \wedge f_n^\star) = \det((a_{i,j})) e_1^\star \wedge \cdots \wedge e_n^\star, \tag{1.8.10}$$

where

$$\det((a_{i,j})) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \tag{1.8.11}$$

summed over $\sigma \in S_n$. The sum on the right is (as most of you know) the *determinant* of the matrix $(a_{i,j})$.

Notice that if $V = W$ and $e_i = f_i$, $i = 1, \ldots, n$, then $\omega = e_1^\star \wedge \cdots \wedge e_n^\star = f_1^\star \wedge \cdots \wedge f_n^\star$, hence by (1.8.6) and (1.8.10),

$$\det(A) = \det((a_{i,j})).$$

Exercises for § 1.8

Exercise 1.8.i. Verify the three assertions of Proposition 1.8.4.

Exercise 1.8.ii. Deduce from Proposition 1.8.9 a well-known fact about determinants of $n \times n$ matrices: If two columns are equal, the determinant is zero.

Exercise 1.8.iii. Deduce from Proposition 1.8.7 another well-known fact about determinants of $n \times n$ matrices: If one interchanges two columns, then one changes the sign of the determinant.

Hint: Let $e_1, \ldots, e_n$ be a basis of V and let $B\colon V \to V$ be the linear mapping: $Be_i = e_j$, $Be_j = e_i$ and $Be_\ell = e_\ell$, $\ell \neq i, j$. What is $B^\star(e_1^\star \wedge \cdots \wedge e_n^\star)$?

Exercise 1.8.iv. Deduce from Propositions 1.8.7 and 1.8.8 another well-known fact about determinants of $n \times n$ matrix: If $(b_{i,j})$ is the inverse of $(a_{i,j})$, its determinant is the inverse of the determinant of $(a_{i,j})$.

Exercise 1.8.v. Extract from (1.8.11) a well-known formula for determinants of 2×2 matrices:

$$\det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1}, & a_{2,2} \end{pmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$

Exercise 1.8.vi. Show that if $A = (a_{i,j})$ is an $n \times n$ matrix and $A^{\intercal} = (a_{j,i})$ is its transpose $\det A = \det A^{\intercal}$.

Hint: You are required to show that the sums

$$\sum_{\sigma \in S_n} (-1)^{\sigma} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

and

$$\sum_{\sigma \in S_n} (-1)^{\sigma} a_{\sigma(1),1} \cdots a_{\sigma(n),n}$$

are the same. Show that the second sum is identical with

$$\sum_{\sigma \in S_n} (-1)^{\sigma^{-1}} a_{\sigma^{-1}(1),1} \cdots a_{\sigma^{-1}(n),n}.$$

Exercise 1.8.vii. Let A be an $n \times n$ matrix of the form

$$A = \begin{pmatrix} B & * \\ 0 & C \end{pmatrix},$$

where B is a $k \times k$ matrix and C an $\ell \times \ell$ matrix and the bottom $\ell \times k$ block is zero. Show that

$$\det(A) = \det(B)\det(C).$$

Hint: Show that in equation (1.8.11) every non-zero term is of the form

$$(-1)^{\sigma\tau} b_{1,\sigma(1)} \cdots b_{k,\sigma(k)} c_{1,\tau(1)} \cdots c_{\ell,\tau(\ell)},$$

where $\sigma \in S_k$ and $\tau \in S_\ell$.

Exercise 1.8.viii. Let V and W be vector spaces and let $A\colon V \to W$ be a linear map. Show that if $Av = w$ then for $\omega \in \Lambda^p(W^\star)$,

$$A^\star \iota_w \omega = \iota_v A^\star \omega.$$

Hint: By equation (1.7.14) and Proposition 1.8.4 it suffices to prove this for $\omega \in \Lambda^1(W^\star)$, i.e., for $\omega \in W^\star$.

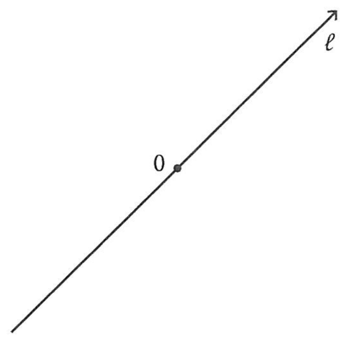

Figure 1.9.1. A line in $\mathbf{R}^2$.

1.9. Orientations

Definition 1.9.1. We recall from freshman calculus that if $\ell \subset \mathbf{R}^2$ is a line through the origin, then $\ell \setminus \{0\}$ has two connected components and an ***orientation*** of ℓ is a choice of one of these components (as in Figure 1.9.1).

More generally, if L is a one-dimensional vector space then $L \setminus \{0\}$ consists of two components: namely if v is an element of $L \setminus \{0\}$, then these two components are

$$L_1 = \{\lambda v \mid \lambda > 0\}$$

and

$$L_2 = \{\lambda v \mid \lambda < 0\}.$$

Definition 1.9.2. Let L be a one-dimensional vector space. An ***orientation*** of L is a choice of one of a connected component of $L \setminus \{0\}$. Usually the component chosen is denoted L_+, and called the ***positive component*** of $L \setminus \{0\}$ and the other component L_- the ***negative component*** of $L \setminus \{0\}$.

Definition 1.9.3. Let (L, L_+) be an oriented one-dimensional vector space. A vector $v \in L$ is ***positively oriented*** if $v \in L_+$.

Definition 1.9.4. Let V be an n-dimensional vector space. An ***orientation*** of V is an orientation of the one-dimensional vector space $\Lambda^n(V^\star)$.

One important way of assigning an orientation to V is to choose a basis, $e_1, \dots, e_n$ of V. Then, if $e_1^\star, \dots, e_n^\star$ is the dual basis, we can orient $\Lambda^n(V^\star)$ by requiring that $e_1^\star \wedge \cdots \wedge e_n^\star$ be in the positive component of $\Lambda^n(V^\star)$.

Definition 1.9.5. Let V be an oriented n-dimensional vector space. We say that an ordered basis $(e_1, \dots, e_n)$ of V is ***positively oriented*** if $e_1^\star \wedge \cdots \wedge e_n^\star$ is in the positive component of $\Lambda^n(V^\star)$.

Suppose that $e_1, \dots, e_n$ and $f_1, \dots, f_n$ are bases of V and that

$$e_j = \sum_{i=1}^{n} a_{i,j,} f_i. \tag{1.9.6}$$

Then by (1.7.10)

$$f_1^\star \wedge \cdots \wedge f_n^\star = \det((a_{i,j}))e_1^\star \wedge \cdots \wedge e_n^\star$$

so we conclude the following.

Proposition 1.9.7. *If $e_1, \ldots, e_n$ is positively oriented, then $f_1, \ldots, f_n$ is positively oriented if and only if $\det((a_{i,j}))$ is positive.*

Corollary 1.9.8. *If $e_1, \ldots, e_n$ is a positively oriented basis of V, then the basis*

$$e_1, \ldots, e_{i-1}, -e_i, e_{i+1}, \ldots, e_n$$

is negatively oriented.

Now let V be a vector space of dimension $n > 1$ and W a subspace of dimension $k < n$. We will use the result above to prove the following important theorem.

Theorem 1.9.9. *Given orientations on V and V/W, one gets from these orientations a natural orientation on W.*

Remark 1.9.10. What we mean by "natural" will be explained in the course of the proof.

Proof. Let $r = n - k$ and let π be the projection of V onto V/W. By Exercises 1.2.i and 1.2.ii we can choose a basis $e_1, \ldots, e_n$ of V such that $e_{r+1}, \ldots, e_n$ is a basis of W and $\pi(e_1), \ldots, \pi(e_r)$ a basis of V/W. Moreover, replacing e_1 by $-e_1$ if necessary we can assume by Corollary 1.9.8 that $\pi(e_1), \ldots, \pi(e_r)$ is a positively oriented basis of V/W and replacing e_n by $-e_n$ if necessary we can assume that $e_1, \ldots, e_n$ is a positively oriented basis of V. Now assign to W the orientation associated with the basis $e_{r+1}, \ldots, e_n$.

Let's show that this assignment is "natural" (i.e., does not depend on our choice of basis $e_1, \ldots, e_n$). To see this let $f_1, \ldots, f_n$ be another basis of V with the properties above and let $A = (a_{i,j})$ be the matrix (1.9.6) expressing the vectors $e_1, \ldots, e_n$ as linear combinations of the vectors $f_1, \ldots f_n$. This matrix has to have the form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}, \tag{1.9.11}$$

where B is the $r \times r$ matrix expressing the basis vectors $\pi(e_1), \ldots, \pi(e_r)$ of V/W as linear combinations of $\pi(f_1), \ldots, \pi(f_r)$ and D the $k \times k$ matrix expressing the basis vectors $e_{r+1}, \ldots, e_n$ of W as linear combinations of $f_{r+1}, \ldots, f_n$. Thus

$$\det(A) = \det(B)\det(D).$$

However, by Proposition 1.9.7, $\det A$ and $\det B$ are positive, so $\det D$ is positive, and hence if $e_{r+1}, \ldots, e_n$ is a positively oriented basis of W so is $f_{r+1}, \ldots, f_n$. □

As a special case of this theorem suppose $\dim W = n - 1$. Then the choice of a vector $v \in V \setminus W$ gives one a basis vector $\pi(v)$ for the one-dimensional space V/W and hence if V is oriented, the choice of v gives one a natural orientation on W.

Definition 1.9.12. Let $A: V_1 \to V_2$ a bijective linear map of oriented n-dimensional vector spaces. We say that A is ***orientation-preserving*** if, for $\omega \in \Lambda^n(V_2^\star)_+$, we have that $A^\star\omega$ is in $\Lambda^n(V_1^\star)_+$.

Example 1.9.13. If $V_1 = V_2$ then $A^\star\omega = \det(A)\omega$ so A is orientation preserving if and only if $\det(A) > 0$.

The following proposition we'll leave as an exercise.

Proposition 1.9.14. *Let V_1, V_2, and V_3 be oriented n-dimensional vector spaces and $A_i: V_i \xrightarrow{\sim} V_{i+1}$ for $i = 1, 2$ be bijective linear maps. Then if A_1 and A_2 are orientation preserving, so is $A_2 \circ A_1$.*

Exercises for § 1.9

Exercise 1.9.i. Prove Corollary 1.9.8.

Exercise 1.9.ii. Show that the argument in the proof of Theorem 1.9.9 can be modified to prove that if V and W are oriented then these orientations induce a natural orientation on V/W.

Exercise 1.9.iii. Similarly show that if W and V/W are oriented these orientations induce a natural orientation on V.

Exercise 1.9.iv. Let V be an n-dimensional vector space and $W \subset V$ a k-dimensional subspace. Let $U = V/W$ and let $\iota: W \to V$ and $\pi: V \to U$ be the inclusion and projection maps. Suppose V and U are oriented. Let μ be in $\Lambda^{n-k}(U^\star)_+$ and let ω be in $\Lambda^n(V^\star)_+$. Show that there exists a ν in $\Lambda^k(V^\star)$ such that $\pi^\star\mu \wedge \nu = \omega$. Moreover show that $\iota^\star\nu$ is *intrinsically* defined (i.e., does not depend on how we choose ν) and sits in the positive part $\Lambda^k(W^\star)_+$ of $\Lambda^k(W^\star)$.

Exercise 1.9.v. Let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbf{R}^n$. The ***standard*** orientation of $\mathbf{R}^n$ is, by definition, the orientation associated with this basis. Show that if W is the subspace of $\mathbf{R}^n$ defined by the equation $x_1 = 0$, and $v = e_1 \notin W$ then the natural orientation of W associated with v and the standard orientation of $\mathbf{R}^n$ coincide with the orientation given by the basis vectors $e_2, \dots, e_n$ of W.

Exercise 1.9.vi. Let V be an oriented n-dimensional vector space and W an $(n-1)$-dimensional subspace. Show that if v and v' are in $V \setminus W$ then $v' = \lambda v + w$, where w is in W and $\lambda \in \mathbf{R} \setminus \{0\}$. Show that v and v' give rise to the same orientation of W if and only if λ is positive.

Exercise 1.9.vii. Prove Proposition 1.9.14.

Exercise 1.9.viii. A key step in the proof of Theorem 1.9.9 was the assertion that the matrix A expressing the vectors, e_i as linear combinations of the vectors f_j, had to have the form (1.9.11). Why is this the case?

Exercise 1.9.ix.

(1) Let V be a vector space, W a subspace of V and $A\colon V \to V$ a bijective linear map which maps W onto W. Show that one gets from A a bijective linear map

$$B\colon V/W \to V/W$$

with the property

$$\pi A = B\pi,$$

where $\pi\colon V \to V/W$ is the quotient map.

(2) Assume that V, W and V/W are compatibly oriented. Show that if A is orientation preserving and its restriction to W is orientation preserving then B is orientation preserving.

Exercise 1.9.x. Let V be an oriented n-dimensional vector space, W an $(n-1)$-dimensional subspace of V and $i\colon W \to V$ the inclusion map. Given $\omega \in \Lambda^n(V^\star)_+$ and $v \in V \smallsetminus W$ show that for the orientation of W described in Exercise 1.9.v, $i^\star(\iota_v\omega) \in \Lambda^{n-1}(W^\star)_+$.

Exercise 1.9.xi. Let V be an n-dimensional vector space, $B\colon V \times V \to \mathbf{R}$ an inner product and $e_1, \ldots, e_n$ a basis of V which is positively oriented and orthonormal. Show that the ***volume element***

$$\mathrm{vol} \coloneqq e_1^\star \wedge \cdots \wedge e_n^\star \in \Lambda^n(V^\star)$$

is intrinsically defined, independent of the choice of this basis.

Hint: Equations (1.2.20) and (1.8.10).

Exercise 1.9.xii.

(1) Let V be an oriented n-dimensional vector space and B an inner product on V. Fix an oriented orthonormal basis, $e_1, \ldots, e_n$, of V and let $A\colon V \to V$ be a linear map. Show that if

$$Ae_i = v_i = \sum_{j=1}^{n} a_{j,i} e_j$$

and $b_{i,j} = B(v_i, v_j)$, the matrices $M_A = (a_{i,j})$ and $M_B = (b_{i,j})$ are related by: $M_B = M_A^\intercal M_A$.

(2) Show that if vol is the volume form $e_1^\star \wedge \cdots \wedge e_n^\star$, and A is orientation preserving

$$A^\star\,\mathrm{vol} = \det(M_B)^{\frac{1}{2}}\,\mathrm{vol}\,.$$

(3) By Theorem 1.5.13 one has a bijective map

$$\Lambda^n(V^\star) \cong \mathcal{A}^n(V).$$

Show that the element, Ω, of $\mathcal{A}^n(V)$ corresponding to the form vol has the property

$$|\Omega(v_1, \ldots, v_n)|^2 = \det((b_{i,j})),$$

where $v_1, \ldots, v_n$ are any n-tuple of vectors in V and $b_{i,j} = B(v_i, v_j)$.