Chapter 2

The Concept of a Differential Form

The goal of this chapter is to generalize to n dimensions the basic operations of three-dimensional vector calculus: divergence, curl, and gradient. The divergence and gradient operations have fairly straightforward generalizations, but the curl operation is more subtle. For vector fields it does not have any obvious generalization, however, if one replaces vector fields by a closely related class of objects, *differential forms,* then not only does it have a natural generalization but it turns out that divergence, curl, and gradient are all special cases of a general operation on differential forms called *exterior differentiation.*

2.1. Vector fields and 1-forms

In this section we will review some basic facts about vector fields in n variables and introduce their dual objects: *1-forms.* We will then take up in § 2.3 the theory of k-forms for $k > 1$. We begin by fixing some notation.

Definition 2.1.1. Let $p \in \mathbf{R}^n$. The ***tangent space to*** $\mathbf{R}^n$ ***at*** p is the set of pairs

$$T_p\mathbf{R}^n := \{(p, v) \mid v \in \mathbf{R}^n\}.$$

The identification

$$T_p\mathbf{R}^n \xrightarrow{\sim} \mathbf{R}^n, \quad (p, v) \mapsto v \tag{2.1.2}$$

makes $T_p\mathbf{R}^n$ into a vector space. More explicitly, for $v, v_1, v_2 \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$ we define the addition and scalar multiplication operations on $T_p\mathbf{R}^n$ by setting

$$(p, v_1) + (p, v_2) := (p, v_1 + v_2)$$

and

$$\lambda(p, v) := (p, \lambda v).$$

Let U be an open subset of $\mathbf{R}^n$ and $f : U \to \mathbf{R}^m$ a C^1 map. We recall that the derivative

$$Df(p) : \mathbf{R}^n \to \mathbf{R}^m$$

of f at p is the linear map associated with the $m \times n$ matrix

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right).$$

It will be useful to have a "base-pointed" version of this definition as well. Namely, if $q = f(p)$ we will define

$$df_p : T_p\mathbf{R}^n \to T_q\mathbf{R}^m$$

to be the map

$$df_p(p, v) := (q, Df(p)v).$$

It is clear from the way we have defined vector space structures on $T_p\mathbf{R}^n$ and $T_q\mathbf{R}^m$ that this map is linear.

Suppose that the image of f is contained in an open set V, and suppose $g : V \to \mathbf{R}^k$ is a C^1 map. Then the "base-pointed" version of the chain rule asserts that

$$dg_q \circ df_p = d(f \circ g)_p.$$

(This is just an alternative way of writing $Dg(q)Df(p) = D(g \circ f)(p)$.)

In three-dimensional vector calculus a ***vector field*** is a function which attaches to each point p of $\mathbf{R}^3$ a base-pointed arrow $(p, v) \in T_p\mathbf{R}^3$. The n-dimensional version of this definition is essentially the same.

Definition 2.1.3. Let U be an open subset of $\mathbf{R}^n$. A ***vector field*** $\boldsymbol{v}$ on U is a function which assigns to each point $p \in U$ a vector $\boldsymbol{v}(p) \in T_p\mathbf{R}^n$.

Thus a vector field is a vector-valued function, but its value at p is an element of a vector space $T_p\mathbf{R}^n$ that itself depends on p.

Examples 2.1.4.

(1) Given a fixed vector $v \in \mathbf{R}^n$, the function

(2.1.5)
$$p \mapsto (p, v)$$

is a vector field on $\mathbf{R}^n$. Vector fields of this type are called ***constant*** vector fields.

(2) In particular let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbf{R}^n$. If $v = e_i$ we will denote the vector field (2.1.5) by $\partial/\partial x_i$. (The reason for this "derivation notation" will be explained below.)

(3) Given a vector field $\boldsymbol{v}$ on U and a function $f : U \to \mathbf{R}$, we denote by $f\boldsymbol{v}$ the vector field on U defined by

$$p \mapsto f(p)\boldsymbol{v}(p).$$

(4) Given vector fields $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ on U, we denote by $\boldsymbol{v}_1 + \boldsymbol{v}_2$ the vector field on U defined by

$$p \mapsto \boldsymbol{v}_1(p) + \boldsymbol{v}_2(p).$$

(5) The vectors, (p, e_i), $i = 1, \dots, n$, are a basis of $T_p\mathbf{R}^n$, so if $\boldsymbol{v}$ is a vector field on U, $\boldsymbol{v}(p)$ can be written uniquely as a linear combination of these vectors with real numbers $g_1(p), \dots, g_n(p)$ as coefficients. In other words, using the notation in example (2) above, $\boldsymbol{v}$ can be written uniquely as a sum

(2.1.6)
$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i},$$

where $g_i : U \to \mathbf{R}$ is the function $p \mapsto g_i(p)$.

Definition 2.1.7. We say that $\boldsymbol{v}$ is a C^∞ vector field if the g_i's are in $C^\infty(U)$.

Definition 2.1.8. A basic vector field operation is ***Lie differentiation***. If $f \in C^1(U)$ we define $L_{\boldsymbol{v}}f$ to be the function on U whose value at p is given by

$$L_{\boldsymbol{v}}f(p) := Df(p)v, \tag{2.1.9}$$

where $\boldsymbol{v}(p) = (p, v)$.

Example 2.1.10. If $\boldsymbol{v}$ is the vector field (2.1.6) then

$$L_{\boldsymbol{v}}f = \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_i}$$

(motivating our "derivation notation" for $\boldsymbol{v}$).

We leave the following generalization of the product rule for differentiation as an exercise.

Lemma 2.1.11. *Let U be an open subset of $\mathbf{R}^n$, $\boldsymbol{v}$ a vector field on U, and $f_1, f_2 \in C^1(U)$. Then*

$$L_{\boldsymbol{v}}(f_1 \cdot f_2) = L_{\boldsymbol{v}}(f_1) \cdot f_2 + f_1 \cdot L_{\boldsymbol{v}}(f_2).$$

We now discuss a class of objects which are in some sense "dual objects" to vector fields.

Definition 2.1.12. Let $p \in \mathbf{R}^n$. The ***cotangent space to $\mathbf{R}^n$ at p*** is the dual vector space

$$T_p^\star \mathbf{R}^n := (T_p\mathbf{R})^\star.$$

An element of $T_p^\star \mathbf{R}^n$ is called a ***cotangent vector*** to $\mathbf{R}^n$ at p.

Definition 2.1.13. Let U be an open subset of $\mathbf{R}^n$. A ***differential 1-form***, or simply ***1-form*** on U, is a function ω which assigns to each point $p \in U$ a vector ω_p in $T_p^\star \mathbf{R}^n$.

Examples 2.1.14.

(1) Let $f : U \to \mathbf{R}$ be a C^1 function. Then for $p \in U$ and $q = f(p)$ one has a linear map

$$df_p : T_p\mathbf{R}^n \to T_q\mathbf{R} \tag{2.1.15}$$

and by making the identification $T_q\mathbf{R} \xrightarrow{\sim} \mathbf{R}$ by sending $(q, v) \mapsto v$, the differential df_p can be regarded as a linear map from $T_p\mathbf{R}^n$ to $\mathbf{R}$, i.e., as an element of $T_p^\star \mathbf{R}^n$. Hence the assignment

$$p \mapsto df_p$$

defines a 1-form on U which we denote by df.

(2) Given a 1-form ω and a function $\phi : U \to \mathbf{R}$, the pointwise product of ϕ with ω defines 1-form $\phi\omega$ on U by $(\phi\omega)_p := \phi(p)\omega_p$.

(3) Given two 1-forms ω_1 and ω_2 their pointwise sum defines a 1-form $\omega_1 + \omega_2$ by $(\omega_1 + \omega_2)_p := (\omega_1)_p + (\omega_2)_p$.

(4) The 1-forms $dx_1, \dots, dx_n$ play a particularly important role. By equation (2.1.15)

$$(dx_i)\left(\frac{\partial}{\partial x_j}\right)_p = \delta_{i,j}, \tag{2.1.16}$$

i.e., it is equal to 1 if $i = j$ and zero if $i \neq j$. Thus $(dx_1)_p, \dots, (dx_n)_p$ are the basis of $T_p^\star \mathbf{R}^n$ *dual* to the basis $(\partial/\partial x_i)_p$. Therefore, if ω is *any* 1-form on U, ω_p can be written uniquely as a sum

$$\omega_p = \sum_{i=1}^n f_i(p)(dx_i)_p, \quad f_i(p) \in \mathbf{R},$$

and ω can be written uniquely as a sum

$$\omega = \sum_{i=1}^n f_i dx_i, \tag{2.1.17}$$

where $f_i \colon U \to \mathbf{R}$ is the function $p \mapsto f_i(p)$. We say that ω is a C^∞ ***1-form*** or ***smooth 1-form*** if the functions $f_1, \dots, f_n$ are C^∞.

We leave the following as an exercise.

Lemma 2.1.18. *Let U be an open subset of $\mathbf{R}^n$. If $f \colon U \to \mathbf{R}$ is a C^∞ function, then*

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.$$

Suppose that $\boldsymbol{v}$ is a vector field and ω a 1-form on $U \subset \mathbf{R}^n$. Then for every $p \in U$ the vectors $\boldsymbol{v}(p) \in T_p\mathbf{R}^n$ and $\omega_p \in (T_p\mathbf{R}^n)^\star$ can be paired to give a number $\iota_{\boldsymbol{v}(p)}\omega_p \in \mathbf{R}$, and hence, as p varies, an $\mathbf{R}$-valued function $\iota_{\boldsymbol{v}}\omega$ which we will call the ***interior product*** of $\boldsymbol{v}$ with ω.

Example 2.1.19. For instance if $\boldsymbol{v}$ is the vector field (2.1.6) and ω the 1-form (2.1.17), then

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^n f_i g_i.$$

Thus if $\boldsymbol{v}$ and ω are C^∞, so is the function $\iota_{\boldsymbol{v}}\omega$. Also notice that if $\phi \in C^\infty(U)$, then as we observed above

$$d\phi = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} \frac{\partial}{\partial x_i}$$

so if $\boldsymbol{v}$ is the vector field (2.1.6) then we have

$$\iota_{\boldsymbol{v}} d\phi = \sum_{i=1}^n g_i \frac{\partial \phi}{\partial x_i} = L_{\boldsymbol{v}}\phi.$$

Exercises for §2.1

Exercise 2.1.i. Prove Lemma 2.1.18.

Exercise 2.1.ii. Prove Lemma 2.1.11.

Exercise 2.1.iii. Let U be an open subset of $\mathbf{R}^n$ and $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ vector fields on U. Show that there is a unique vector field $\boldsymbol{w}$, on U with the property

$$L_{\boldsymbol{w}}\phi = L_{\boldsymbol{v}_1}(L_{\boldsymbol{v}_2}\phi) - L_{\boldsymbol{v}_2}(L_{\boldsymbol{v}_1}\phi)$$

for all $\phi \in C^\infty(U)$.

Exercise 2.1.iv. The vector field $\boldsymbol{w}$ in Exercise 2.1.iii is called the ***Lie bracket*** of the vector fields $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and is denoted by $[\boldsymbol{v}_1, \boldsymbol{v}_2]$. Verify that the Lie bracket is *skew-symmetric*, i.e.,

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] = -[\boldsymbol{v}_2, \boldsymbol{v}_1],$$

and satisfies the *Jacobi identity*

$$[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] + [\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]] + [\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]] = 0.$$

(And thus defines the structure of a *Lie algebra*.)

Hint: Prove analogous identities for $L_{\boldsymbol{v}_1}$, $L_{\boldsymbol{v}_2}$ and $L_{\boldsymbol{v}_3}$.

Exercise 2.1.v. Let $\boldsymbol{v}_1 = \partial/\partial x_i$ and $\boldsymbol{v}_2 = \sum_{j=1}^n g_j \partial/\partial x_j$. Show that

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] = \sum_{j=1}^n \frac{\partial g_j}{\partial x_i}\frac{\partial}{\partial x_j}.$$

Exercise 2.1.vi. Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be vector fields and f a C^∞ function. Show that

$$[\boldsymbol{v}_1, f\boldsymbol{v}_2] = L_{\boldsymbol{v}_1} f \boldsymbol{v}_2 + f[\boldsymbol{v}_1, \boldsymbol{v}_2].$$

Exercise 2.1.vii. Let U be an open subset of $\mathbf{R}^n$ and let $\gamma\colon [a,b] \to U$, $t \mapsto (\gamma_1(t), \ldots, \gamma_n(t))$ be a C^1 curve. Given a C^∞ 1-form $\omega = \sum_{i=1}^n f_i dx_i$ on U, define the ***line integral*** of ω over γ to be the integral

$$\int_\gamma \omega := \sum_{i=1}^n \int_a^b f_i(\gamma(t)) \frac{d\gamma_i}{dt}\, dt.$$

Show that if $\omega = df$ for some $f \in C^\infty(U)$

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a)).$$

In particular conclude that if γ is a closed curve, i.e., $\gamma(a) = \gamma(b)$, this integral is zero.

Exercise 2.1.viii. Let ω be the C^∞ 1-form on $\mathbf{R}^2 \smallsetminus \{0\}$ defined by

$$\omega = \frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2},$$

and let $\gamma\colon [0, 2\pi] \to \mathbf{R}^2 \smallsetminus \{0\}$ be the closed curve $t \mapsto (\cos t, \sin t)$. Compute the line integral $\int_\gamma \omega$, and note that $\int_\gamma \omega \neq 0$. Conclude that ω is not of the form df for $f \in C^\infty(\mathbf{R}^2 \smallsetminus \{0\})$.

Exercise 2.1.ix. Let f be the function

$$f(x_1, x_2) = \begin{cases} \arctan \dfrac{x_2}{x_1} & x_1 > 0, \\ \dfrac{\pi}{2}, & x_2 > 0 \text{ or } x_1 = 0, \\ \arctan \dfrac{x_2}{x_1} + \pi & x_1 < 0\,. \end{cases}$$

Recall that $-\frac{\pi}{2} < \arctan(t) < \frac{\pi}{2}$. Show that f is C^∞ and that df is the 1-form ω in Exercise 2.1.viii. Why does not this contradict what you proved in Exercise 2.1.viii?

2.2. Integral curves for vector fields

In this section we'll discuss some properties of vector fields which we'll need for the manifold segment of this text. We'll begin by generalizing to n-variables the calculus notion of an "integral curve" of a vector field.

Definition 2.2.1. Let $U \subset \mathbf{R}^n$ be open and $\boldsymbol{v}$ a vector field on U. A C^1 curve $\gamma\colon (a, b) \to U$ is an ***integral curve*** of $\boldsymbol{v}$ if for all $t \in (a, b)$ we have

$$\boldsymbol{v}(\gamma(t)) = \left(\gamma(t), \frac{d\gamma}{dt}(t)\right).$$

Remark 2.2.2. If $\boldsymbol{v}$ is the vector field (2.1.6) and $g\colon U \to \mathbf{R}^n$ is the function $(g_1, \dots, g_n)$, the condition for $\gamma(t)$ to be an integral curve of $\boldsymbol{v}$ is that it satisfies the system of differential equations

$$\frac{d\gamma}{dt}(t) = g(\gamma(t)). \tag{2.2.3}$$

We will quote without proof a number of basic facts about systems of ordinary differential equations of the type (2.2.3).[1]

Theorem 2.2.4 (existence of integral curves). *Let U be an open subset of $\mathbf{R}^n$ and $\boldsymbol{v}$ a vector field on U. Given a point $p_0 \in U$ and $a \in \mathbf{R}$, there exist an interval $I = (a - T, a + T)$, a neighborhood U_0 of p_0 in U, and for every $p \in U_0$ an integral curve $\gamma_p\colon I \to U$ for $\boldsymbol{v}$ such that $\gamma_p(a) = p$.*

Theorem 2.2.5 (uniqueness of integral curves). *Let U be an open subset of $\mathbf{R}^n$ and $\boldsymbol{v}$ a vector field on U. For $i = 1, 2$, let $\gamma_i\colon I_i \to U$, $i = 1, 2$ be integral curves for $\boldsymbol{v}$. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$ then $\gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}$ and the curve $\gamma\colon I_1 \cup I_2 \to U$ defined by*

$$\gamma(t) := \begin{cases} \gamma_1(t), & t \in I_1, \\ \gamma_2(t), & t \in I_2 \end{cases}$$

is an integral curve for $\boldsymbol{v}$.

[1] A source for these results that we highly recommend is [2, Chapter 6].

Theorem 2.2.6 (smooth dependence on initial data). *Let $V \subset U \subset \mathbf{R}^n$ be open subsets. Let $\boldsymbol{v}$ be a C^∞ vector field on V, $I \subset \mathbf{R}$ an open interval, $a \in I$, and $h: V \times I \to U$ a map with the following properties.*

(1) *$h(p, a) = p$.*
(2) *For all $p \in V$ the curve*

$$\gamma_p : I \to U, \qquad \gamma_p(t) := h(p,t)$$

is an integral curve of $\boldsymbol{v}$.

Then the map h is C^∞.

One important feature of the system (2.2.3) is that it is an *autonomous* system of differential equations: the function $g(x)$ is a function of x alone (it does *not* depend on t). One consequence of this is the following.

Theorem 2.2.7. *Let $I = (a, b)$ and for $c \in \mathbf{R}$ let $I_c = (a-c, b-c)$. Then if $\gamma: I \to U$ is an integral curve, the reparameterized curve*

$$\gamma_c : I_c \to U, \quad \gamma_c(t) := \gamma(t+c) \tag{2.2.8}$$

is an integral curve.

We recall that C^1 function $\phi: U \to \mathbf{R}$ is an ***integral*** of the system (2.2.3) if for every integral curve $\gamma(t)$, the function $t \mapsto \phi(\gamma(t))$ is constant. This is true if and only if for all t

$$0 = \frac{d}{dt}\phi(\gamma(t)) = (D\phi)_{\gamma(t)}\left(\frac{d\gamma}{dt}\right) = (D\phi)_{\gamma(t)}(v),$$

where $\boldsymbol{v}(p) = (p, v)$. By equation (2.1.6) the term on the right is $L_{\boldsymbol{v}}\phi(p)$. Hence we conclude the following.

Theorem 2.2.9. *Let U be an open subset of $\mathbf{R}^n$ and $\phi \in C^1(U)$. Then ϕ is an integral of the system (2.2.3) if and only if $L_{\boldsymbol{v}}\phi = 0$.*

Definition 2.2.10. Let U be an open subset of $\mathbf{R}^n$ and $\boldsymbol{v}$ a vector field on U. We say that $\boldsymbol{v}$ is ***complete*** if for every $p \in U$ there exists an integral curve $\gamma: \mathbf{R} \to U$ with $\gamma(0) = p$, i.e., for every p there exists an integral curve that starts at p and *exists for all time.*

To see what completeness involves, we recall that an integral curve

$$\gamma: [0, b) \to U,$$

with $\gamma(0) = p$ is called ***maximal*** if it cannot be extended to an interval $[0, b')$, where $b' > b$. (See for instance [2, § 6.11].) For such curves it is known that either

(1) $b = +\infty$,
(2) $|\gamma(t)| \to +\infty$ as $t \to b$,
(3) or the limit set of

$$\{\gamma(t) \mid 0 \le t < b\}$$

contains points on the boundary of U.

Hence if we can exclude (2) and (3), we have shown that an integral curve with $\gamma(0) = p$ exists for all positive time. A simple criterion for excluding (2) and (3) is the following.

Lemma 2.2.11. *The scenarios* (2) *and* (3) *cannot happen if there exists a proper* C^1 *function* $\phi: U \to \mathbf{R}$ *with* $L_{\boldsymbol{v}}\phi = 0$.

Proof. The identity $L_{\boldsymbol{v}}\phi = 0$ implies that ϕ is constant on $\gamma(t)$, but if $\phi(p) = c$ this implies that the curve $\gamma(t)$ lies on the compact subset $\phi^{-1}(c) \subset U$, hence it cannot "run off to infinity" as in scenario (2) or "run off the boundary" as in scenario (3). □

Applying a similar argument to the interval $(-b, 0]$ we conclude the following.

Theorem 2.2.12. *Suppose there exists a proper* C^1 *function* $\phi: U \to \mathbf{R}$ *with the property* $L_{\boldsymbol{v}}\phi = 0$. *Then the vector field* $\boldsymbol{v}$ *is complete.*

Example 2.2.13. Let $U = \mathbf{R}^2$ and let v be the vector field

$$v = x^3 \frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Then $\phi(x, y) = 2y^2 + x^4$ is a proper function with the property above.

Another hypothesis on $\boldsymbol{v}$ which excludes (2) and (3) is the following.

Definition 2.2.14. The ***support*** of $\boldsymbol{v}$ is set

$$\mathrm{supp}(\boldsymbol{v}) = \overline{\{q \in U \mid \boldsymbol{v}(q) \neq 0\}}.$$

We say that $\boldsymbol{v}$ is ***compactly supported*** if $\mathrm{supp}(\boldsymbol{v})$ is compact.

We now show that compactly supported vector fields are complete.

Theorem 2.2.15. *If a vector field* $\boldsymbol{v}$ *is compactly supported, then* $\boldsymbol{v}$ *is complete.*

Proof. Notice first that if $\boldsymbol{v}(p) = 0$, the constant curve, $\gamma_0(t) = p$, $-\infty < t < \infty$, satisfies the equation

$$\frac{d}{dt}\gamma_0(t) = 0 = \boldsymbol{v}(p),$$

so it is an integral curve of $\boldsymbol{v}$. Hence if $\gamma(t)$, $-a < t < b$, is any integral curve of $\boldsymbol{v}$ with the property $\gamma(t_0) = p$ for some t_0, it has to coincide with γ_0 on the interval $(-a, a)$, and hence has to be the constant curve $\gamma(t) = p$ on $(-a, a)$.

Now suppose the support $\mathrm{supp}(\boldsymbol{v})$ of $\boldsymbol{v}$ is compact. Then either $\gamma(t)$ is in $\mathrm{supp}(\boldsymbol{v})$ for all t or is in $U \setminus \mathrm{supp}(\boldsymbol{v})$ for some t_0. But if this happens, and $p = \gamma(t_0)$ then $\boldsymbol{v}(p) = 0$, so $\gamma(t)$ has to coincide with the constant curve $\gamma_0(t) = p$, for all t. In neither case can it go off to ∞ or off to the boundary of U as $t \to b$. □

One useful application of this result is the following. Suppose $\boldsymbol{v}$ is a vector field on $U \subset \mathbf{R}^n$, and one wants to see what its integral curves look like on some compact set $A \subset U$. Let $\rho \in C_0^\infty(U)$ be a bump function which is equal to one on a neighborhood of A. Then the vector field $\boldsymbol{w} = \rho \boldsymbol{v}$ is compactly supported and hence complete, but it is identical with $\boldsymbol{v}$ on A, so its integral curves on A coincide with the integral curves of $\boldsymbol{v}$.

If $\boldsymbol{v}$ is complete then for every p, one has an integral curve $\gamma_p \colon \mathbf{R} \to U$ with $\gamma_p(0) = p$, so one can define a map

$$f_t \colon U \to U$$

by setting $f_t(p) := \gamma_p(t)$. If $\boldsymbol{v}$ is C^∞, this mapping is C^∞ by the smooth dependence on initial data theorem, and, by definition, $f_0 = \mathrm{id}_U$. We claim that the f_t's also have the property

$$f_t \circ f_a = f_{t+a}. \tag{2.2.16}$$

Indeed if $f_a(p) = q$, then by the reparameterization theorem, $\gamma_q(t)$ and $\gamma_p(t+a)$ are both integral curves of $\boldsymbol{v}$, and since $q = \gamma_q(0) = \gamma_p(a) = f_a(p)$, they have the same initial point, so

$$\begin{aligned} \gamma_q(t) &= f_t(q) = (f_t \circ f_a)(p) \\ &= \gamma_p(t+a) = f_{t+a}(p) \end{aligned}$$

for all t. Since f_0 is the identity it follows from (2.2.16) that $f_t \circ f_{-t}$ is the identity, i.e.,

$$f_{-t} = f_t^{-1},$$

so f_t is a C^∞ diffeomorphism. Hence if $\boldsymbol{v}$ is complete it generates a ***one-parameter group*** f_t, $-\infty < t < \infty$, of C^∞ diffeomorphisms of U.

If $\boldsymbol{v}$ is not complete there is an analogous result, but it is trickier to formulate precisely. Roughly speaking $\boldsymbol{v}$ generates a one-parameter group of diffeomorphisms f_t but these diffeomorphisms are not defined on all of U nor for all values of t. Moreover, the identity (2.2.16) only holds on the open subset of U where both sides are well-defined.

We devote the remainder of this section to discussing some "functorial" properties of vector fields and 1-forms.

Definition 2.2.17. Let $U \subset \mathbf{R}^n$ and $W \subset \mathbf{R}^m$ be open, and let $f \colon U \to W$ be a C^∞ map. If $\boldsymbol{v}$ is a C^∞ vector field on U and $\boldsymbol{w}$ a C^∞ vector field on W we say that $\boldsymbol{v}$ and $\boldsymbol{w}$ are ***f-related*** if, for all $p \in U$ we have

$$df_p(\boldsymbol{v}(p)) = \boldsymbol{w}(f(p)).$$

Writing

$$\boldsymbol{v} = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \quad v_i \in C^k(U)$$

and

$$\boldsymbol{w} = \sum_{j=1}^m w_j \frac{\partial}{\partial y_j}, \quad w_j \in C^k(V)$$

this equation reduces, in coordinates, to the equation

$$w_i(q) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(p) v_j(p).$$

In particular, if $m = n$ and f is a C^∞ diffeomorphism, the formula (3.2) defines a C^∞ vector field on W, i.e.,

$$\boldsymbol{w} = \sum_{j=1}^{n} w_i \frac{\partial}{\partial y_j}$$

is the vector field defined by the equation

$$w_i = \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} v_j \right) \circ f^{-1}.$$

Hence we have proved the following.

Theorem 2.2.18. *If $f : U \xrightarrow{\simeq} W$ is a C^∞ diffeomorphism and $\boldsymbol{v}$ a C^∞ vector field on U, there exists a* unique *C^∞ vector field $\boldsymbol{w}$ on W having the property that $\boldsymbol{v}$ and $\boldsymbol{w}$ are f-related.*

Definition 2.2.19. In the setting of Theorem 2.2.18, we denote the vector field $\boldsymbol{w}$ by $f_\star \boldsymbol{v}$, and call $f_\star \boldsymbol{v}$ the ***pushforward*** of $\boldsymbol{v}$ by f.

We leave the following assertions as easy exercises (but provide some hints).

Theorem 2.2.20. *For $i = 1, 2$, let U_i be an open subset of $\mathbf{R}^{n_i}$, $\boldsymbol{v}_i$ a vector field on U_i, and $f : U_1 \to U_2$ a C^∞ map. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are f-related, every integral curve*

$$\gamma : I \to U_1$$

of $\boldsymbol{v}_1$ gets mapped by f onto an integral curve $f \circ \gamma : I \to U_2$ of $\boldsymbol{v}_2$.

Proof sketch. Theorem 2.2.20 follows from the chain rule: if $p = \gamma(t)$ and $q = f(p)$

$$df_p\left(\frac{d}{dt}\gamma(t)\right) = \frac{d}{dt} f(\gamma(t)). \qquad \square$$

Corollary 2.2.21. *In the setting of Theorem 2.2.20, suppose $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are complete. Let $(f_{i,t})_{t\in\mathbf{R}} : U_i \xrightarrow{\simeq} U_i$ be the one-parameter group of diffeomorphisms generated by $\boldsymbol{v}_i$. Then $f \circ f_{1,t} = f_{2,t} \circ f$.*

Proof sketch. To deduce Corollary 2.2.21 from Theorem 2.2.20 note that for $p \in U$, $f_{1,t}(p)$ is just the integral curve $\gamma_p(t)$ of $\boldsymbol{v}_1$ with initial point $\gamma_p(0) = p$.

The notion of f-relatedness can be very succinctly expressed in terms of the Lie differentiation operation. For $\phi \in C^\infty(U_2)$ let $f^\star\phi$ be the composition, $\phi \circ f$, viewed as a C^∞ function on U_1, i.e., for $p \in U_1$ let $f^\star\phi(p) = \phi(f(p))$. Then

$$f^\star L_{\boldsymbol{v}_2}\phi = L_{\boldsymbol{v}_1} f^\star \phi.$$

To see this, note that if $f(p) = q$ then at the point p the right-hand side is

$$(d\phi)_q \circ df_p(\boldsymbol{v}_1(p))$$

by the chain rule and by definition the left-hand side is

$$d\phi_q(v_2(q)).$$

Moreover, by definition

$$v_2(q) = df_p(v_1(p))$$

so the two sides are the same. □

Another easy consequence of the chain rule is the following theorem.

Theorem 2.2.22. *For $i = 1, 2, 3$, let U_i be an open subset of $\mathbf{R}^{n_i}$, $\boldsymbol{v}_i$ a vector field on U_i, and for $i = 1, 2$ let $f_i : U_i \to U_{i+1}$ be a C^∞ map. Suppose that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are f_1-related and that $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are f_2-related. Then $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$ are $(f_2 \circ f_1)$-related.*

In particular, if f_1 and f_2 are diffeomorphisms writing $\boldsymbol{v} \coloneqq \boldsymbol{v}_1$ we have

$$(f_2)_\star (f_1)_\star \boldsymbol{v} = (f_2 \circ f_1)_\star \boldsymbol{v}.$$

The results we described above have "dual" analogues for 1-forms.

Construction 2.2.23. Let $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$ be open, and let $f : U \to V$ be a C^∞ map. Given a 1-form μ on V one can define a ***pullback*** 1-form $f^\star\mu$ on U by the following method. For $p \in U$, by definition $\mu_{f(p)}$ is a linear map

$$\mu_{f(p)} : T_{f(p)}\mathbf{R}^m \to \mathbf{R}$$

and by composing this map with the linear map

$$df_p : T_p\mathbf{R}^n \to T_{f(p)}\mathbf{R}^n$$

we get a linear map

$$\mu_{f(p)} \circ df_p : T_p\mathbf{R}^n \to \mathbf{R},$$

i.e., an element $\mu_{f(p)} \circ df_p$ of $T_p^\star\mathbf{R}^n$.

The pullback 1-form $f^\star\mu$ is defined by the assignment $p \mapsto (\mu_{f(p)} \circ df_p)$.

In particular, if $\phi : V \to \mathbf{R}$ is a C^∞ function and $\mu = d\phi$ then

$$\mu_q \circ df_p = d\phi_q \circ df_p = d(\phi \circ f)_p,$$

i.e.,

$$f^\star\mu = d\phi \circ f.$$

In particular if μ is a 1-form of the form $\mu = d\phi$, with $\phi \in C^\infty(V)$, $f^\star\mu$ is C^∞. From this it is easy to deduce the following theorem.

Theorem 2.2.24. *If μ is any C^∞ 1-form on V, its pullback $f^\star\omega$ is C^∞. (See Exercise 2.2.ii.)*

Notice also that the pullback operation on 1-forms and the pushforward operation on vector fields are somewhat different in character. The former is defined for *all* C^∞ maps, but the latter is only defined for diffeomorphisms.

Exercises for §2.2

Exercise 2.2.i. Prove the reparameterization result Theorem 2.2.7.

Exercise 2.2.ii. Let U be an open subset of $\mathbf{R}^n$, V an open subset of $\mathbf{R}^n$ and $f : U \to V$ a C^k map.

(1) Show that for $\phi \in C^\infty(V)$ (2.2) the chain the rule for f and ϕ

$$f^\star d\phi = df^\star\phi.$$

(2) Let μ be the 1-form

$$\mu = \sum_{i=1}^{m} \phi_i dx_i, \quad \phi_i \in C^\infty(V)$$

on V. Show that if $f = (f_1, \ldots, f_m)$ then

$$f^\star\mu = \sum_{i=1}^{m} f^\star\phi_i\, df_i.$$

(3) Show that if μ is C^∞ and f is C^∞, $f^\star\mu$ is C^∞.

Exercise 2.2.iii. Let $\boldsymbol{v}$ be a complete vector field on U and $f_t : U \to U$ the one parameter group of diffeomorphisms generated by $\boldsymbol{v}$. Show that if $\phi \in C^1(U)$

$$L_v\phi = \frac{d}{dt} f_t^\star\phi \bigg|_{t=0}.$$

Exercise 2.2.iv.

(1) Let $U = \mathbf{R}^2$ and let $\boldsymbol{v}$ be the vector field, $x_1\partial/\partial x_2 - x_2\partial/\partial x_1$. Show that the curve

$$t \mapsto (r\cos(t+\theta), r\sin(t+\theta)),$$

for $t \in \mathbf{R}$, is the unique integral curve of $\boldsymbol{v}$ passing through the point, $(r\cos\theta, r\sin\theta)$, at $t = 0$.

(2) Let $U = \mathbf{R}^n$ and let $\boldsymbol{v}$ be the constant vector field: $\sum_{i=1}^n c_i\partial/\partial x_i$. Show that the curve

$$t \mapsto a + t(c_1, \ldots, c_n),$$

for $t \in \mathbf{R}$, is the unique integral curve of $\boldsymbol{v}$ passing through $a \in \mathbf{R}^n$ at $t = 0$.

(3) Let $U = \mathbf{R}^n$ and let $\boldsymbol{v}$ be the vector field, $\sum_{i=1}^n x_i\partial/\partial x_i$. Show that the curve

$$t \mapsto e^t(a_1, \ldots, a_n),$$

for $t \in \mathbf{R}$, is the unique integral curve of $\boldsymbol{v}$ passing through a at $t = 0$.

Exercise 2.2.v. Show that the following are one-parameter groups of diffeomorphisms:

(1) $f_t : \mathbf{R} \to \mathbf{R}$, $f_t(x) = x + t$;
(2) $f_t : \mathbf{R} \to \mathbf{R}$, $f_t(x) = e^t x$;
(3) $f_t : \mathbf{R}^2 \to \mathbf{R}^2$, $f_t(x, y) = (x\cos(t) - y\sin(t), x\sin(t) + y\cos(t))$.

Exercise 2.2.vi. Let $A: \mathbf{R}^n \to \mathbf{R}^n$ be a linear mapping. Show that the series

$$\exp(tA) := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \mathrm{id}_n + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

converges and defines a one-parameter group of diffeomorphisms of $\mathbf{R}^n$.

Exercise 2.2.vii.

(1) What are the infinitesimal generators of the one-parameter groups in Exercise 2.2.v?
(2) Show that the infinitesimal generator of the one-parameter group in Exercise 2.2.vi is the vector field

$$\sum_{1 \le i,j \le n} a_{i,j} x_j \frac{\partial}{\partial x_i},$$

where $(a_{i,j})$ is the defining matrix of A.

Exercise 2.2.viii. Let $\boldsymbol{v}$ be the vector field on $\mathbf{R}$ given by $x^2 \frac{d}{dx}$. Show that the curve

$$x(t) = \frac{a}{a - at}$$

is an integral curve of $\boldsymbol{v}$ with initial point $x(0) = a$. Conclude that for $a > 0$ the curve

$$x(t) = \frac{a}{1 - at}, \quad 0 < t < \frac{1}{a}$$

is a maximal integral curve. (In particular, conclude that $\boldsymbol{v}$ is not complete.)

Exercise 2.2.ix. Let U and V be open subsets of $\mathbf{R}^n$ and $f: U \xrightarrow{\sim} V$ a diffeomorphism. If $\boldsymbol{w}$ is a vector field on V, define the ***pullback*** of $\boldsymbol{w}$ to U to be the vector field

$$f^\star \boldsymbol{w} := (f_\star^{-1} \boldsymbol{w}).$$

Show that if ϕ is a C^∞ function on V

$$f^\star L_{\boldsymbol{w}} \phi = L_{f^\star \boldsymbol{w}} f^\star \phi.$$

Hint: Equation (2.2.25).

Exercise 2.2.x. Let U be an open subset of $\mathbf{R}^n$ and $\boldsymbol{v}$ and $\boldsymbol{w}$ vector fields on U. Suppose $\boldsymbol{v}$ is the infinitesimal generator of a one-parameter group of diffeomorphisms

$$f_t: U \xrightarrow{\sim} U, \quad -\infty < t < \infty.$$

Let $\boldsymbol{w}_t = f_t^\star \boldsymbol{w}$. Show that for $\phi \in C^\infty(U)$ we have

$$L_{[\boldsymbol{v},\boldsymbol{w}]}\phi = L_{\dot{\boldsymbol{w}}}\phi,$$

where

$$\dot{\boldsymbol{w}} = \frac{d}{dt} f_t^\star \boldsymbol{w} \bigg|_{t=0}.$$

Hint: Differentiate the identity

$$f_t^\star L_w \phi = L_{w_t} f_t^\star \phi$$

with respect to t and show that at $t = 0$ the derivative of the left-hand side is $L_v L_w \phi$ by Exercise 2.2.iii, and the derivative of the right-hand side is

$$L_{\dot{w}} + L_w(L_v \phi).$$

Exercise 2.2.xi. Conclude from Exercise 2.2.x that

$$[v, w] = \frac{d}{dt} f_t^\star w \bigg|_{t=0} . \tag{2.2.25}$$

2.3. Differential k-forms

1-Forms are the bottom tier in a pyramid of objects whose kth tier is the space of *k-forms*. More explicitly, given $p \in \mathbf{R}^n$ we can, as in § 1.5, form the kth exterior powers

$$\Lambda^k(T_p^\star \mathbf{R}^n) , \quad k = 1, 2, 3, \dots, n$$

of the vector space $T_p^\star \mathbf{R}^n$, and since

$$\Lambda^1(T_p^\star \mathbf{R}^n) = T_p^\star \mathbf{R}^n \tag{2.3.1}$$

one can think of a 1-form as a function which takes its value at p in the space (2.3.1). This leads to an obvious generalization.

Definition 2.3.2. Let U be an open subset of $\mathbf{R}^n$. A ***k-form*** ω on U is a function which assigns to each point $p \in U$ an element $\omega_p \in \Lambda^k(T_p^\star \mathbf{R}^n)$.

The wedge product operation gives us a way to construct lots of examples of such objects.

Example 2.3.3. Let $\omega_1, \dots, \omega_k$ be 1-forms. Then $\omega_1 \wedge \cdots \wedge \omega_k$ is the k-form whose value at p is the wedge product

$$(\omega_1 \wedge \cdots \wedge \omega_k)_p := (\omega_1)_p \wedge \cdots \wedge (\omega_k)_p. \tag{2.3.4}$$

Notice that since $(\omega_i)_p$ is in $\Lambda^1(T_p^\star \mathbf{R}^n)$ the wedge product (2.3.4) makes sense and is an element of $\Lambda^k(T_p^\star \mathbf{R}^n)$.

Example 2.3.5. Let $f_1, \dots, f_k$ be a real-valued C^∞ functions on U. Letting $\omega_i = df_i$ we get from (2.3.4) a k-form

$$df_1 \wedge \cdots \wedge df_k$$

whose value at p is the wedge product

$$(df_1)_p \wedge \cdots \wedge (df_k)_p. \tag{2.3.6}$$

Since $(dx_1)_p, \dots, (dx_n)_p$ are a basis of $T_p^\star \mathbf{R}^n$, the wedge products

$$(dx_{i_1})_p \wedge \cdots \wedge (dx_{1_k})_p, \quad 1 \le i_1 < \cdots < i_k \le n \tag{2.3.7}$$

are a basis of $\Lambda^k(T_p^\star)$. To keep our multi-index notation from getting out of hand, we denote these basis vectors by $(dx_I)_p$, where $I = (i_1, \dots, i_k)$ and the multi-indices

I range over multi-indices of length k which are *strictly increasing*. Since these wedge products are a basis of $\Lambda^k(T_p^\star \mathbf{R}^n)$, every element of $\Lambda^k(T_p^\star \mathbf{R}^n)$ can be written uniquely as a sum

$$\sum_I c_I (dx_I)_p, \quad c_I \in \mathbf{R}$$

and every k-form ω on U can be written uniquely as a sum

$$\omega = \sum_I f_I dx_I, \tag{2.3.8}$$

where dx_I is the k-form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and f_I is a real-valued function $f_I : U \to \mathbf{R}$.

Definition 2.3.9. The k-form (2.3.8) is ***of class*** C^r if each function f_I is in $C^r(U)$.

Henceforth we assume, unless otherwise stated, that *all the k-forms we consider are of class* C^∞. We denote the set of k-forms of class C^∞ on U by $\Omega^k(U)$.

We will conclude this section by discussing a few simple operations on k-forms. Let U be an open subset of $\mathbf{R}^n$.

(1) Given a function $f \in C^\infty(U)$ and a k-form $\omega \in \Omega^k(U)$, we define $f\omega \in \Omega^k(U)$ to be the k-form defined by

$$p \mapsto f(p)\omega_p.$$

(2) Given $\omega_1, \omega_2 \in \Omega^k(U)$, we define $\omega_1 + \omega_2 \in \Omega^k(U)$ to be the k-form

$$p \mapsto (\omega_1)_p + (\omega_2)_p.$$

(Notice that this sum makes sense since each summand is in $\Lambda^k(T_p^\star \mathbf{R}^n)$.)

(3) Given $\omega_1 \in \Omega^{k_1}(U)$ and $\omega_2 \in \Omega^{k_2}(U)$ we define their ***wedge product***, $\omega_1 \wedge \omega_2 \in \Omega^{k_1+k_2}(U)$, to be the $(k_1 + k_2)$-form

$$p \mapsto (\omega_1)_p \wedge (\omega_2)_p.$$

We recall that $\Lambda^0(T_p^\star \mathbf{R}^n) = \mathbf{R}$, so a 0-form is an $\mathbf{R}$-valued function and a 0-form of class C^∞ is a C^∞ function, i.e., $\Omega^0(U) = C^\infty(U)$.

The addition and multiplication by functions operations naturally give the sets $\Omega^k(U)$ the structures of $\mathbf{R}$-vector spaces — we always regard $\Omega^k(U)$ as a vector space in this manner.

A fundamental operation on forms is the exterior differentiation operation which associates to a function $f \in C^\infty(U)$ the 1-form df. It is clear from the identity (2.2.3) that df is a 1-form of class C^∞, so the d-operation can be viewed as a map

$$d : \Omega^0(U) \to \Omega^1(U).$$

In the next section we show that an analogue of this map exists for every $\Omega^k(U)$.

Exercises for §2.3

Exercise 2.3.i. Let $\omega \in \Omega^2(\mathbf{R}^4)$ be the 2-form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Compute $\omega \wedge \omega$.

Exercise 2.3.ii. Let $\omega_1, \omega_2, \omega_3 \in \Omega^1(\mathbf{R}^3)$ be the 1-forms

$$\omega_1 = x_2 dx_3 - x_3 dx_2,$$
$$\omega_2 = x_3 dx_1 - x_1 dx_3,$$
$$\omega_3 = x_1 dx_2 - x_2 dx_1.$$

Compute the following.

(1) $\omega_1 \wedge \omega_2$.
(2) $\omega_2 \wedge \omega_3$.
(3) $\omega_3 \wedge \omega_1$.
(4) $\omega_1 \wedge \omega_2 \wedge \omega_3$.

Exercise 2.3.iii. Let U be an open subset of $\mathbf{R}^n$ and $f_1, \dots, f_n \in C^\infty(U)$. Show that

$$df_1 \wedge \cdots \wedge df_n = \det\left[\frac{\partial f_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n.$$

Exercise 2.3.iv. Let U be an open subset of $\mathbf{R}^n$. Show that every $(n-1)$-form $\omega \in \Omega^{n-1}(U)$ can be written uniquely as a sum

$$\sum_{i=1}^{n} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where $f_i \in C^\infty(U)$ and $\widehat{dx_i}$ indicates that dx_i is to be omitted from the wedge product $dx_1 \wedge \cdots \wedge dx_n$.

Exercise 2.3.v. Let $\mu = \sum_{i=1}^{n} x_i dx_i$. Show that there exists an $(n-1)$-form, $\omega \in \Omega^{n-1}(\mathbf{R}^n \smallsetminus \{0\})$ with the property

$$\mu \wedge \omega = dx_1 \wedge \cdots \wedge dx_n.$$

Exercise 2.3.vi. Let J be the multi-index $(j_1, \dots, j_k)$ and let $dx_J = dx_{j_1} \wedge \cdots \wedge dx_{j_k}$. Show that $dx_J = 0$ if $j_r = j_s$ for some $r \neq s$ and show that if the numbers $j_1, \dots, j_k$ are all distinct, then

$$dx_J = (-1)^\sigma dx_I,$$

where $I = (i_1, \dots, i_k)$ is the strictly increasing rearrangement of $(j_1, \dots, j_k)$ and σ is the permutation

$$(j_1, \dots, j_k) \mapsto (i_1, \dots, i_k).$$

Exercise 2.3.vii. Let I be a strictly increasing multi-index of length k and J a strictly increasing multi-index of length ℓ. What can one say about the wedge product $dx_I \wedge dx_J$?

2.4. Exterior differentiation

Let U be an open subset of $\mathbf{R}^n$. In this section we define an *exterior differentiation* operation

$$d\colon \Omega^k(U) \to \Omega^{k+1}(U). \tag{2.4.1}$$

This operation is the fundamental operation in n-dimensional vector calculus.

For $k = 0$ we already defined the operation (2.4.1) in §2.1. Before defining it for $k > 0$, we list some properties that we require to this operation to satisfy.

Properties 2.4.2 (desired properties of exterior differentiation). Let U be an open subset of $\mathbf{R}^n$.

(1) For ω_1 and ω_2 in $\Omega^k(U)$, we have

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2.$$

(2) For $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$, we have

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2. \tag{2.4.3}$$

(3) For all $\omega \in \Omega^k(U)$, we have

$$d(d\omega) = 0. \tag{2.4.4}$$

Let us point out a few consequences of these properties.

Lemma 2.4.5. *Let U be an open subset of $\mathbf{R}^n$. For any functions $f_1, \dots, f_k \in C^\infty(U)$, we have*

$$d(df_1 \wedge \cdots \wedge df_k) = 0.$$

Proof. We prove this by induction on k. For the base case note that by (3)

$$d(df_1) = 0 \tag{2.4.6}$$

for every function $f_1 \in C^\infty(U)$.

For the induction step, suppose that we know the result for $k-1$ functions and that we are given functions $f_1, \dots, f_k \in C^\infty(U)$. Let $\mu = df_2 \wedge \cdots \wedge df_k$. Then, by the induction hypothesis $d\mu = 0$, combining (2.4.3) with (2.4.6) we see that

$$\begin{aligned} d(df_1 \wedge df_2 \wedge \cdots \wedge df_k) &= d(df_1 \wedge \mu) \\ &= d(df_1) \wedge \mu + (-1) df_1 \wedge d\mu \\ &= 0. \end{aligned}$$

□

Example 2.4.7. As a special case of Lemma 2.4.5, given a multi-index $I = (i_1, \dots, i_k)$ with $1 \le i_r \le n$ we have

$$d(dx_I) = d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = 0. \tag{2.4.8}$$

Recall that every k-form $\omega \in \Omega^k(U)$ can be written uniquely as a sum

$$\omega = \sum_I f_I dx_I\,, \quad f_I \in C^\infty(U),$$

where the multi-indices I are strictly increasing. Thus by (2.4.3) and (2.4.8)

$$d\omega = \sum_I df_I \wedge dx_I. \tag{2.4.9}$$

This shows that if there exists an operator $d\colon \Omega^*(U) \to \Omega^{*+1}(U)$ with Properties 2.4.2, then d is necessarily given by the formula (2.4.9). Hence all we have to show is that the operator defined by this equation (2.4.9) has these properties.

Proposition 2.4.10. *Let U be an open subset of $\mathbf{R}^n$. There is a unique operator $d\colon \Omega^*(U) \to \Omega^{*+1}(U)$ satisfying Properties 2.4.2.*

Proof. The property (1) is obvious.

To verify (2) we first note that for I strictly increasing equation (2.4.8) is a special case of equation (2.4.9) (take $f_I = 1$ and $f_J = 0$ for $J \neq I$). Moreover, if I is not strictly increasing it is either repeating, in which case $dx_I = 0$, or non-repeating in which case there exists a permutation $\sigma \in S_k$ such that I^σ is strictly increasing. Moreover,

$$dx_I = (-1)^\sigma dx_{I^\sigma}. \tag{2.4.11}$$

Hence (2.4.9) implies (2.4.8) for *all* multi-indices I. The same argument shows that for any sum $\sum_I f_I dx_I$ over multi-indices I of length k, we have

$$d(\textstyle\sum_I f_I dx_I) = \displaystyle\sum_I df_I \wedge dx_I. \tag{2.4.12}$$

(As above we can ignore the repeating multi-indices $dx_I = 0$ if I is repeating, and by (2.4.11) we can replace the non-repeating multi-indices by strictly increasing multi-indices.)

Suppose now that $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$. Write $\omega_1 := \sum_I f_I dx_I$ and $\omega_2 := \sum_J g_J dx_J$, where $f_I, g_J \in C^\infty(U)$. We see that the wedge product $\omega_1 \wedge \omega_2$ is given by

$$\omega_1 \wedge \omega_2 = \sum_{I,J} f_I g_J dx_I \wedge dx_J, \tag{2.4.13}$$

and by equation (2.4.12)

$$d(\omega_1 \wedge \omega_2) = \sum_{I,J} d(f_I g_J) \wedge dx_I \wedge dx_J. \tag{2.4.14}$$

(Notice that if $I = (i_1, \ldots, i_k)$ and $J = (j_i, \ldots, i_\ell)$, we have $dx_I \wedge dx_J = dx_K$, where $K := (i_1, \ldots, i_k, j_1, \ldots, j_\ell)$. Even if I and J are strictly increasing, K is not necessarily strictly increasing. However, in deducing (2.4.14) from (2.4.13) we have observed that this does not matter.)

Now note that by equation (2.2.8)

$$d(f_I g_J) = g_J df_I + f_I dg_J,$$

and by the wedge product identities of §1.6,

$$dg_J \wedge dx_I = dg_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = (-1)^k dx_I \wedge dg_J,$$

so the sum (2.4.14) can be rewritten:

$$\sum_{I,J} df_I \wedge dx_I \wedge g_J dx_J + (-1)^k \sum_{I,J} f_I dx_I \wedge dg_J \wedge dx_J,$$

or

$$\left(\sum_I df_I \wedge dx_I\right) \wedge \left(\sum_J g_J dx_J\right) + (-1)^k \left(\sum_J dg_J \wedge dx_J\right) \wedge \left(\sum_I df_I \wedge dx_I\right),$$

or, finally,

$$d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2.$$

Thus the operator d defined by equation (2.4.9) satisfies (2).

Let us now check that d satisfies (3). If $\omega = \sum_I f_I dx_I$, where $f_I \in C^\infty(U)$, then by definition, $d\omega = \sum_I df_I \wedge dx_I$ and by (2.4.8) and (2.4.3)

$$d(d\omega) = \sum_I d(df_I) \wedge dx_I,$$

so it suffices to check that $d(df_I) = 0$, i.e., it suffices to check (2.4.6) for 0-forms $f \in C^\infty(U)$. However, by (2.2.3) we have

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j$$

so by equation (2.4.9)

$$d(df) = \sum_{j=1}^n d\left(\frac{\partial f}{\partial x_j}\right) dx_j = \sum_{j=1}^n \left(\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i\right) \wedge dx_j$$

$$= \sum_{1 \le i,j \le n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j.$$

Notice, however, that in this sum, $dx_i \wedge dx_j = -dx_j \wedge dx_i$ and

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

so the (i, j) term cancels the (j, i) term, and the total sum is zero. □

Definition 2.4.15. Let U be an open subset of $\mathbf{R}^n$. A k-form $\omega \in \Omega^k(U)$ is ***closed*** if $d\omega = 0$ and is ***exact*** if $\omega = d\mu$ for some $\mu \in \Omega^{k-1}(U)$.

By (3) every exact form is closed, but the converse is not true even for 1-forms (see Exercise 2.1.iii). In fact it is a very interesting (and hard) question to determine if an open set U has the following property: *For $k > 0$ every closed k-form is exact.*[2]

Some examples of spaces with this property are described in the exercises at the end of §2.6. We also sketch below a proof of the following result (and ask you to fill in the details).

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[2]For $k = 0$, $df = 0$ *does not* imply that f is exact. In fact "exactness" does not make much sense for zero forms since there are not any "(-1)-forms". However, if $f \in C^\infty(U)$ and $df = 0$ then f is constant on connected components of U (see Exercise 2.2.iii).

Lemma 2.4.16 (Poincaré lemma). *If ω is a closed form on U of degree $k > 0$, then for every point $p \in U$, there exists a neighborhood of p on which ω is exact.*

Proof. See Exercises 2.4.v and 2.4.vi. □

Exercises for § 2.4

Exercise 2.4.i. Compute the exterior derivatives of the following differential forms.

(1) $x_1 dx_2 \wedge dx_3$;
(2) $x_1 dx_2 - x_2 dx_1$;
(3) $e^{-f} df$ where $f = \sum_{i=1}^n x_i^2$;
(4) $\sum_{i=1}^n x_i dx_i$;
(5) $\sum_{i=1}^n (-1)^i x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$.

Exercise 2.4.ii. Solve the equation $d\mu = \omega$ for $\mu \in \Omega^1(\mathbf{R}^3)$, where ω is the 2-form:

(1) $dx_2 \wedge dx_3$;
(2) $x_2 dx_2 \wedge dx_3$;
(3) $(x_1^2 + x_2^2) dx_1 \wedge dx_2$;
(4) $\cos(x_1) dx_1 \wedge dx_3$.

Exercise 2.4.iii. Let U be an open subset of $\mathbf{R}^n$.

(1) Show that if $\mu \in \Omega^k(U)$ is exact and $\omega \in \Omega^\ell(U)$ is closed then $\mu \wedge \omega$ is exact.
Hint: Equation (2.4.3).
(2) In particular, dx_1 is exact, so if $\omega \in \Omega^\ell(U)$ is closed $dx_1 \wedge \omega = d\mu$. What is μ?

Exercise 2.4.iv. Let Q be the rectangle $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Show that if ω is in $\Omega^n(Q)$, then ω is exact.

Hint: Let $\omega = f dx_1 \wedge \cdots \wedge dx_n$ with $f \in C^\infty(Q)$ and let g be the function

$$g(x_1, \dots, x_n) = \int_{a_1}^{x_1} f(t, x_2, \dots, x_n) dt.$$

Show that $\omega = d(g dx_2 \wedge \cdots \wedge dx_n)$.

Exercise 2.4.v. Let U be an open subset of $\mathbf{R}^{n-1}$, $A \subset \mathbf{R}$ an open interval and (x, t) product coordinates on $U \times A$. We say that a form $\mu \in \Omega^\ell(U \times A)$ is ***reduced*** if μ can be written as a sum

$$\mu = \sum_I f_I(x, t) dx_I \tag{2.4.17}$$

(i.e., with no terms involving dt).

(1) Show that every form, $\omega \in \Omega^k(U \times A)$ can be written uniquely as a sum:

$$\omega = dt \wedge \alpha + \beta, \tag{2.4.18}$$

where α and β are reduced.

(2) Let μ be the reduced form (2.4.17) and let

$$\frac{d\mu}{dt} = \sum \frac{d}{dt} f_I(x, t) dx_I$$

and

$$d_U\mu = \sum_I \left(\sum_{i=1}^n \frac{\partial}{\partial x_i} f_I(x,t) dx_i \right) \wedge dx_I.$$

Show that

$$d\mu = dt \wedge \frac{d\mu}{dt} + d_U\mu.$$

(3) Let ω be the form (2.4.18). Show that

$$d\omega = -dt \wedge d_U\alpha + dt \wedge \frac{d\beta}{dt} + d_U\beta$$

and conclude that ω is closed if and only if

(2.4.19)
$$\begin{cases} \dfrac{d\beta}{dt} = d_U\alpha, \\ d_U\beta = 0. \end{cases}$$

(4) Let α be a reduced $(k-1)$-form. Show that there exists a reduced $(k-1)$-form ν such that

(2.4.20)
$$\frac{d\nu}{dt} = \alpha.$$

Hint: Let $\alpha = \sum_I f_I(x,t)dx_I$ and $\nu = \sum g_I(x,t)dx_I$. Equation (2.4.20) reduces to the system of equations

(2.4.21)
$$\frac{d}{dt} g_I(x,t) = f_I(x,t).$$

Let c be a point on the interval, A, and using calculus show that equation (2.4.21) has a unique solution, $g_I(x,t)$, with $g_I(x,c) = 0$.

(5) Show that if ω is the form (2.4.18) and ν a solution of (2.4.20) then the form

(2.4.22)
$$\omega - d\nu$$

is reduced.

(6) Let

$$\gamma = \sum h_I(x,t)dx_I$$

be a reduced k-form. Deduce from (2.4.19) that if γ is closed then $\frac{d\gamma}{dt} = 0$ and $d_U\gamma = 0$. Conclude that $h_I(x,t) = h_I(x)$ and that

$$\gamma = \sum_I h_I(x)dx_I$$

is effectively a closed k-form on U. Prove that if every closed k-form on U is exact, then every closed k-form on $U \times A$ is exact.

Hint: Let ω be a closed k-form on $U \times A$ and let γ be the form (2.4.22).

Exercise 2.4.vi. Let $Q \subset \mathbf{R}^n$ be an open rectangle. Show that every closed form on Q of degree $k > 0$ is exact.

Hint: Let $Q = (a_1, b_1) \times \cdots \times (a_n, b_n)$. Prove this assertion by induction, at the nth stage of the induction letting $U = (a_1, b_1) \times \cdots \times (a_{n-1}, b_{n-1})$ and $A = (a_n, b_n)$.

2.5. The interior product operation

In § 2.1 we explained how to pair a 1-form ω and a vector field $\boldsymbol{v}$ to get a function $\iota_{\boldsymbol{v}}\omega$. This pairing operation generalizes.

Definition 2.5.1. Let U be an open subset of $\mathbf{R}^n$, $\boldsymbol{v}$ a vector field on U, and $\omega \in \Omega^k(U)$. The ***interior product*** of $\boldsymbol{v}$ with ω is $(k-1)$-form $\iota_{\boldsymbol{v}}\omega$ on U defined by declaring the value of $\iota_{\boldsymbol{v}}\omega$ value at $p \in U$ to be the interior product

$$\iota_{\boldsymbol{v}(p)}\omega_p. \tag{2.5.2}$$

Note that $\boldsymbol{v}(p)$ is in $T_p\mathbf{R}^n$ and ω_p in $\Lambda^k(T_p^\star\mathbf{R}^n)$, so by definition of interior product (see § 1.7), the expression (2.5.2) is an element of $\Lambda^{k-1}(T_p^\star\mathbf{R}^n)$.

From the properties of interior product on vector spaces which we discussed in § 1.7, one gets analogous properties for this interior product on forms. We list these properties, leaving their verification as an easy exercise.

Properties 2.5.3. Let U be an open subset of $\mathbf{R}^n$, $\boldsymbol{v}$ and $\boldsymbol{w}$ vector fields on U, ω_1 and ω_2 both k-forms on U, and ω a k-form and μ an ℓ-form on U.

(1) *Linearity in the form:* we have

$$\iota_{\boldsymbol{v}}(\omega_1 + \omega_2) = \iota_{\boldsymbol{v}}\omega_1 + \iota_{\boldsymbol{v}}\omega_2.$$

(2) *Linearity in the vector field:* we have

$$\iota_{\boldsymbol{v}+\boldsymbol{w}}\omega = \iota_{\boldsymbol{v}}\omega + \iota_{\boldsymbol{w}}\omega.$$

(3) *Derivation property:* we have

$$\iota_{\boldsymbol{v}}(\omega \wedge \mu) = \iota_{\boldsymbol{v}}\omega \wedge \mu + (-1)^k\omega \wedge \iota_{\boldsymbol{v}}\mu.$$

(4) The interior product satisfies the identity

$$\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega) = -\iota_{\boldsymbol{w}}(\iota_{\boldsymbol{v}}\omega).$$

(5) As a special case of (4), the interior product satisfies the identity

$$\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) = 0.$$

(6) Moreover, if ω is decomposable, i.e., is a wedge product of 1-forms

$$\omega = \mu_1 \wedge \cdots \wedge \mu_k,$$

then

$$\iota_{\boldsymbol{v}}\omega = \sum_{r=1}^{k}(-1)^{r-1}\iota_{\boldsymbol{v}}(\mu_r)\mu_1 \wedge \cdots \wedge \hat{\mu}_r \wedge \cdots \wedge \mu_k.$$

We also leave for you to prove the following two assertions, both of which are special cases of (6).

Example 2.5.4. If $\boldsymbol{v} = \partial/\partial x_r$ and $\omega = dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, then

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^{k}(-1)^{i-1}\delta_{i,i_r}dx_{I_r}, \tag{2.5.5}$$

where

$$\delta_{i,i_r} := \begin{cases} 1, & i = i_r, \\ 0, & i \neq i_r, \end{cases}$$

and $I_r = (i_1, \ldots, \hat{\imath}_r, \ldots, i_k)$.

Example 2.5.6. If $\boldsymbol{v} = \sum_{i=1}^n f_i \partial/\partial x_i$ and $\omega = dx_1 \wedge \cdots \wedge dx_n$, then

$$\iota_{\boldsymbol{v}}\omega = \sum_{r=1}^{n} (-1)^{r-1} f_r\, dx_1 \wedge \cdots \widehat{dx_r} \cdots \wedge dx_n. \tag{2.5.7}$$

By combining exterior differentiation with the interior product operation one gets another basic operation of vector fields on forms: the *Lie differentiation* operation. For 0-forms, i.e., for C^∞ functions, we defined this operation by the formula (2.1.16). For k-forms we define it by a slightly more complicated formula.

Definition 2.5.8. Let U be an open subset of $\mathbf{R}^n$, $\boldsymbol{v}$ a vector field on U, and $\omega \in \Omega^k(U)$. The ***Lie derivative*** of ω with respect to $\boldsymbol{v}$ is the k-form

$$L_{\boldsymbol{v}}\omega := \iota_{\boldsymbol{v}}(d\omega) + d(\iota_{\boldsymbol{v}}\omega). \tag{2.5.9}$$

Notice that for 0-forms the second summand is zero, so (2.5.9) and (2.1.16) agree.

Properties 2.5.10. Let U be an open subset of $\mathbf{R}^n$, $\boldsymbol{v}$ a vector field on U, $\omega \in \Omega^k(U)$, and $\mu \in \Omega^\ell(U)$. The Lie derivative enjoys the following properties:

(1) *Commutativity with exterior differentiation:* we have

$$d(L_{\boldsymbol{v}}\omega) = L_{\boldsymbol{v}}(d\omega). \tag{2.5.11}$$

(2) *Interaction with wedge products:* we have

$$L_{\boldsymbol{v}}(\omega \wedge \mu) = L_{\boldsymbol{v}}\omega \wedge \mu + \omega \wedge L_{\boldsymbol{v}}\mu. \tag{2.5.12}$$

From Properties 2.5.10 it is fairly easy to get an explicit formula for $L_{\boldsymbol{v}}\omega$. Namely let ω be the k-form

$$\omega = \sum_I f_I dx_I, \quad f_I \in C^\infty(U)$$

and $\boldsymbol{v}$ the vector field

$$\boldsymbol{v} = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i}, \quad g_i \in C^\infty(U).$$

By equation (2.5.12)

$$L_{\boldsymbol{v}}(f_I dx_I) = (L_{\boldsymbol{v}} f_I) dx_I + f_I (L_{\boldsymbol{v}} dx_I)$$

and

$$L_{\boldsymbol{v}} dx_I = \sum_{r=1}^k dx_{i_1} \wedge \cdots \wedge L_{\boldsymbol{v}} dx_{i_r} \wedge \cdots \wedge dx_{i_k},$$

and by equation (2.5.11)

$$L_{\boldsymbol{v}} dx_{i_r} = dL_{\boldsymbol{v}} x_{i_r}$$

so to compute $L_{\boldsymbol{v}}\omega$ one is reduced to computing $L_{\boldsymbol{v}} x_{i_r}$ and $L_{\boldsymbol{v}} f_I$.

However by equation (2.5.12)

$$L_v x_{i_r} = g_{i_r}$$

and

$$L_v f_I = \sum_{i=1}^{n} g_i \frac{\partial f_I}{\partial x_i}.$$

We leave the verification of Properties 2.5.10 as exercises, and also ask you to prove (by the method of computation that we have just sketched) the following *divergence formula.*

Lemma 2.5.13. *Let $U \subset \mathbf{R}^n$ be open, let $g_1, \dots, g_n \in C^\infty(U)$, and set $v := \sum_{i=1}^n g_i\, \partial/\partial x_i$. Then*

$$L_v(dx_1 \wedge \cdots \wedge dx_n) = \sum_{i=1}^{n} \left(\frac{\partial g_i}{\partial x_i}\right) dx_1 \wedge \cdots \wedge dx_n. \tag{2.5.14}$$

Exercises for §2.5

Exercise 2.5.i. Verify Properties 2.5.3(1)–(6).

Exercise 2.5.ii. Show that if ω is the k-form dx_I and v the vector field $\partial/\partial x_r$, then $\iota_v\omega$ is given by equation (2.5.5).

Exercise 2.5.iii. Show that if ω is the n-form $dx_1 \wedge \cdots \wedge dx_n$ and v the vector field $\sum_{i=1}^n f_i \partial/\partial x_i$, then $\iota_v\omega$ is given by equation (2.5.7).

Exercise 2.5.iv. Let U be an open subset of $\mathbf{R}^n$ and v a C^∞ vector field on U. Show that for $\omega \in \Omega^k(U)$

$$dL_v\omega = L_v d\omega$$

and

$$\iota_v(L_v\omega) = L_v(\iota_v\omega).$$

Hint: Deduce the first of these identities from the identity $d(d\omega) = 0$ and the second from the identity $\iota_v(\iota_v\omega) = 0$.

Exercise 2.5.v. Given $\omega_i \in \Omega^{k_i}(U)$, for $i = 1, 2$ show that

$$L_v(\omega_1 \wedge \omega_2) = L_v\omega_1 \wedge \omega_2 + \omega_1 \wedge L_v\omega_2.$$

Hint: Plug $\omega = \omega_1 \wedge \omega_2$ into equation (2.5.9) and use equation (2.4.3) and the derivation property of the interior product to evaluate the resulting expression.

Exercise 2.5.vi. Let v_1 and v_2 be vector fields on U and let w be their Lie bracket. Show that for $\omega \in \Omega^k(U)$

$$L_w\omega = L_{v_1}(L_{v_2}\omega) - L_{v_2}(L_{v_1}\omega).$$

Hint: By definition this is true for 0-forms and by equation (2.5.11) for exact 1-forms. Now use the fact that every form is a sum of wedge products of 0-forms and 1-forms and the fact that L_v satisfies the product identity (2.5.12).

Exercise 2.5.vii. Prove the divergence formula (2.5.14).

Exercise 2.5.viii.

(1) Let $\omega = \Omega^k(\mathbf{R}^n)$ be the form

$$\omega = \sum_I f_I(x_1, \ldots, x_n) dx_I$$

and $\boldsymbol{v}$ the vector field $\partial/\partial x_n$. Show that

$$L_{\boldsymbol{v}}\omega = \sum \frac{\partial}{\partial x_n} f_I(x_1, \ldots, x_n) dx_I.$$

(2) Suppose $\iota_{\boldsymbol{v}}\omega = L_{\boldsymbol{v}}\omega = 0$. Show that ω only depends on $x_1, \ldots, x_{n-1}$ and $dx_1, \ldots, dx_{n-1}$, i.e., is effectively a k-form on $\mathbf{R}^{n-1}$.

(3) Suppose $\iota_{\boldsymbol{v}}\omega = d\omega = 0$. Show that ω is effectively a closed k-form on $\mathbf{R}^{n-1}$.

(4) Use these results to give another proof of the Poincaré lemma for $\mathbf{R}^n$. Prove by induction on n that every closed form on $\mathbf{R}^n$ is exact.

Hints:

- Let ω be the form in part (1) and let

$$g_I(x_1, \ldots, x_n) = \int_0^{x_n} f_I(x_1, \ldots, x_{n-1}, t) dt.$$

Show that if $\nu = \sum_I g_I dx_I$, then $L_{\boldsymbol{v}}\nu = \omega$.

- Conclude that

(2.5.15) $$\omega - d\iota_{\boldsymbol{v}}\nu = \iota_{\boldsymbol{v}} d\nu.$$

- Suppose $d\omega = 0$. Conclude from equation (2.5.15) and (5) of Properties 2.5.3 that the form $\beta = \iota_{\boldsymbol{v}} d\nu$ satisfies $d\beta = \iota(\boldsymbol{v})\beta = 0$.
- By part (3), β is effectively a closed form on $\mathbf{R}^{n-1}$, and by induction, $\beta = d\alpha$. Thus by equation (2.5.15)

$$\omega = d\iota_{\boldsymbol{v}}\nu + d\alpha.$$

2.6. The pullback operation on forms

Let U be an open subset of $\mathbf{R}^n$, V an open subset of $\mathbf{R}^m$, and $f : U \to V$ a C^∞ map. Then for $p \in U$ and the derivative of f at p

$$df_p : T_p\mathbf{R}^n \to T_{f(p)}\mathbf{R}^m$$

is a linear map, so (as explained in §1.7) we get from df_p a pullback map

(2.6.1) $$df_p^\star := (df_p)^\star : \Lambda^k(T_{f(p)}^\star\mathbf{R}^m) \to \Lambda^k(T_p^\star\mathbf{R}^n).$$

In particular, let ω be a k-form on V. Then at $f(p) \in V$, ω takes the value

$$\omega_{f(p)} \in \Lambda^k(T_q^\star\mathbf{R}^m),$$

so we can apply to it the operation (2.6.1), and this gives us an element:

(2.6.2) $$df_p^\star \omega_{f(p)} \in \Lambda^k(T_p^\star\mathbf{R}^n).$$

In fact we can do this for every point $p \in U$, and the assignment

$$p \mapsto (df_p)^\star \omega_{f(p)} \tag{2.6.3}$$

defines a k-form on U, which we denote by $f^\star\omega$. We call $f^\star\omega$ the ***pullback*** of ω along the map f. A few of its basic properties are described below.

Properties 2.6.4. Let U be an open subset of $\mathbf{R}^n$, V an open subset of $\mathbf{R}^m$, and $f : U \to V$ a C^∞ map.

(1) Let ϕ be a 0-form, i.e., a function $\phi \in C^\infty(V)$. Since

$$\Lambda^0(T_p^\star) = \Lambda^0(T_{f(p)}^\star) = \mathbf{R}$$

the map (2.6.1) is just the identity map of $\mathbf{R}$ onto $\mathbf{R}$ when $k = 0$. Hence for 0-forms

$$(f^\star\phi)(p) = \phi(f(p)), \tag{2.6.5}$$

that is, $f^\star\phi$ is just the composite function, $\phi \circ f \in C^\infty(U)$.

(2) Let $\phi \in \Omega^0(U)$ and let $\mu \in \Omega^1(V)$ be the 1-form $\mu = d\phi$. By the chain rule (2.6.2) unwinds to:

$$(df_p)^\star d\phi_q = (d\phi)_q \circ df_p = d(\phi \circ f)_p \tag{2.6.6}$$

and hence by (2.6.5) we have

$$f^\star\, d\phi = df^\star\phi. \tag{2.6.7}$$

(3) If $\omega_1, \omega_2 \in \Omega^k(V)$ from equation (2.6.2) we see that

$$(df_p)^\star(\omega_1 + \omega_2)_q = (df_p)^\star(\omega_1)_q + (df_p)^\star(\omega_2)_q,$$

and hence by (2.6.3) we have

$$f^\star(\omega_1 + \omega_2) = f^\star\omega_1 + f^\star\omega_2.$$

(4) We observed in §1.7 that the operation (2.6.1) commutes with wedge product, hence if $\omega_1 \in \Omega^k(V)$ and $\omega_2 \in \Omega^\ell(V)$

$$df_p^\star(\omega_1)_q \wedge (\omega_2)_q = df_p^\star(\omega_1)_q \wedge df_p^\star(\omega_2)_q.$$

In other words

$$f^\star\omega_1 \wedge \omega_2 = f^\star\omega_1 \wedge f^\star\omega_2. \tag{2.6.8}$$

(5) Let W be an open subset of $\mathbf{R}^k$ and $g : V \to W$ a C^∞ map. Given a point $p \in U$, let $q = f(p)$ and $w = g(q)$. Then the composition of the map

$$(df_p)^\star : \Lambda^k(T_q^\star) \to \Lambda^k(T_p^\star)$$

and the map

$$(dg_q)^\star : \Lambda^k(T_w^\star) \to \Lambda^k(T_q^\star)$$

is the map

$$(dg_q \circ df_p)^\star : \Lambda^k(T_w^\star) \to \Lambda^k(T_p^\star)$$

by formula (1.7.5). However, by the chain rule

$$(dg_q) \circ (df)_p = d(g \circ f)_p$$

so this composition is the map

$$d(g \circ f)_p^\star : \Lambda^k(T_w^\star) \to \Lambda^k(T_p^\star).$$

Thus if $\omega \in \Omega^k(W)$

$$f^\star(g^\star\omega) = (g \circ f)^\star\omega.$$

Let us see what the pullback operation looks like in coordinates. Using multi-index notation we can express every k-form $\omega \in \Omega^k(V)$ as a sum over multi-indices of length k

$$\omega = \sum_I \phi_I dx_I, \tag{2.6.9}$$

the coefficient ϕ_I of dx_I being in $C^\infty(V)$. Hence by (2.6.5)

$$f^\star\omega = \sum f^\star\phi_I f^\star(dx_I),$$

where $f^\star\phi_I$ is the function of $\phi \circ f$.

What about $f^\star dx_I$? If I is the multi-index, $(i_1, \dots, i_k)$, then by definition

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

so

$$f^\star(dx_I) = f^\star(dx_{i_1}) \wedge \cdots \wedge f^\star(dx_{i_k})$$

by (2.6.8), and by (2.6.7), we have

$$f^\star dx_i = df^\star x_i = df_i,$$

where f_i is the ith coordinate function of the map f. Thus, setting

$$df_I := df_{i_1} \wedge \cdots \wedge df_{i_k},$$

we see that for each multi-index I we have

$$f^\star(dx_I) = df_I, \tag{2.6.10}$$

and for the pullback of the form (2.6.9)

$$f^\star\omega = \sum_I f^\star\phi_I \, df_I. \tag{2.6.11}$$

We will use this formula to prove that pullback commutes with exterior differentiation:

$$d(f^\star\omega) = f^\star(d\omega). \tag{2.6.12}$$

To prove this we recall that by (2.3.6) we have $d(df_I) = 0$, hence by (2.3.1) and (2.6.10) we have

$$\begin{aligned} d(f^\star\omega) &= \sum_I d(f^\star\phi_I) \wedge df_I = \sum_I f^\star(d\phi_I) \wedge f^\star(dx_I) \\ &= f^\star \sum_I d\phi_I \wedge dx_I = f^\star(d\omega). \end{aligned}$$

A special case of formula (2.6.10) will be needed in Chapter 4: Let U and V be open subsets of $\mathbf{R}^n$ and let $\omega = dx_1 \wedge \cdots \wedge dx_n$. Then by (2.6.10) we have

$$f^\star \omega_p = (df_1)_p \wedge \cdots \wedge (df_n)_p$$

for all $p \in U$. However,

$$(df_i)_p = \sum \frac{\partial f_i}{\partial x_j}(p)(dx_j)_p$$

and hence by formula (1.7.10)

$$f^\star \omega_p = \det\left[\frac{\partial f_i}{\partial x_j}(p)\right](dx_1 \wedge \cdots \wedge dx_n)_p.$$

In other words

$$f^\star(dx_1 \wedge \cdots \wedge dx_n) = \det\left[\frac{\partial f_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n. \quad (2.6.13)$$

In Exercise 2.6.iv and equation (2.6.6) we outline the proof of an important topological property of the pullback operation.

Definition 2.6.14. Let U be an open subset of $\mathbf{R}^n$, V an open subset of $\mathbf{R}^m$, $A \subset \mathbf{R}$ an open interval containing 0 and 1, and $f_0, f_1 : U \to V$ two C^∞ maps. A C^∞ map $F : U \times A \to V$ is a C^∞ ***homotopy*** between f_0 and f_1 if $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

If there exists a homotopy between f_0 and f_1, we say that f_0 and f_1 are ***homotopic*** and write $f_0 \simeq f_1$.

Intuitively, f_0 and f_1 are homotopic if there exists a family of C^∞ maps $f_t : U \to V$, where $f_t(x) = F(x,t)$, which "smoothly deform f_0 into f_1". In Exercise 2.6.iv and equation (2.6.6) you will be asked to verify that for f_0 and f_1 to be homotopic they have to satisfy the following criteria.

Theorem 2.6.15. *Let U be an open subset of $\mathbf{R}^n$, V an open subset of $\mathbf{R}^m$, and $f_0, f_1 : U \to V$ two C^∞ maps. If f_0 and f_1 are homotopic then for every closed form $\omega \in \Omega^k(V)$ the form $f_1^\star\omega - f_0^\star\omega$ is exact.*

This theorem is closely related to the Poincaré lemma, and, in fact, one gets from it a slightly stronger version of the Poincaré lemma than that described in Exercise 2.3.iv and equation (2.3.7).

Definition 2.6.16. An open subset U of $\mathbf{R}^n$ is ***contractible*** if, for some point $p_0 \in U$, the identity map $\mathrm{id}_U : U \to U$ is homotopic to the constant map

$$f_0 : U \to U, \quad f_0(p) = p_0$$

at p_0.

From Theorem 2.6.15 it is easy to see that the Poincaré lemma holds for contractible open subsets of $\mathbf{R}^n$. If U is contractible, every closed k-form on U of degree $k > 0$ is exact. To see this, note that if $\omega \in \Omega^k(U)$, then for the identity map $\mathrm{id}_U^\star\,\omega = \omega$ and if f is the constant map at a point of U, then $f^\star\omega = 0$.

Exercises for §2.6

Exercise 2.6.i. Let $f: \mathbf{R}^3 \to \mathbf{R}^3$ be the map

$$f(x_1, x_2, x_3) = (x_1x_2, x_2x_3^2, x_3^3).$$

Compute the pullback, $f^\star\omega$ for the following forms:

(1) $\omega = x_2dx_3$;
(2) $\omega = x_1dx_1 \wedge dx_3$;
(3) $\omega = x_1dx_1 \wedge dx_2 \wedge dx_3$.

Exercise 2.6.ii. Let $f: \mathbf{R}^2 \to \mathbf{R}^3$ be the map

$$f(x_1, x_2) = (x_1^2, x_2^2, x_1x_2).$$

Complete the pullback, $f^\star\omega$, for the following forms:

(1) $\omega = x_2dx_2 + x_3dx_3$;
(2) $\omega = x_1dx_2 \wedge dx_3$;
(3) $\omega = dx_1 \wedge dx_2 \wedge dx_3$.

Exercise 2.6.iii. Let U be an open subset of $\mathbf{R}^n$, V an open subset of $\mathbf{R}^m$, $f: U \to V$ a C^∞ map, and $\gamma: [a,b] \to U$ a C^∞ curve. Show that for $\omega \in \Omega^1(V)$

$$\int_\gamma f^\star\omega = \int_{\gamma_1} \omega,$$

where $\gamma_1: [a,b] \to V$ is the curve, $\gamma_1(t) = f(\gamma(t))$. (See Exercise 2.2.viii.)

Exercise 2.6.iv. Let U be an open subset of $\mathbf{R}^n$, $A \subset \mathbf{R}$ an open interval containing the points, 0 and 1, and (x,t) product coordinates on $U \times A$. Recall from Exercise 2.3.v that a form $\mu \in \Omega^\ell(U \times A)$ is *reduced* if it can be written as a sum

$$\mu = \sum_I f_I(x,t)dx_I$$

(i.e., none of the summands involve dt). For a reduced form μ, let $Q\mu \in \Omega^\ell(U)$ be the form

$$Q\mu := \left(\sum_I \int_0^1 f_I(x,t)dt\right)dx_I \tag{2.6.17}$$

and let $\mu_1, \mu_2 \in \Omega^\ell(U)$ be the forms

$$\mu_0 = \sum_I f_I(x,0)dx_I$$

and

$$\mu_1 = \sum f_I(x,1)dx_I.$$

Now recall from Exercise 2.4.v that every form $\omega \in \Omega^k(U \times A)$ can be written uniquely as a sum

$$\omega = dt \wedge \alpha + \beta, \tag{2.6.18}$$

where α and β are reduced.

(1) Prove the following theorem.

Theorem 2.6.19. *If the form* (2.6.18) *is closed then*

$$\beta_1 - \beta_0 = dQ\alpha. \tag{2.6.20}$$

Hint: Equation (2.4.19).

(2) Let ι_0 and ι_1 be the maps of U into $U \times A$ defined by $\iota_0(x) = (x, 0)$ and $\iota_1(x) = (x, 1)$. Show that (2.6.20) can be rewritten

$$\iota_1^\star \omega - \iota_0^\star \omega = dQ\alpha. \tag{2.6.21}$$

Exercise 2.6.v. Let V be an open subset of $\mathbf{R}^m$ and let $f_0, f_1 : U \to V$ be C^∞ maps. Suppose f_0 and f_1 are homotopic. Show that for every closed form, $\mu \in \Omega^k(V)$, $f_1^\star \mu - f_0^\star \mu$ is exact.

Hint: Let $F: U \times A \to V$ be a homotopy between f_0 and f_1 and let $\omega = F^\star \mu$. Show that ω is closed and that $f_0^\star \mu = \iota_0^\star \omega$ and $f_1^\star \mu = \iota_1^\star \omega$. Conclude from equation (2.6.21) that

$$f_1^\star \mu - f_0^\star \mu = dQ\alpha,$$

where $\omega = dt \wedge \alpha + \beta$ and α and β are reduced.

Exercise 2.6.vi. Show that if $U \subset \mathbf{R}^n$ is a contractible open set, then the Poincaré lemma holds: every closed form of degree $k > 0$ is exact.

Exercise 2.6.vii. An open subset, U, of $\mathbf{R}^n$ is said to be ***star-shaped*** if there exists a point $p_0 \in U$, with the property that for every point $p \in U$, the line segment,

$$\{ tp + (1-t)p_0 \mid 0 \le t \le 1 \},$$

joining p to p_0 is contained in U. Show that if U is star-shaped it is contractible.

Exercise 2.6.viii. Show that the following open sets are star-shaped.

(1) The open unit ball

$$\{ x \in \mathbf{R}^n \mid |x| < 1 \}.$$

(2) The open rectangle, $I_1 \times \cdots \times I_n$, where each I_k is an open subinterval of $\mathbf{R}$.
(3) $\mathbf{R}^n$ itself.
(4) Product sets

$$U_1 \times U_2 \subset \mathbf{R}^n \cong \mathbf{R}^{n_1} \times \mathbf{R}^{n_2},$$

where U_i is a star-shaped open set in $\mathbf{R}^{n_i}$.

Exercise 2.6.ix. Let U be an open subset of $\mathbf{R}^n$, $f_t : U \to U$, $t \in \mathbf{R}$, a one-parameter group of diffeomorphisms and v its infinitesimal generator. Given $\omega \in \Omega^k(U)$ show that at $t = 0$

$$\frac{d}{dt} f_t^\star \omega = L_v \omega. \tag{2.6.22}$$

Here is a sketch of a proof:

(1) Let $\gamma(t)$ be the curve, $\gamma(t) = f_t(p)$, and let ϕ be a 0-form, i.e., an element of $C^\infty(U)$. Show that

$$f_t^\star \phi(p) = \phi(\gamma(t))$$

and by differentiating this identity at $t = 0$ conclude that (2.6.22) holds for 0-forms.

(2) Show that if (2.6.22) holds for ω it holds for $d\omega$.
Hint: Differentiate the identity

$$f_t^\star d\omega = d f_t^\star \omega$$

at $t = 0$.

(3) Show that if (2.6.22) holds for ω_1 and ω_2 it holds for $\omega_1 \wedge \omega_2$.
Hint: Differentiate the identity

$$f_t^\star(\omega_1 \wedge \omega_2) = f_t^\star \omega_1 \wedge f_t^\star \omega_2$$

at $t = 0$.

(4) Deduce (2.6.22) from (1)–(3).
Hint: Every k-form is a sum of wedge products of 0-forms and exact 1-forms.

Exercise 2.6.x. In Exercise 2.6.ix show that for *all t*

$$\frac{d}{dt} f_t^\star \omega = f_t^\star L_v \omega = L_v f_t^\star \omega. \tag{2.6.23}$$

Hint: By the definition of a one-parameter group, $f_{s+t} = f_s \circ f_t = f_r \circ f_s$, hence:

$$f_{s+t}^\star \omega = f_t^\star(f_s^\star \omega) = f_s^\star(f_t^\star \omega).$$

Prove the first assertion by differentiating the first of these identities with respect to s and then setting $s = 0$, and prove the second assertion by doing the same for the second of these identities.

In particular conclude that

$$f_t^\star L_v \omega = L_v f_t^\star \omega.$$

Exercise 2.6.xi.

(1) By massaging the result above show that

$$\frac{d}{dt} f_t^\star \omega = dQ_t \omega + Q_t d\omega,$$

where

$$Q_t \omega = f_t^\star \iota_v \omega.$$

Hint: Equation (2.5.9).

(2) Let

$$Q\omega = \int_0^1 f_t^\star \iota_v \omega dt.$$

Prove the homotopy identity

$$f_1^\star \omega - f_0^\star \omega = dQ\omega + Qd\omega.$$

Exercise 2.6.xii. Let U be an open subset of $\mathbf{R}^n$, V an open subset of $\mathbf{R}^m$, $\boldsymbol{v}$ a vector field on U, $\boldsymbol{w}$ a vector field on V, and $f : U \to V$ a C^∞ map. Show that if $\boldsymbol{v}$ and $\boldsymbol{w}$ are f-related

$$\iota_v f^\star \omega = f^\star \iota_w \omega.$$

2.7. Divergence, curl, and gradient

The basic operations in three-dimensional vector calculus: gradient, curl, and divergence are, by definition, operations on *vector fields*. As we see below these operations are closely related to the operations

$$d\colon \Omega^k(\mathbf{R}^3) \to \Omega^{k+1}(\mathbf{R}^3) \tag{2.7.1}$$

in degrees $k = 0, 1, 2$. However, only the gradient and divergence generalize to n dimensions. (They are essentially the d-operations in degrees zero and $n-1$.) And, unfortunately, there is no simple description in terms of vector fields for the other d-operations. This is one of the main reasons why an adequate theory of vector calculus in n-dimensions forces on one the differential form approach that we have developed in this chapter.

Even in three dimensions, however, there is a good reason for replacing the gradient, divergence, and curl by the three operations (2.7.1). A problem that physicists spend a lot of time worrying about is the problem of *general covariance*: formulating the laws of physics in such a way that they admit as large a set of symmetries as possible, and frequently these formulations involve differential forms. An example is Maxwell's equations, the fundamental laws of electromagnetism. These are usually expressed as identities involving div and curl. However, as we explain below, there is an alternative formulation of Maxwell's equations based on the operations (2.7.1), and from the point of view of general covariance, this formulation is much more satisfactory: the only symmetries of $\mathbf{R}^3$ which preserve div and curl are translations and rotations, whereas the operations (2.7.1) admit all diffeomorphisms of $\mathbf{R}^3$ as symmetries.

To describe how the gradient, divergence, and curl are related to the operations (2.7.1) we first note that there are two ways of converting vector fields into forms.

The first makes use of the natural inner product $B(v, w) = \sum v_i w_i$ on $\mathbf{R}^n$. From this inner product one gets by Exercise 1.2.ix a bijective linear map

$$L\colon \mathbf{R}^n \xrightarrow{\sim} (\mathbf{R}^n)^*$$

with the defining property: $L(v) = \ell$ if and only if $\ell(w) = B(v, w)$. Via the identification (2.1.2) B and L can be transferred to $T_p\mathbf{R}^n$, giving one an inner product B_p on $T_p\mathbf{R}^n$ and a bijective linear map

$$L_p\colon T_p\mathbf{R}^n \xrightarrow{\sim} T_p^*\mathbf{R}^n.$$

Hence if we are given a vector field $\boldsymbol{v}$ on U we can convert it into a 1-form $\boldsymbol{v}^\sharp$ by setting

$$\boldsymbol{v}^\sharp(p) := L_p\boldsymbol{v}(p) \tag{2.7.2}$$

and this sets up a bijective correspondence between vector fields and 1-forms.

For instance

$$\boldsymbol{v} = \frac{\partial}{\partial x_i} \iff \boldsymbol{v}^\sharp = dx_i$$

(see Exercise 2.7.iii) and, more generally,

$$v = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \iff v^\sharp = \sum_{i=1}^{n} f_i dx_i. \tag{2.7.3}$$

Example 2.7.4. In particular if f is a C^∞ function on $U \subset \mathbf{R}^n$ the ***gradient*** of f is the vector field

$$\operatorname{grad}(f) := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i},$$

and this gets converted by (2.7.5) into the 1-form df. Thus the gradient operation in vector calculus is basically just the exterior derivative operation $d \colon \Omega^0(U) \to \Omega^1(U)$.

The second way of converting vector fields into forms is via the interior product operation. Namely let Ω be the n-form $dx_1 \wedge \cdots \wedge dx_n$. Given an open subset U of $\mathbf{R}^n$ and a C^∞ vector field

$$v = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \tag{2.7.5}$$

on U the interior product of v with Ω is the $(n-1)$-form

$$\iota_v \Omega = \sum_{r=1}^{n} (-1)^{r-1} f_r dx_1 \wedge \cdots \wedge \widehat{dx_r} \cdots \wedge dx_n. \tag{2.7.6}$$

Moreover, every $(n-1)$-form can be written uniquely as such a sum, so (2.7.5) and (2.7.6) set up a bijective correspondence between vector fields and $(n-1)$-forms. Under this correspondence the d-operation gets converted into an operation on vector fields

$$v \mapsto d\iota_v \Omega.$$

Moreover, by (2.5.9)

$$d\iota_v \Omega = L_v \Omega$$

and by (2.5.14)

$$L_v \Omega = \operatorname{div}(v) \Omega,$$

where

$$\operatorname{div}(v) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}. \tag{2.7.7}$$

In other words, this correspondence between $(n-1)$-forms and vector fields converts the d-operation into the divergence operation (2.7.7) on vector fields.

Notice that divergence and gradient are well-defined as vector calculus operations in n dimensions, even though one usually thinks of them as operations in three-dimensional vector calculus. The curl operation, however, is intrinsically a three-dimensional vector calculus operation. To define it we note that by (2.7.6) every 2-form μ on an open subset $U \subset \mathbf{R}^3$ can be written uniquely as an interior product,

$$\mu = \iota_w dx_1 \wedge dx_2 \wedge dx_3, \tag{2.7.8}$$

for some vector field w, and the left-hand side of this formula determines w uniquely.

Definition 2.7.9. Let U be an open subset of $\mathbf{R}^3$ and v a vector field on U. From v we get by (2.7.3) a 1-form $v^\sharp$, and hence by (2.7.8) a vector field w satisfying

$$dv^\sharp = \iota_w dx_1 \wedge dx_2 \wedge dx_3.$$

The ***curl*** of v is the vector field

$$\operatorname{curl}(v) := w.$$

We leave for you to check that this definition coincides with the definition one finds in calculus books. More explicitly we leave for you to check that if v is the vector field

$$v = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + f_3 \frac{\partial}{\partial x_3},$$

then

$$\operatorname{curl}(v) = g_1 \frac{\partial}{\partial x_1} + g_2 \frac{\partial}{\partial x_2} + g_3 \frac{\partial}{\partial x_3},$$

where

$$\begin{aligned} g_1 &= \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}, \\ g_2 &= \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}, \\ g_3 &= \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}. \end{aligned} \tag{2.7.10}$$

To summarize: the gradient, curl, and divergence operations in three-dimensions are basically just the three operations (2.7.1). The gradient operation is the operation (2.7.1) in degree zero, curl is the operation (2.7.1) in degree one, and divergence is the operation (2.7.1) in degree two. However, to define gradient we had to assign an inner product B_p to the tangent space $T_p\mathbf{R}^n$ for each $p \in U$; to define divergence we had to equip U with the 3-form Ω and to define curl, the most complicated of these three operations, we needed the inner products B_p *and* the form Ω. This is why diffeomorphisms preserve the three operations (2.7.1) but do not preserve gradient, curl, and divergence. The additional structures which one needs to define grad, curl and div are only preserved by translations and rotations.

Maxwell's equations and differential forms

We conclude this section by showing how Maxwell's equations, which are usually formulated in terms of divergence and curl, can be reset into "form" language. The discussion below is an abbreviated version of [5, §1.20].

Maxwell's equations assert:

$$\operatorname{div}(\boldsymbol{v}_E) = q, \tag{2.7.11}$$

$$\operatorname{curl}(\boldsymbol{v}_E) = -\frac{\partial}{\partial t}\boldsymbol{v}_M, \tag{2.7.12}$$

$$\operatorname{div}(\boldsymbol{v}_M) = 0, \tag{2.7.13}$$

$$c^2 \operatorname{curl}(\boldsymbol{v}_M) = \boldsymbol{w} + \frac{\partial}{\partial t}\boldsymbol{v}_E, \tag{2.7.14}$$

where $\boldsymbol{v}_E$ and $\boldsymbol{v}_M$ are the ***electric*** and ***magnetic*** fields, q is the ***scalar charge density***, $\boldsymbol{w}$ is the ***current density*** and c is the velocity of light. (To simplify (2.7.15) slightly we assume that our units of space–time are chosen so that $c = 1$.) As above let $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ and let

$$\mu_E = \iota(\boldsymbol{v}_E)\Omega$$

and

$$\mu_M = \iota(\boldsymbol{v}_M)\Omega.$$

We can then rewrite equations (2.7.11) and (2.7.13) in the form

$$d\mu_E = q\Omega \tag{2.7.11$'$}$$

and

$$d\mu_M = 0. \tag{2.7.13$'$}$$

What about (2.7.12) and (2.7.14)? We leave the following "form" versions of these equations as an exercise:

$$d\boldsymbol{v}_E^\sharp = -\frac{\partial}{\partial t}\mu_M \tag{2.7.12$'$}$$

and

$$d\boldsymbol{v}_M^\sharp = \iota_{\boldsymbol{w}}\Omega + \frac{\partial}{\partial t}\mu_E, \tag{2.7.14$'$}$$

where the 1-forms, $\boldsymbol{v}_E^\sharp$ and $\boldsymbol{v}_M^\sharp$, are obtained from $\boldsymbol{v}_E$ and $\boldsymbol{v}_M$ by the operation (2.7.2).

These equations can be written more compactly as differential form identities in (3 + 1) dimensions. Let ω_M and ω_E be the 2-forms

$$\omega_M = \mu_M - \boldsymbol{v}_E^\sharp \wedge dt$$

and

$$\omega_E = \mu_E - \boldsymbol{v}_M^\sharp \wedge dt \tag{2.7.15}$$

and let Λ be the 3-form

$$\Lambda := q\Omega + \iota_{\boldsymbol{w}}\Omega \wedge dt.$$

We will leave for you to show that the four equations (2.7.11)–(2.7.14) are equivalent to two elegant and compact (3 + 1)-dimensional identities

$$d\omega_M = 0$$

and

$$d\omega_E = \Lambda.$$

Exercises for § 2.7

Exercise 2.7.i. Verify that the curl operation is given in coordinates by equation (2.7.10).

Exercise 2.7.ii. Verify that Maxwell's equations (2.7.11) and (2.7.12) become equations (2.7.13) and (2.7.14) when rewritten in differential form notation.

Exercise 2.7.iii. Show that in (3+1)-dimensions Maxwell's equations take the form of equations (2.7.10) and (2.7.11).

Exercise 2.7.iv. Let U be an open subset of $\mathbf{R}^3$ and $\boldsymbol{v}$ a vector field on U. Show that if $\boldsymbol{v}$ is the gradient of a function, its curl has to be zero.

Exercise 2.7.v. If U is simply connected prove the converse: If the curl of $\boldsymbol{v}$ vanishes, $\boldsymbol{v}$ is the gradient of a function.

Exercise 2.7.vi. Let $\boldsymbol{w} = \mathrm{curl}(\boldsymbol{v})$. Show that the divergence of $\boldsymbol{w}$ is zero.

Exercise 2.7.vii. Is the converse to Exercise 2.7.vi true? Suppose the divergence of $\boldsymbol{w}$ is zero. Is $\boldsymbol{w} = \mathrm{curl}(\boldsymbol{v})$ for some vector field $\boldsymbol{v}$?

2.8. Symplectic geometry and classical mechanics

In this section we describe some other applications of the theory of differential forms to physics. Before describing these applications, however, we say a few words about the geometric ideas that are involved.

Definition 2.8.1. Let $x_1, \dots, x_{2n}$ be the standard coordinate functions on $\mathbf{R}^{2n}$ and for $i = 1, \dots, n$ let $y_i := x_{n+i}$. The 2-form

$$\omega := \sum_{i=1}^{n} dx_i \wedge dy_i = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}$$

is called the ***Darboux form*** on $\mathbf{R}^{2n}$.

From the identity

$$\omega = -d\left(\sum_{i=1}^{n} y_i dx_i\right), \tag{2.8.2}$$

it follows that ω is exact. Moreover computing the n-fold wedge product of ω with itself we get

$$\begin{aligned}\omega^n &= \left(\sum_{i_i=1}^{n} dx_{i_1} \wedge dy_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_n=1}^{n} dx_{i_n} \wedge dy_{i_n}\right) \\ &= \sum_{i_1,\dots,i_n} dx_{i_1} \wedge dy_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dy_{i_n}.\end{aligned}$$

We can simplify this sum by noting that if the multi-index $I = (i_1, \dots, i_n)$ is repeating, the wedge product

$$dx_{i_1} \wedge dy_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dx_{i_n} \tag{2.8.3}$$

involves two repeating dx_{i_j} and hence is zero, and if I is non-repeating we can permute the factors and rewrite (2.8.3) in the form

$$dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

(See Exercise 1.6.v.)

Hence since these are exactly $n!$ non-repeating multi-indices

$$\omega^n = n!\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

i.e.,

$$\frac{1}{n!}\omega^n = \Omega, \tag{2.8.4}$$

where

$$\Omega = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \tag{2.8.5}$$

is the ***symplectic volume form*** on $\mathbf{R}^{2n}$.

Definition 2.8.6. Let U and V be open subsets of $\mathbf{R}^{2n}$. A diffeomorphism $f : U \xrightarrow{\sim} V$ is a ***symplectic diffeomorphism*** or ***symplectomorphism*** if $f^\star\omega = \omega$, where ω denotes the Darboux form on $\mathbf{R}^{2n}$.

Definition 2.8.7. Let U be an open subset of $\mathbf{R}^{2n}$, let

$$f_t : U \xrightarrow{\sim} U, \quad -\infty < t < \infty \tag{2.8.8}$$

be a one-parameter group of diffeomorphisms of U, and $\boldsymbol{v}$ be the vector field generating (2.8.8). We say that $\boldsymbol{v}$ is a ***symplectic vector field*** if the diffeomorphisms (2.8.8) are symplectomorphisms, i.e., for all t we have $f_t^\star\omega = \omega$.

Let us see what such vector fields have to look like. Note that by (2.6.23)

$$\frac{d}{dt} f_t^\star \omega = f_t^\star L_{\boldsymbol{v}}\omega, \tag{2.8.9}$$

hence if $f_t^\star\omega = \omega$ for all t, the left-hand side of (2.8.9) is zero, so

$$f_t^\star L_{\boldsymbol{v}}\omega = 0.$$

In particular, for $t = 0$, f_t is the identity map so $f_t^\star L_{\boldsymbol{v}}\omega = L_{\boldsymbol{v}}\omega = 0$. Conversely, if $L_{\boldsymbol{v}}\omega = 0$, then $f_t^\star L_{\boldsymbol{v}}\omega = 0$ so by (2.8.9) $f_t^\star\omega$ does not depend on t. However, since

$f_t^\star \omega = \omega$ for $t = 0$ we conclude that $f_t^\star \omega = \omega$ for all t. To summarize, we have proved the following.

Theorem 2.8.10. *Let $f_t : U \to U$ be a one-parameter group of diffeomorphisms and $\boldsymbol{v}$ the infinitesimal generator of this group. Then $\boldsymbol{v}$ is symplectic of and only if $L_{\boldsymbol{v}}\omega = 0$.*

There is an equivalent formulation of Theorem 2.8.10 in terms of the interior product $\iota_{\boldsymbol{v}}\omega$. By (2.5.9)

$$L_{\boldsymbol{v}}\omega = d\iota_{\boldsymbol{v}}\omega + \iota_{\boldsymbol{v}}\, d\omega.$$

But by (2.8.2) we have $d\omega = 0$, so

$$L_{\boldsymbol{v}}\omega = d\iota_{\boldsymbol{v}}\omega.$$

Thus we have shown the following.

Theorem 2.8.11. *A vector field $\boldsymbol{v}$ on an open subset of $\mathbf{R}^{2n}$ is symplectic if and only if the form $\iota_{\boldsymbol{v}}\omega$ is closed.*

Let U be an open subset of $\mathbf{R}^{2n}$ and $\boldsymbol{v}$ a vector field on U. If $\iota_{\boldsymbol{v}}\omega$ is not only closed but is exact we say that $\boldsymbol{v}$ is a ***Hamiltonian vector field***. In other words, $\boldsymbol{v}$ is Hamiltonian if

$$\iota_{\boldsymbol{v}}\omega = dH \tag{2.8.12}$$

for some C^∞ function $H \in C^\infty(U)$.

Let us see what this condition looks like in coordinates. Let

$$\boldsymbol{v} = \sum_{i=1}^{n} \left(f_i \frac{\partial}{\partial x_i} + g_i \frac{\partial}{\partial y_i} \right). \tag{2.8.13}$$

Then

$$\iota_{\boldsymbol{v}}\omega = \sum_{1 \le i,j \le n} f_i \iota_{\partial/\partial x_i}(dx_j \wedge dy_j) + \sum_{1 \le i,j \le n} g_i \iota_{\partial/\partial y_i}(dx_j \wedge dy_j).$$

But

$$\iota_{\partial/\partial x_i} dx_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases}$$

and

$$\iota_{\partial/\partial x_i} dy_j = 0$$

so the first summand above is $\sum_{i=1}^n f_i\, dy_i$, and a similar argument shows that the second summand is $-\sum_{i=1}^n g_i dx_i$.

Hence if $\boldsymbol{v}$ is the vector field (2.8.13), then

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^{n} (f_i\, dy_i - g_i dx_i). \tag{2.8.14}$$

Thus since

$$dH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x_i} dx_i + \frac{\partial H}{\partial y_i}\, dy_i \right)$$

we get from (2.8.12)–(2.8.14)

$$f_i = \frac{\partial H}{\partial y_i} \quad \text{and} \quad g_i = -\frac{\partial H}{\partial x_i}$$

so $\boldsymbol{v}$ has the form:

$$\boldsymbol{v} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} \right). \tag{2.8.15}$$

In particular if $\gamma(t) = (x(t), y(t))$ is an integral curve of $\boldsymbol{v}$ it has to satisfy the system of differential equations

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial y_i}(x(t), y(t)), \\ \dfrac{dy_i}{dt} = -\dfrac{\partial H}{\partial x_i}(x(t), y(t)). \end{cases} \tag{2.8.16}$$

The formulas (2.8.13) and (2.8.14) exhibit an important property of the Darboux form ω. Every 1-form on U can be written uniquely as a sum

$$\sum_{i=1}^{n} (f_i dy_i - g_i dx_i)$$

with f_i and g_i in $C^\infty(U)$ and hence (2.8.13) and (2.8.14) imply the following theorem.

Theorem 2.8.17. *The map $\boldsymbol{v} \mapsto \iota_{\boldsymbol{v}}\omega$ defines a bijective between vector field and 1-forms.*

In particular, for every C^∞ function H, we get by correspondence a unique vector field $\boldsymbol{v} = \boldsymbol{v}_H$ with the property (2.8.12). We next note that by (1.7.9)

$$L_{\boldsymbol{v}} H = \iota_{\boldsymbol{v}} dH = \iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) = 0.$$

Thus

$$L_{\boldsymbol{v}} H = 0 \tag{2.8.18}$$

i.e., H is an integral of motion of the vector field $\boldsymbol{v}$. In particular, if the function $H : U \to \mathbf{R}$ is proper, then by Theorem 2.2.12 the vector field, $\boldsymbol{v}$, is complete and hence by Theorem 2.8.10 generates a one-parameter group of symplectomorphisms.

Remark 2.8.19. If the one-parameter group (2.8.8) is a group of symplectomorphisms, then $f_t^\star \omega^n = f_t^\star \omega \wedge \cdots \wedge f_t^\star \omega = \omega^n$, so by (2.8.4)

$$f_t^\star \Omega = \Omega, \tag{2.8.20}$$

where Ω is the symplectic volume form (2.8.5).

Application 2.8.21. The application we want to make of these ideas concerns the description, in Newtonian mechanics, of a physical system consisting of N interacting point-masses. The ***configuration space*** of such a system is

$$\mathbf{R}^n = \mathbf{R}^3 \times \cdots \times \mathbf{R}^3,$$

where there are N copies of $\mathbf{R}^3$ in the product, with position coordinates $x_1, \dots, x_n$ and the ***phase space*** is $\mathbf{R}^{2n}$ with position coordinates $x_1, \dots, x_n$ and momentum coordinates, $y_1, \dots, y_n$. The ***kinetic energy*** of this system is a quadratic function of the momentum coordinates

$$\frac{1}{2}\sum_{i=1}^{n} \frac{1}{m_i} y_i^2,$$

and for simplicity we assume that the potential energy is a function $V(x_1, \dots, x_n)$ of the position coordinates alone, i.e., it does not depend on the momenta and is time independent as well. Let

$$H = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{m_i} y_i^2 + V(x_1, \dots, x_n) \tag{2.8.22}$$

be the ***total energy*** of the system.

We now show that Newton's second law of motion in classical mechanics reduces to the assertion.

Proposition 2.8.23. *The trajectories in phase space of the system above are just the integral curves of the Hamiltonian vector field* $\boldsymbol{v}_H$.

Proof. For the function (2.8.22), equations (2.8.16) become

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{1}{m_i} y_i, \\ \dfrac{dy_i}{dt} = -\dfrac{\partial V}{\partial x_i}. \end{cases}$$

The first set of equation are essentially just the definitions of momentum, however, if we plug them into the second set of equations we get

$$m_i \frac{d^2 x_i}{dt^2} = -\frac{\partial V}{\partial x_i} \tag{2.8.24}$$

and interpreting the term on the right as the force exerted on the ith point-mass and the term on the left as mass times acceleration this equation becomes Newton's second law. □

In classical mechanics, equations (2.8.16) are known as the *Hamilton–Jacobi equations.* For a more detailed account of their role in classical mechanics, we highly recommend [1]. Historically, these equations came up for the first time, not in Newtonian mechanics, but in geometric optics and a brief description of their origins there and of their relation to Maxwell's equations can be found in [5].

Exercise 2.8.v. The expression (2.8.25) is known as the ***Poisson bracket*** of H_1 and H_2, denoted by $\{H_1, H_2\}$. Show that it is anti-symmetric

$$\{H_1, H_2\} = -\{H_2, H_1\}$$

and satisfies the *Jacobi identity*

$$0 = \{H_1, \{H_2, H_3\}\} + \{H_2, \{H_3, H_1\}\} + \{H_3, \{H_1, H_2\}\}.$$

Exercise 2.8.vi. Show that

$$\{H_1, H_2\} = L_{\boldsymbol{v}_{H_1}} H_2 = -L_{\boldsymbol{v}_{H_2}} H_1.$$

Exercise 2.8.vii. Prove that the following three properties are equivalent.

(1) $\{H_1, H_2\} = 0$.
(2) H_1 is an integral of motion of $\boldsymbol{v}_2$.
(3) H_2 is an integral of motion of $\boldsymbol{v}_1$.

Exercise 2.8.viii. Verify Noether's principle.

Exercise 2.8.ix (conservation of linear momentum). Suppose the potential V in equation (2.8.22) is invariant under the one-parameter group of translations

$$T_t(x_1, \dots, x_n) = (x_1 + t, \dots, x_n + t).$$

(1) Show that the function (2.8.22) is invariant under the group of diffeomorphisms

$$\gamma_t(x, y) = (T_t x, y).$$

(2) Show that the infinitesimal generator of this group is the Hamiltonian vector field $\boldsymbol{v}_G$ where $G = \sum_{i=1}^{n} y_i$.
(3) Conclude from Noether's principle that this function is an integral of the vector field $\boldsymbol{v}_H$, i.e., that "total linear momentum" is conserved.
(4) Show that "total linear momentum" is conserved if V is the Coulomb potential

$$\sum_{i \neq j} \frac{m_i}{|x_i - x_j|}.$$

Exercise 2.8.x. Let $R_t^i \colon \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ be the rotation which fixes the variables, (x_k, y_k), $k \neq i$ and rotates (x_i, y_i) by the angle, t:

$$R_t^i(x_i, y_i) = (\cos t\, x_i + \sin t\, y_i, -\sin t\, x_i + \cos t\, y_i).$$

(1) Show that R_t^i, $-\infty < t < \infty$, is a one-parameter group of symplectomorphisms.
(2) Show that its generator is the Hamiltonian vector field, $\boldsymbol{v}_{H_i}$, where $H_i = (x_i^2 + y_i^2)/2$.
(3) Let H be the harmonic oscillator Hamiltonian from Exercise 2.8.iii. Show that the R_t^j's preserve H.
(4) What does Noether's principle tell one about the classical mechanical system with energy function H?

Exercise 2.8.xi. Show that if U is an open subset of $\mathbf{R}^{2n}$ and $\boldsymbol{v}$ is a symplectic vector field on U then for every point $p_0 \in U$ there exists a neighborhood U_0 of p_0 on which $\boldsymbol{v}$ is Hamiltonian.

Exercise 2.8.xii. Deduce from Exercises 2.8.iv and 2.8.xi that if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are symplectic vector fields on an open subset U of $\mathbf{R}^{2n}$ their Lie bracket $[\boldsymbol{v}_1, \boldsymbol{v}_2]$ is a Hamiltonian vector field.

Exercise 2.8.xiii. Let $\alpha = \sum_{i=1}^n y_i dx_i$.

(1) Show that $\omega = -d\alpha$.

(2) Show that if α_1 is any 1-form on $\mathbf{R}^{2n}$ with the property $\omega = -d\alpha_1$, then

$$\alpha = \alpha_1 + dF$$

for some C^∞ function F.

(3) Show that $\alpha = \iota_{\boldsymbol{w}}\omega$ where $\boldsymbol{w}$ is the vector field

$$\boldsymbol{w} := -\sum_{i=1}^n y_i \frac{\partial}{\partial y_i}.$$

Exercise 2.8.xiv. Let U be an open subset of $\mathbf{R}^{2n}$ and $\boldsymbol{v}$ a vector field on U. Show that $\boldsymbol{v}$ has the property, $L_{\boldsymbol{v}}\alpha = 0$, if and only if

$$\iota_{\boldsymbol{v}}\omega = d\iota_{\boldsymbol{v}}\alpha.$$

In particular conclude that if $L_{\boldsymbol{v}}\alpha = 0$ then $\boldsymbol{v}$ is Hamiltonian.

Hint: Equation (2.8.2).

Exercise 2.8.xv. Let H be the function

$$H(x, y) = \sum_{i=1}^n f_i(x) y_i, \tag{2.8.26}$$

where the f_i's are C^∞ functions on $\mathbf{R}^n$. Show that

$$L_{\boldsymbol{v}_H}\alpha = 0. \tag{2.8.27}$$

Exercise 2.8.xvi. Conversely show that if H is any C^∞ function on $\mathbf{R}^{2n}$ satisfying equation (2.8.27) it has to be a function of the form (2.8.26).

Hints:

(1) Let $\boldsymbol{v}$ be a vector field on $\mathbf{R}^{2n}$ satisfying $L_{\boldsymbol{v}}\alpha = 0$. By equation (2.8.16) we have $\boldsymbol{v} = \boldsymbol{v}_H$, where $H = \iota_{\boldsymbol{v}}\alpha$.

(2) Show that H has to satisfy the equation

$$\sum_{i=1}^n y_i \frac{\partial H}{\partial y_i} = H.$$

(3) Conclude that if $H_r = \frac{\partial H}{\partial y_r}$ then H_r has to satisfy the equation

$$\sum_{i=1}^n y_i \frac{\partial}{\partial y_i} H_r = 0.$$

(4) Conclude that H_r has to be constant along the rays (x, ty), for $0 \le t < \infty$.

(5) Conclude finally that H_r has to be a function of x alone, i.e., does not depend on y.

Exercise 2.8.xvii. Show that if $\boldsymbol{v}_{\mathbf{R}^n}$ is a vector field

$$\sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i}$$

on configuration space there is a unique lift of $\boldsymbol{v}_{\mathbf{R}^n}$ to phase space

$$\boldsymbol{v} = \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i} + g_i(x,y)\frac{\partial}{\partial y_i}$$

satisfying $L_{\boldsymbol{v}}\alpha = 0$.