CHAPTER 3

# INTEGRATION OF FORMS

## 3.1 Introduction

The change of variables formula asserts that if $U$ and $V$ are open subsets of $\mathbb{R}^n$ and $f : U \to V$ a $C^1$ diffeomorphism then, for every continuous function, $\varphi : V \to \mathbb{R}$ the integral

$$\int_V \varphi(y)\, dy$$

exists if and only if the integral

$$\int_U \varphi \circ f(x) |\det Df(x)|\, dx$$

exists, and if these integrals exist they are equal. Proofs of this can be found in [**?**], [**?**] or [**?**]. This chapter contains an alternative proof of this result. This proof is due to Peter Lax. Our version of his proof in §3.5 below makes use of the theory of differential forms; but, as Lax shows in the article [**?**] (which we strongly recommend as collateral reading for this course), references to differential forms can be avoided, and the proof described in§3.5 can be couched entirely in the language of elementary multivariable calculus.

The virtue of Lax's proof is that is allows one to prove a version of the change of variables theorem for other mappings besides diffeomorphisms, and involves a topological invariant, the *degree of a mapping*, which is itself quite interesting. Some properties of this invariant, and some topological applications of the change of variables formula will be discussed in §3.6 of these notes.

**Remark 3.1.1.** *The proof we are about to describe is somewhat simpler and more transparent if we assume that $f$ is a $\mathcal{C}^\infty$ diffeomorphism. We'll henceforth make this assumption.*

## 3.2 The Poincaré lemma for compactly supported forms on rectangles

Let $\nu$ be a $k$-form on $\mathbb{R}^n$. We define the *support* of $\nu$ to be the closure of the set

$$\{x \in \mathbb{R}^n \,,\, \nu_x \neq 0\}$$

and we say that $\nu$ is *compactly* supported if $\operatorname{supp} \nu$ is compact. We will denote by $\Omega^k_c(\mathbb{R}^n)$ the set of all $\mathcal{C}^\infty$ $k$-forms which are compactly supported, and if $U$ is an open subset of $\mathbb{R}^n$, we will denote by $\Omega^k_c(U)$ the set of all compactly supported $k$-forms whose support is contained in $U$ .

Let $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ be a compactly supported $n$-form with $f \in \mathcal{C}^\infty_0(\mathbb{R}^n)$. We will define the integral of $\omega$ over $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \omega$$

to be the usual integral of $f$ over $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} f\, dx\,.$$

(Since $f$ is $\mathcal{C}^\infty$ and compactly supported this integral is well-defined.)

Now let $Q$ be the rectangle

$$[a_1, b_1] \times \cdots \times [a_n, b_n]\,.$$

The Poincaré lemma for rectangles asserts:

**Theorem 3.2.1.** *Let $\omega$ be a compactly supported $n$-form, with* $\operatorname{supp} \omega \subseteq \operatorname{Int} Q$*. Then the following assertions are equivalent:*

*a.* $\int \omega = 0$.

*b.* *There exists a compactly supported $(n-1)$-form, $\mu$, with* $\operatorname{supp} \mu \subseteq \operatorname{Int} Q$ *satisfying $d\mu = \omega$.*

We will first prove that (b)⇒( a). Let

$$\mu = \sum_{i=1}^{n} f_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n\,,$$

(the "hat" over the $dx_i$ meaning that $dx_i$ has to be omitted from the wedge product). Then

$$d\mu = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n ,$$

and to show that the integral of $d\mu$ is zero it suffices to show that each of the integrals

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}\, dx \tag{2.1$_i$}$$

is zero. By Fubini we can compute $(2.1)_i$ by first integrating with respect to the variable, $x_i$, and then with respect to the remaining variables. But

$$\int \frac{\partial f}{\partial x_i}\, dx_i = f(x) \Big|_{x_i=a_i}^{x_i=b_i} = 0$$

since $f_i$ is supported on $U$.

We will prove that (a) $\Rightarrow$ (b) by proving a somewhat stronger result. Let $U$ be an open subset of $\mathbb{R}^m$. We'll say that $U$ has *property* $P$ if every form, $\omega \in \Omega_c^m(U)$ whose integral is zero in $d\,\Omega_c^{m-1}(U)$.

We will prove

**Theorem 3.2.2.** *Let $U$ be an open subset of $\mathbb{R}^{n-1}$ and $A \subseteq \mathbb{R}$ an open interval. Then if $U$ has property $P$, $U \times A$ does as well.*

**Remark 3.2.3.** *It's very easy to see that the open interval $A$ itself has property $P$. (See exercise 1 below.) Hence it follows by induction from Theorem 3.2.2 that*

$$\operatorname{Int} Q = A_1 \times \cdots \times A_n, \quad A_i = (a_i, b_i)$$

*has property $P$, and this proves "(a) $\Rightarrow$ (b)".*

To prove Theorem 3.2.2 let $(x, t) = (x_1, \ldots, x_{n-1}, t)$ be product coordinates on $U \times A$. Given $\omega \in \Omega_c^n(U \times A)$ we can express $\omega$ as a wedge product, $dt \wedge \alpha$ with $\alpha = f(x, t)\, dx_1 \wedge \cdots \wedge dx_{n-1}$ and $f \in \mathcal{C}_0^\infty(U \times A)$. Let $\theta \in \Omega_c^{n-1}(U)$ be the form

$$\theta = \left( \int_A f(x, t)\, dt \right) dx_1 \wedge \cdots \wedge dx_{n-1} . \tag{3.2.1}$$

Then

$$\int_{\mathbb{R}^{n-1}} \theta = \int_{\mathbb{R}^n} f(x, t)\, dx\, dt = \int_{\mathbb{R}^n} \omega$$

so if the integral of $\omega$ is zero, the integral of $\theta$ is zero. Hence since $U$ has property $P$, $\beta = d\nu$ for some $\nu \in \Omega_c^{n-1}(U)$. Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a bump function which is supported on $A$ and whose integral over $A$ is one. Setting

$$\kappa = -\rho(t)\, dt \wedge \nu$$

we have

$$d\kappa = \rho(t)\, dt \wedge\, d\nu = \rho(t)\, dt \wedge \theta\,,$$

and hence

$$\omega - d\kappa \quad = \quad dt \wedge (\alpha - \rho(t)\theta) = dt \wedge u(x,t)\, dx_1 \wedge \cdots \wedge\, dx_{n-1}$$

where

$$u(x,t) \quad = \quad f(x,t) - \rho(t) \int_A f(x,t)\, dt$$

by (3.2.1). Thus

$$\int u(x,t)\, dt = 0\,. \tag{3.2.2}$$

Let $a$ and $b$ be the end points of $A$ and let

$$v(x,t) = \int_a^t i(x,s)\, ds\,. \tag{3.2.3}$$

By (3.2.2) $v(a,x) = v(b,x) = 0$, so $v$ is in $\mathcal{C}_0^\infty(U \times A)$ and by (3.2.3), $\partial v/\partial t = u$. Hence if we let $\gamma$ be the form, $v(x,t)\, dx_1 \wedge \cdots \wedge dx_{n-1}$, we have:

$$d\gamma \quad = \quad u(x,t)\, dx \wedge \cdots \wedge dx_{n-1} = \omega - d\kappa$$

and

$$\omega \quad = \quad d(\gamma + \kappa)\,.$$

Since $\gamma$ and $\kappa$ are both in $\Omega_c^{n-1}(U \times A)$ this proves that $\omega$ is in $d\,\Omega_c^{n-1}(U \times A)$ and hence that $U \times A$ has property $P$.

## Exercises for §3.2.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function of class $C^r$ with support on the interval, $(a,b)$. Show that the following are equivalent.

(a) $\int_a^b f(x)\,dx = 0$.

(b) There exists a function, $g : \mathbb{R} \to \mathbb{R}$ of class $C^{r+1}$ with support on $(a,b)$ with $\frac{dg}{dx} = f$.

*Hint:* Show that the function

$$g(x) = \int_a^x f(s)\,ds$$

is compactly supported.

2. Let $f = f(x,y)$ be a compactly supported function on $\mathbb{R}^k \times \mathbb{R}^\ell$ with the property that the partial derivatives

$$\frac{\partial f}{\partial x_i}(x,y)\,,\; i = 1,\ldots,k\,,$$

and are continuous as functions of $x$ and $y$. Prove the following "differentiation under the integral sign" theorem (which we implicitly used in our proof of Theorem 3.2.2).

**Theorem 3.2.4.** *The function*

$$g(x) = \int f(x,y)\,dy$$

*is of class $C^1$ and*

$$\frac{\partial g}{\partial x_i}(x) = \int \frac{\partial f}{\partial x_i}(x,y)\,dy\,.$$

*Hints:* For $y$ fixed and $h \in \mathbb{R}^k$,

$$f_i(x+h,y) - f_i(x,y) = D_x f_i(c)h$$

for some point, $c$, on the line segment joining $x$ to $x+c$. Using the fact that $D_x f$ is continuous as a function of $x$ and $y$ and compactly supported, conclude:

**Lemma 3.2.5.** *Given $\epsilon > 0$ there exists a $\delta > 0$ such that for $|h| \leq \delta$*

$$|f(x+h,y) - f(x,y) - D_x f(x,c)h| \leq \epsilon |h|\,.$$

Now let $Q \subseteq \mathbb{R}^\ell$ be a rectangle with $\operatorname{supp} f \subseteq \mathbb{R}^k \times Q$ and show that

$$|g(x+h) - g(x) - \left(\int D_x f(x,y)\,dy\right) h| \le \epsilon \operatorname{vol}(Q)|h|\,.$$

Conclude that $g$ is differentiable at $x$ and that its derivative is

$$\int D_x f(x,y)\,dy\,.$$

3. Let $f : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}$ be a compactly supported continuous function. Prove

**Theorem 3.2.6.** *If all the partial derivatives of $f(x,y)$ with respect to $x$ of order $\le r$ exist and are continuous as functions of $x$ and $y$ the function*

$$g(x) = \int f(x,y)\,dy$$

*is of class $C^r$.*

4. Let $U$ be an open subset of $\mathbb{R}^{n-1}$, $A \subseteq \mathbb{R}$ an open interval and $(x,t)$ product coordinates on $U \times A$. Recall (§2.2) exercise 5) that every form, $\omega \in \Omega^k(U \times A)$, can be written uniquely as a sum, $\omega = dt \wedge \alpha + \beta$ where $\alpha$ and $\beta$ are *reduced*, i.e., don't contain a factor of $dt$.

(a) Show that if $\omega$ is compactly supported on $U \times A$ then so are $\alpha$ and $\beta$.

(b) Let $\alpha = \sum_I f_I(x,t)\,dx_I$. Show that the form

$$\theta = \sum_I \left(\int_A f_I(x,t)\,dt\right) dx_I \tag{3.2.4}$$

is in $\Omega_c^{k-1}(U)$.

(c) Show that if $d\omega = 0$, then $d\theta = 0$. *Hint:* By (3.2.4)

$$\begin{aligned} d\theta &= \sum_{I,i} \left(\int_A \frac{\partial f_I}{\partial x_i}(x,t)\,dt\right) dx_i \wedge dx_I \\ &= \int_A (d_U \alpha)\,dt \end{aligned}$$

and by (**??**) $d_U \alpha = \dfrac{d\beta}{dt}$.

5. In exercise 4 show that if $\theta$ is in $d\,\Omega^{k-1}(U)$ then $\omega$ is in $d\,\Omega^k_c(U)$.
*Hints:*

(a) Let $\theta = d\nu$, with $\nu = \Omega^{k-2}_c(U)$ and let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a bump function which is supported on $A$ and whose integral over $A$ is one. Setting $k = -\rho(t)\,dt \wedge \nu$ show that

$$\begin{aligned} \omega - d\kappa &= dt \wedge (\alpha - \rho(t)\theta) + \beta \\ &= dt \wedge (\sum_I u_I(x,t)\,dx_I) + \beta \end{aligned}$$

where

$$u_I(x,t) = f_I(x,t) - \rho(t) \int_A f_I(x,t)\,dt\,.$$

(b) Let $a$ and $b$ be the end points of $A$ and let

$$v_I(x,t) = \int_a^t u_I(x,t)\,dt\,.$$

Show that the form $\sum v_I(x,t)\,dx_I$ is in $\Omega^{k-1}_c(U \times A)$ and that

$$d\gamma = \omega - d\kappa - \beta - d_U\gamma\,.$$

(c) Conclude that the form $\omega - d(\kappa + \gamma)$ is reduced.

(d) Prove: If $\lambda \in \Omega^k_c(U \times A)$ is reduced and $d\lambda = 0$ then $\lambda = 0$.
*Hint:* Let $\lambda = \sum g_I(x,t)\,dx_I$. Show that $d\lambda = 0 \Rightarrow \dfrac{\partial}{\partial t} g_I(x,t) = 0$ and exploit the fact that for fixed $x$, $g_I(x,t)$ is compactly supported in $t$.

6. Let $U$ be an open subset of $\mathbb{R}^m$. We'll say that $U$ has *property* $P_k$, for $k < n$, if every closed $k$-form, $\omega \in \Omega^k_c(U)$, is in $d\,\Omega^{k-1}_c(U)$. Prove that if the open set $U \subseteq \mathbb{R}^{n-1}$ in exercise 3 has property $P_k$ then so does $U \times A$.

7. Show that if $Q$ is the rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ and $U = \operatorname{Int} Q$ then $u$ has property $P_k$.

8. Let $\mathbb{H}^n$ be the half-space

$$\{(x_1, \dots, x_n)\,; \quad x_1 \le 0\} \tag{3.2.5}$$

and let $\omega \in \Omega^n_c(\mathbb{R})$ be the $n$-form, $f\, dx_1 \wedge \cdots \wedge dx_n$ with $f \in \mathcal{C}^\infty_0(\mathbb{R}^n)$. Define:

$$\int_{\mathbb{H}^n} \omega = \int_{\mathbb{H}^n} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n \tag{3.2.6}$$

where the right hand side is the usual Riemann integral of $f$ over $\mathbb{H}^n$. (This integral makes sense since $f$ is compactly supported.) Show that if $\omega = d\mu$ for some $\mu \in \Omega^{n-1}_c(\mathbb{R}^n)$ then

$$\int_{\mathbb{H}^n} \omega = \int_{\mathbb{R}^{n-1}} \iota^* \mu \tag{3.2.7}$$

where $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is the inclusion map

$$(x_2, \dots, x_n) \to (0, x_2, \dots, x_n)\,.$$

*Hint:* Let $\mu = \sum_i f_i\, dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$. Mimicking the "(b) $\Rightarrow$ (a)" part of the proof of Theorem 3.2.1 show that the integral (3.2.6) is the integral over $\mathbb{R}^{n-1}$ of the function

$$\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1}(x_1, x_2, \dots, x_n)\, dx_1\,.$$

## 3.3 The Poincaré lemma for compactly supported forms on open subsets of $\mathbb{R}^n$

In this section we will generalize Theorem 3.2.1 to arbitrary connected open subsets of $\mathbb{R}^n$.

**Theorem 3.3.1.** *Let $U$ be a connected open subset of $\mathbb{R}^n$ and let $\omega$ be a compactly supported $n$-form with* $\operatorname{supp}\omega \subset U$. *The the following assertions are equivalent,*

*a.* $\int \omega = 0$.

*b.* *There exists a compactly supported $(n-1)$-form, $\mu$, with* $\operatorname{supp}\mu \subseteq U$ *and $\omega = d\mu$.*

Proof that (b) $\Rightarrow$ (a). The support of $\mu$ is contained in a large rectangle, so the integral of $d\mu$ is zero by Theorem 3.2.1.

Proof that (a) $\Rightarrow$ (b): Let $\omega_1$ and $\omega_2$ be compactly supported $n$-forms with support in $U$. We will write

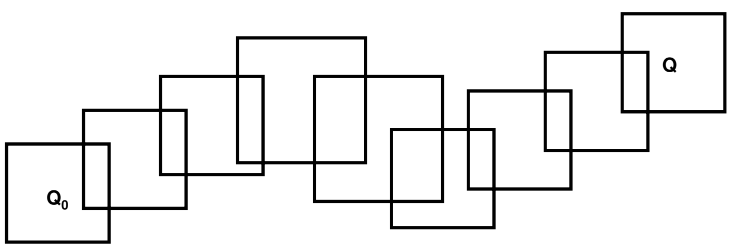
$$\omega_1 \sim \omega_2$$

as shorthand notation for the statement: "There exists a compactly supported $(n-1)$-form, $\mu$, with support in $U$ and with $\omega_1 - \omega_2 = d\mu$.", We will prove that (a) $\Rightarrow$ (b) by proving an equivalent statement: Fix a rectangle, $Q_0 \subset U$ and an $n$-form, $\omega_0$, with $\operatorname{supp}\omega_0 \subseteq Q_0$ and integral equal to one.

**Theorem 3.3.2.** *If $\omega$ is a compactly supported $n$-form with $\operatorname{supp}\omega \subseteq U$ and $c = \int \omega$ then $\omega \sim c\omega_0$.*

Thus in particular if $c = 0$, Theorem 3.3.2 says that $\omega \sim 0$ proving that (a) $\Rightarrow$ (b).

To prove Theorem 3.3.2 let $Q_i \subseteq U$, $i = 1, 2, 3, \ldots$, be a collection of rectangles with $U = \cup \operatorname{Int} Q_i$ and let $\varphi_i$ be a partition of unity with $\operatorname{supp}\varphi_i \subseteq \operatorname{Int} Q_i$. Replacing $\omega$ by the finite sum $\sum_{i=1}^{m} \varphi_i \omega$, $m$ large, it suffices to prove Theorem 3.3.2 for each of the summands $\varphi_i \omega$. In other words we can assume that $\operatorname{supp}\omega$ is contained in one of the open rectangles, $\operatorname{Int} Q_i$. Denote this rectangle by $Q$. We claim that one can join $Q_0$ to $Q$ by a sequence of rectangles as in the figure below.




**Lemma 3.3.3.** *There exists a sequence of rectangles, $R_i$, $i = 0, \ldots, N+1$ such that $R_0 = Q_0$, $R_{N+1} = Q$ and $\operatorname{Int} R_i \cap \operatorname{Int} R_{i+1}$ is non-empty.*

*Proof.* Denote by $A$ the set of points, $x \in U$, for which there exists a sequence of rectangles, $R_i$, $i = 0, \ldots, N+1$ with $R_0 = Q_0$, with $x \in \operatorname{Int} R_{N+1}$ and with $\operatorname{Int} R_i \cap \operatorname{Int} R_{i+1}$ non-empty. It is clear that this set is open and that its complement is open; so, by the connectivity of $U$, $U = A$. $\square$

To prove Theorem 3.3.2 with $\operatorname{supp} \omega \subseteq Q$, select, for each $i$, a compactly supported $n$-form, $\nu_i$, with $\operatorname{supp} \nu_i \subseteq \operatorname{Int} R_i \cap \operatorname{Int} R_{i+1}$ and with $\int \nu_i = 1$. The difference, $\nu_i - \nu_{i+1}$ is supported in $\operatorname{Int} R_{i+1}$, and its integral is zero; so by Theorem 3.2.1, $\nu_i \sim \nu_{i+1}$. Similarly, $\omega_0 \sim \nu_1$ and, if $c = \int \omega$, $\omega \sim c\nu_N$. Thus

$$c\omega_0 \sim c\nu_0 \sim \cdots \sim c\nu_N = \omega$$

proving the theorem.

## 3.4 The degree of a differentiable mapping

Let $U$ and $V$ be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$. A continuous mapping, $f : U \to V$, is *proper* if, for every compact subset, $B$, of $V$, $f^{-1}(B)$ is compact. Proper mappings have a number of nice properties which will be investigated in the exercises below. One obvious property is that if $f$ is a $\mathcal{C}^\infty$ mapping and $\omega$ is a compactly supported $k$-form with support on $V$, $f^*\omega$ is a compactly supported $k$-form with support on $U$. Our goal in this section is to show that if $U$ and $V$ are connected open subsets of $\mathbb{R}^n$ and $f : U \to V$ is a proper $\mathcal{C}^\infty$ mapping then there exists a topological invariant of $f$, which we will call its *degree* (and denote by $\deg(f)$), such that the "change of variables" formula:

$$\int_U f^*\omega = \deg(f) \int_V \omega \tag{3.4.1}$$

holds for all $\omega \in \Omega_c^n(V)$.

Before we prove this assertion let's see what this formula says in coordinates. If

$$\omega = \varphi(y)\, dy_1 \wedge \cdots \wedge dy_n$$

then at $x \in U$

$$f^*\omega = (\varphi \circ f)(x) \det(Df(x))\, dx_1 \wedge \cdots \wedge dx_n\,;$$

so, in coordinates, (3.4.1) takes the form

$$\int_V \varphi(y)\, dy = \deg(f) \int_U \varphi \circ f(x) \det(Df(x))\, dx\,. \tag{3.4.2}$$

*Proof of 3.4.1.* Let $\omega_0$ be an $n$-form of compact support with $\operatorname{supp}\omega_0 \subset V$ and with $\int \omega_0 = 1$. If we set $\deg f = \int_U f^*\omega_0$ then (3.4.1) clearly holds for $\omega_0$. We will prove that (3.4.1) holds for every compactly supported $n$-form, $\omega$, with $\operatorname{supp}\omega \subseteq V$. Let $c = \int_V \omega$. Then by Theorem 3.1 $\omega - c\omega_0 = d\mu$, where $\mu$ is a completely supported $(n-1)$-form with $\operatorname{supp}\mu \subseteq V$. Hence

$$f^*\omega - cf^*\omega_0 = f^*\, d\mu = d\, f^*\mu\,,$$

and by part (a) of Theorem 3.1

$$\int_U f^*\omega = c\int f^*\omega_0 = \deg(f)\int_V \omega\,.$$

□

We will show in § 3.6 that the degree of $f$ is always an integer and explain why it is a "topological" invariant of $f$. For the moment, however, we'll content ourselves with pointing out a simple but useful property of this invariant. Let $U$, $V$ and $W$ be connected open subsets of $\mathbb{R}^n$ and $f : U \to V$ and $g : V \to W$ proper $\mathcal{C}^\infty$ mappings. Then

(3.4.3) $$\deg(g \circ f) = \deg(g)\deg(f)\,.$$

*Proof.* Let $\omega$ be a compactly supported $n$-form with support on $W$. Then

$$(g \circ f)^*\omega = g^* f^*\omega\,;$$

so

$$\begin{aligned} \int_U (g \circ f)^*\omega &= \int_U g^*(f^*\omega) = \deg(g)\int_V f^*\omega \\ &= \deg(g)\deg(f)\int_W \omega\,. \end{aligned}$$

□

From this multiplicative property it is easy to deduce the following result (which we will need in the next section).

**Theorem 3.4.1.** *Let $A$ be a non-singular $n \times n$ matrix and $f_A : \mathbb{R}^n \to \mathbb{R}^n$ the linear mapping associated with $A$. Then* $\deg(f_A) = +1$ *if* $\det A$ *is positive and* $-1$ *if* $\det A$ *is negative.*

A proof of this result is outlined in exercises 5–9 below.

## Exercises for §3.4.

1. Let $U$ be an open subset of $\mathbb{R}^n$ and $\varphi_i$, $i = 1, 2, 3, \ldots$, a partition of unity on $U$. Show that the mapping, $f : U \to \mathbb{R}$ defined by

$$f = \sum_{k=1}^{\infty} k\varphi_k$$

is a proper $\mathcal{C}^\infty$ mapping.

2. Let $U$ and $V$ be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$ and let $f : U \to V$ be a proper continuous mapping. Prove:

**Theorem 3.4.2.** *If $B$ is a compact subset of $V$ and $A = f^{-1}(B)$ then for every open subset, $U_0$, with $A \subseteq U_0 \subseteq U$, there exists an open subset, $V_0$, with $B \subseteq V_0 \subseteq V$ and $f^{-1}(V_0) \subseteq U_0$.*

*Hint:* Let $C$ be a compact subset of $V$ with $B \subseteq \operatorname{Int} C$. Then the set, $W = f^{-1}(C) - U_0$ is compact; so its image, $f(W)$, is compact. Show that $f(W)$ and $B$ are disjoint and let

$$V_0 = \operatorname{Int} C - f(W)\,.$$

3. Show that if $f : U \to V$ is a proper continuous mapping and $X$ is a closed subset of $U$, $f(X)$ is closed.

*Hint:* Let $U_0 = U - X$. Show that if $p$ is in $V - f(X)$, $f^{-1}(p)$ is contained in $U_0$ and conclude from the previous exercise that there exists a neighborhood, $V_0$, of $p$ such that $f^{-1}(V_0)$ is contained in $U_0$. Conclude that $V_0$ and $f(X)$ are disjoint.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the translation, $f(x) = x + a$. Show that $\deg(f) = 1$.

*Hint:* Let $\psi : \mathbb{R} \to \mathbb{R}$ be a compactly supported $\mathcal{C}^\infty$ function. For $a \in \mathbb{R}$, the identity

$$\int \psi(t)\,dt = \int \psi(t-a)\,dt \tag{3.4.4}$$

is easy to prove by elementary calculus, and this identity proves the assertion above in dimension one. Now let

$$\varphi(x) = \psi(x_1)\ldots\varphi(x_n) \tag{3.4.5}$$

and compute the right and left sides of (3.4.2) by Fubini's theorem.

5. Let $\sigma$ be a permutation of the numbers, $1, \dots, n$ and let $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be the diffeomorphism, $f_\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Prove that $\deg f_\sigma = \operatorname{sgn}(\sigma)$.

*Hint:* Let $\varphi$ be the function (3.4.5). Show that if $\omega$ is equal to $\varphi(x)\, dx_1 \wedge \cdots \wedge dx_n$, $f^*\omega = (\operatorname{sgn} \sigma)\omega$.

6. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$f(x_1, \dots, x_n) = (x_1 + \lambda x_2, x_2, \dots, x_n).$$

Prove that $\deg(f) = 1$.

*Hint:* Let $\omega = \varphi(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$ where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is compactly supported and of class $\mathcal{C}^\infty$. Show that

$$\int f^*\omega = \int \varphi(x_1 + \lambda x_2, x_2, \dots, x_n)\, dx_1 \dots dx_n$$

and evaluate the integral on the right by Fubini's theorem; i.e., by first integrating with respect to the $x_1$ variable and then with respect to the remaining variables. Note that by (3.4.4)

$$\int f(x_1 + \lambda x_2, x_2, \dots, x_n)\, dx_1 = \int f(x_1, x_2, \dots, x_n)\, dx_1 \,.$$

7. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$f(x_1, \dots, x_n) = (\lambda x_1, x_2, \dots, x_n)$$

with $\lambda \neq 0$. Show that $\deg f = +1$ if $\lambda$ is positive and $-1$ if $\lambda$ is negative.

*Hint:* In dimension 1 this is easy to prove by elementary calculus techniques. Prove it in $d$-dimensions by the same trick as in the previous exercise.

8. (a) Let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbb{R}^n$ and $A$, $B$ and $C$ the linear mappings

$$\begin{aligned} Ae_1 &= e, \quad Ae_i = \sum_j a_{j,i} e_j \,, \qquad i > 1 \\ Be_i &= e_i \,, \quad i > 1 \,, \quad Be_1 = \sum_{j=1}^n b_j e_j \\ Ce_1 &= e_1 \,, \quad Ce_i = e_i + c_i e_1 \,, \quad i > 1 \,. \end{aligned} \tag{3.4.6}$$

Show that

$$BACe_1 = \sum b_j e_j$$

and

$$BACe_i = \sum_j^n = (a_{j,i} + c_i b_j)e_j + c_i b_1 e_1$$

for $i > 1$.

(b)

$$Le_i = \sum_{j=1}^n \ell_{j,i} e_j \,, \quad i = 1, \dots, n \,. \tag{3.4.7}$$

Show that if $\ell_{1,1} \neq 0$ one can write $L$ as a product, $L = BAC$, where $A$, $B$ and $C$ are linear mappings of the form (3.4.6).

*Hint:* First solve the equations

$$\ell_{j,1} = b_j$$

for $j = 1, \dots, n$, then the equations

$$\ell_{1,i} = b_1 c_i$$

for $i > 1$, then the equations

$$\ell_{j,i} = a_{j,i} + c_i b_j$$

for $i, j > 1$.

(c) Suppose $L$ is invertible. Conclude that $A$, $B$ and $C$ are invertible and verify that Theorem 3.4.1 holds for $B$ and $C$ using the previous exercises in this section.

(d) Show by an inductive argument that Theorem 3.4.1 holds for $A$ and conclude from (3.4.3) that it holds for $L$.

9. To show that Theorem 3.4.1 holds for an arbitrary linear mapping, $L$, of the form (3.4.7) we'll need to eliminate the assumption: $\ell_{1,1} \neq 0$. Show that for some $j$, $\ell_{j,1}$ is non-zero, and show how to eliminate this assumption by considering $f_\sigma \circ L$ where $\sigma$ is the transposition, $1 \leftrightarrow j$.

10. Here is an alternative proof of Theorem 4.3.1 which is shorter than the proof outlined in exercise 9 but uses some slightly more sophisticated linear algebra.

(a) Prove Theorem 3.4.1 for linear mappings which are *orthogonal*, i.e., satisfy $L^tL = I$.

*Hints:*

i. Show that $L^*(x_1^2 + \cdots + x_n^2) = x_1^2 + \cdots + x_n^2$.

ii. Show that $L^*(dx_1 \wedge \cdots \wedge dx_n)$ is equal to $dx_1 \wedge \cdots \wedge dx_n$ or $-dx_1 \wedge \cdots \wedge dx_n$ depending on whether $L$ is orientation preserving or orinetation reversing. (See § 1.2, exercise 10.)

iii. Let $\psi$ be as in exercise 4 and let $\omega$ be the form

$$\omega = \psi(x_1^2 + \cdots + x_n^2)\, dx_1 \wedge \cdots \wedge \, dx_n \,.$$

Show that $L^*\omega = \omega$ if $L$ is orientation preserving and $L^*\omega = -\omega$ if $L$ is orientation reversing.

(b) Prove Theorem 3.4.1 for linear mappings which are *self-adjoint* (satisfy $L^t = L$). *Hint:* A self-adjoint linear mapping is diagonizable: there exists an intervertible linear mapping, $M : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$M^{-1}LMe_i = \lambda_i e_i \,, \quad i = 1, \ldots, n \,. \tag{3.4.8}$$

(c) Prove that every invertible linear mapping, $L$, can be written as a product, $L = BC$ where $B$ is orthogonal and $C$ is self-adjoint.

*Hints:*

i. Show that the mapping, $A = L^tL$, is self-adjoint and that it's eigenvalues, the $\lambda_i$'s in 3.4.8, are positive.

ii. Show that there exists an invertible self-adjoint linear mapping, $C$, such that $A = C^2$ and $AC = CA$.

iii. Show that the mapping $B = LC^{-1}$ is orthogonal.

## 3.5 The change of variables formula

Let $U$ and $V$ be connected open subsets of $\mathbb{R}^n$. If $f : U \to V$ is a diffeomorphism, the determinant of $Df(x)$ at $x \in U$ is non-zero, and hence, since it is a continuous function of $x$, its sign is the same at every point. We will say that $f$ is *orientation preserving* if this sign is positive and *orientation reversing* if it is negative. We will prove below:

**Theorem 3.5.1.** *The degree of $f$ is $+1$ if $f$ is orientation preserving and $-1$ if $f$ is orientation reversing.*

We will then use this result to prove the following change of variables formula for diffeomorphisms.

**Theorem 3.5.2.** *Let $\varphi : V \to \mathbb{R}$ be a compactly supported continuous function. Then*

$$\int_U \varphi \circ f(x) |\det(Df)(x)| = \int_V \varphi(y)\, dy\,. \tag{3.5.1}$$

*Proof of Theorem 3.5.1.* Given a point, $a_1 \in U$, let $a_2 = -f(a_1)$ and for $i = 1, 2$, let $g_i : \mathbb{R}^n \to \mathbb{R}^n$ be the translation, $g_i(x) = x + a_i$. By (3.4.1) and exercise 4 of § 4 the composite diffeomorphism

$$g_2 \circ f \circ g_1 \tag{3.5.2}$$

has the same degree as $f$, so it suffices to prove the theorem for this mapping. Notice however that this mapping maps the origin onto the origin. Hence, replacing $f$ by this mapping, we can, without loss of generality, assume that 0 is in the domain of $f$ and that $f(0) = 0$.

Next notice that if $A : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective linear mapping the theorem is true for $A$ (by exercise 9 of § 3.4), and hence if we can prove the theorem for $A^{-1} \circ f$, (3.4.1) will tell us that the theorem is true for $f$. In particular, letting $A = Df(0)$, we have

$$D(A^{-1} \circ f)(0) = A^{-1} Df(0) = I$$

where $I$ is the identity mapping. Therefore, replacing $f$ by $A^{-1}f$, we can assume that the mapping, $f$, for which we are attempting to prove Theorem 3.5.1 has the properties: $f(0) = 0$ and $Df(0) = I$. Let $g(x) = f(x) - x$. Then these properties imply that $g(0) = 0$ and $Dg(0) = 0$.

$\square$

**Lemma 3.5.3.** *There exists a $\delta > 0$ such that $|g(x)| \le \frac{1}{2}|x|$ for $|x| \le \delta$.*

*Proof.* Let $g(x) = (g_1(x), \dots, g_n(x))$. Then

$$\frac{\partial g_i}{\partial x_j}(0) = 0\,;$$

so there exists a $\delta > 0$ such that

$$\left|\frac{\partial g_i}{\partial x_j}(x)\right| \leq \frac{1}{2}$$

for $|x| \leq \delta$. However, by the mean value theorem,

$$g_i(x) = \sum \frac{\partial g_i}{\partial x_j}(c) x_j$$

for $c = t_0 x$, $0 < t_0 < 1$. Thus, for $|x| < \delta$,

$$|g_i(x)| \leq \frac{1}{2} \sup |x_i| = \frac{1}{2} |x| ,$$

so

$$|g(x)| = \sup |g_i(x)| \leq \frac{1}{2} |x| .$$

□

Let $\rho$ be a compactly supported $\mathcal{C}^\infty$ function with $0 \leq \rho \leq 1$ and with $\rho(x) = 0$ for $|x| \geq \delta$ and $\rho(x) = 1$ for $|x| \leq \frac{\delta}{2}$ and let $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$\widetilde{f}(x) = x + \rho(x) g(x) . \tag{3.5.3}$$

It's clear that

$$\widetilde{f}(x) = x \text{ for } |x| \geq \delta \tag{3.5.4}$$

and, since $f(x) = x + g(x)$,

$$\widetilde{f}(x) = f(x) \text{ for } |x| \leq \frac{\delta}{2} . \tag{3.5.5}$$

In addition, for all $x \in \mathbb{R}^n$:

$$|\widetilde{f}(x)| \geq \frac{1}{2} |x| . \tag{3.5.6}$$

Indeed, by (3.5.4), $|\widetilde{f}(x)| \geq |x|$ for $|x| \geq \delta$, and for $|x| \leq \delta$

$$\begin{aligned} |\widetilde{f}(x)| &\geq |x| - \rho(x)|g(x)| \\ &\geq |x| - |g(x)| \geq |x| - \frac{1}{2} |x| = \frac{1}{2} |x| \end{aligned}$$

by Lemma 3.5.3.

Now let $\mathcal{Q}_r$ be the cube, $\{x \in \mathbb{R}^n\,,\ |x| \le r\}$, and let $\mathcal{Q}_r^c = \mathbb{R}^n - \mathcal{Q}_r$.

From (3.5.6) we easily deduce that

$$\widetilde{f}^{-1}(\mathcal{Q}_r) \subseteq \mathcal{Q}_{2r} \tag{3.5.7}$$

for all $r$, and hence that $\widetilde{f}$ is *proper*. Also notice that for $x \in \mathcal{Q}_\delta$,

$$|\widetilde{f}(x)| \le |x| + |g(x)| \le \frac{3}{2}\,|x|$$

by Lemma 3.5.3 and hence

$$\widetilde{f}^{-1}(\mathcal{Q}^c_{\frac{3}{2}\delta}) \subseteq \mathcal{Q}^c_\delta\,. \tag{3.5.8}$$

We will now prove Theorem 3.5.1. Since $f$ is a diffeomorphism mapping 0 to 0, it maps a neighborhood, $U_0$, of 0 in $U$ diffeomorphically onto a neighborhood, $V_0$, of 0 in $V$, and by shrinking $U_0$ if necessary we can assume that $U_0$ is contained in $\mathcal{Q}_{\delta/2}$ and $V_0$ contained in $\mathcal{Q}_{\delta/4}$. Let $\omega$ be an $n$-form with support in $V_0$ whose integral over $\mathbb{R}^n$ is equal to one. Then $f^*\omega$ is supported in $U_0$ and hence in $\mathcal{Q}_{\delta/2}$. Also by (3.5.7) $\widetilde{f}^*\omega$ is supported in $\mathcal{Q}_{\delta/2}$. Thus both of these forms are zero outside $\mathcal{Q}_{\delta/2}$. However, on $\mathcal{Q}_{\delta/2}$, $\widetilde{f} = f$ by (3.5.5), so these forms are equal everywhere, and hence

$$\deg(f) = \int f^*\omega = \int \widetilde{f}^*\omega = \deg(\widetilde{f})\,.$$

Next let $\omega$ be a compactly supported $n$-form with support in $\mathcal{Q}^c_{3\delta/2}$ and with integral equal to one. Then $\widetilde{f}^*\omega$ is supported in $\mathcal{Q}^c_\delta$ by (3.5.8), and hence since $f(x) = x$ on $\mathcal{Q}^c_\delta$ $\widetilde{f}^*\omega = \omega$. Thus

$$\deg(\widetilde{f}) = \int f^*\omega = \int \omega = 1\,.$$

Putting these two identities together we conclude that $\deg(f) = 1$. Q.E.D.

If the function, $\varphi$, in Theorem 3.5.2 is a $\mathcal{C}^\infty$ function, the identity (3.5.1) is an immediate consequence of the result above and the identity (3.4.2). If $\varphi$ is not $\mathcal{C}^\infty$, but is just continuous, we will deduce Theorem 3.5.2 from the following result.

**Theorem 3.5.4.** *Let $V$ be an open subset of $\mathbb{R}^n$. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function of compact support with* $\operatorname{supp}\varphi \subseteq V$*; then for every $\epsilon > 0$ there exists a $\mathcal{C}^\infty$ function of compact support, $\psi : \mathbb{R}^n \to \mathbb{R}$ with* $\operatorname{supp}\psi \subseteq V$ *and*

$$\sup|\psi(x) - \varphi(x)| < \epsilon\,.$$

*Proof.* Let $A$ be the support of $\varphi$ and let $d$ be the distance in the sup norm from $A$ to the complement of $V$. Since $\varphi$ is continuous and compactly supported it is uniformly continuous; so for every $\epsilon > 0$ there exists a $\delta > 0$ with $\delta < \frac{d}{2}$ such that $|\varphi(x) - \varphi(y)| < \epsilon$ when $|x - y| \le \delta$. Now let $Q$ be the cube: $|x| < \delta$ and let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a non-negative $\mathcal{C}^\infty$ function with $\operatorname{supp}\rho \subseteq Q$ and

$$\int \rho(y)\,dy = 1\,. \tag{3.5.9}$$

Set

$$\psi(x) = \int \rho(y - x)\varphi(y)\,dy\,.$$

By Theorem 3.2.5 $\psi$ is a $\mathcal{C}^\infty$ function. Moreover, if $A_\delta$ is the set of points in $\mathbb{R}^d$ whose distance in the sup norm from $A$ is $\le \delta$ then for $x \notin A_\delta$ and $y \in A$, $|x - y| > \delta$ and hence $\rho(y - x) = 0$. Thus for $x \notin A_\delta$

$$\int \rho(y - x)\varphi(y)\,dy = \int_A \rho(y - x)\varphi(y)\,dy = 0\,,$$

so $\psi$ is supported on the compact set $A_\delta$. Moreover, since $\delta < \frac{d}{2}$, $\operatorname{supp}\psi$ is contained in $V$. Finally note that by (3.5.9) and exercise 4 of §3.4:

$$\int \rho(y - x)\,dy = \int \rho(y)\,dy = 1 \tag{3.5.10}$$

and hence

$$\varphi(x) \quad = \quad \int \varphi(x)\rho(y - x)\,dy$$

so

$$\varphi(x) - \psi(x) \quad = \quad \int (\varphi(x) - \varphi(y))\rho(y - x)\,dy$$

and

$$|\varphi(x) - \psi(x)| \quad \leq \quad \int |\varphi(x) - \varphi(y)|\, \rho(y - x)\, dy\,.$$

But $\rho(y-x) = 0$ for $|x-y| \geq \delta$; and $|\varphi(x)-\varphi(y)| < \epsilon$ for $|x-y| \leq \delta$, so the integrand on the right is less than

$$\epsilon \int \rho(y - x)\, dy\,,$$

and hence by (3.5.10)

$$|\varphi(x) - \psi(x)| \leq \epsilon\,.$$

$\square$

To prove the identity (3.5.1), let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ cut-off function which is one on a neighborhood, $V_1$, of the support of $\varphi$, is non-negative, and is compactly supported with $\operatorname{supp}\gamma \subseteq V$, and let

$$c = \int \gamma(y)\, dy\,.$$

By Theorem 3.5.4 there exists, for every $\epsilon > 0$, a $\mathcal{C}^\infty$ function $\psi$, with support on $V_1$ satisfying

$$|\varphi - \psi| \leq \tfrac{\epsilon}{2c}\,. \tag{3.5.11}$$

Thus

$$\begin{aligned} \Big|\int_V (\varphi - \psi)(y)\, dy\Big| \quad &\leq \quad \int_V |\varphi - \psi|(y)\, dy \\ &\leq \quad \int_V \gamma|\varphi - \psi|(xy)\, dy \\ &\leq \quad \tfrac{\epsilon}{2c}\int \gamma(y)\, dy \leq \tfrac{\epsilon}{2} \end{aligned}$$

so

$$\Big|\int_V \varphi(y)\, dy - \int_V \psi(y)\, dy\Big| \leq \tfrac{\epsilon}{2}\,. \tag{3.5.12}$$

Similarly, the expression

$$\Big|\int_U (\varphi - \psi)\circ f(x) |\det Df(x)|\, dx\Big|$$

is less than or equal to the integral

$$\int_U \gamma \circ f(x)|(\varphi - \psi) \circ f(x)| \, |\det Df(x)| \, dx$$

and by (3.5.11), $|(\varphi - \psi) \circ f(x)| \leq \frac{\epsilon}{2c}$, so this integral is less than or equal to

$$\tfrac{\epsilon}{2c} \int \gamma \circ f(x)| \det Df(x)| \, dx$$

and hence by (3.5.1) is less than or equal to $\frac{\epsilon}{2}$. Thus
(3.5.13)

$$\left| \int_U \varphi \circ f(x) \, |\det Df(x)| dx - \int_U \psi \circ f(x) | \det Df(x)| \; dx \right| \leq \tfrac{\epsilon}{2} \, .$$

Combining (3.5.12), (3.5.13) and the identity

$$\int_V \psi(y) \, dy = \int \psi \circ f(x)| \det Df(x)| \, dx$$

we get, for all $\epsilon > 0$,

$$\left| \int_V \varphi(y) \, dy - \int_U \varphi \circ f(x)| \det Df(x)| \, dx \right| \leq \epsilon$$

and hence

$$\int \varphi(y) \, dy = \int \varphi \circ f(x)| \det Df(x)| \, dx \, .$$

## Exercises for §3.5

1. Let $h : V \to \mathbb{R}$ be a non-negative continuous function. Show that if the improper integral

$$\int_V h(y) \, dy$$

is well-defined, then the improper integral

$$\int_U h \circ f(x)| \det Df(x)| \, dx$$

is well-defined and these two integrals are equal.

*Hint:* If $\varphi_i$, $i = 1, 2, 3, \ldots$ is a partition of unity on $V$ then $\psi_i = \varphi_i \circ f$ is a partition of unity on $U$ and

$$\int \varphi_i h \, dy = \int \psi_i (h \circ f(x)) |\det Df(x)| \, dx \, .$$

Now sum both sides of this identity over $i$.

2. Show that the result above is true without the assumption that $h$ is non-negative.

*Hint:* $h = h_+ - h_-$, where $h_+ = \max(h, 0)$ and $h_- = \max(-h, 0)$.

3. Show that, in the formula (3.4.2), one can allow the function, $\varphi$, to be a *continuous* compactly supported function rather than a $\mathcal{C}^\infty$ compactly supported function.

4. Let $\mathbb{H}^n$ be the half-space (**??**) and $U$ and $V$ open subsets of $\mathbb{R}^n$. Suppose $f : U \to V$ is an orientation preserving diffeomorphism mapping $U \cap \mathbb{H}^n$ onto $V \cap \mathbb{H}^n$. Show that for $\omega \in \Omega^n_c(V)$

$$\int_{U \cap \mathbb{H}^n} f^* \omega = \int_{V \cap \mathbb{H}^n} \omega \, . \tag{3.5.14}$$

*Hint:* Interpret the left and right hand sides of this formula as improper integrals over $U \cap \operatorname{Int} \mathbb{H}^n$ and $V \cap \operatorname{Int} \mathbb{H}^n$.

5. The boundary of $\mathbb{H}^n$ is the set

$$b\mathbb{H}^n = \{(0, x_2, \ldots, x_n) \, , \quad (x_2, \ldots, x_n) \in \mathbb{R}^n\}$$

so the map

$$\iota : \mathbb{R}^{n-1} \to \mathbb{H}^n \, , \quad (x_2, \ldots, x_n) \to (0, x_2, \ldots, x_n)$$

in exercise 9 in §3.2 maps $\mathbb{R}^{n-1}$ bijectively onto $b\mathbb{H}^n$.

(a) Show that the map $f : U \to V$ in exercise 4 maps $U \cap b\mathbb{H}^n$ onto $V \cap b\mathbb{H}^n$.

(b) Let $U' = \iota^{-1}(U)$ and $V' = \iota^{-1}(V)$. Conclude from part (a) that the restriction of $f$ to $U \cap b\mathbb{H}^n$ gives one a diffeomorphism

$$g : U' \to V'$$

satisfying:

$$\iota \cdot g = f \cdot \iota \, . \tag{3.5.15}$$

(c) Let $\mu$ be in $\Omega_c^{n-1}(V)$. Conclude from (3.2.7) and (3.5.14):

$$\int_{U'} g^*\iota^*\mu = \int_{V'} \iota^*\mu \tag{3.5.16}$$

and in particular show that the diffeomorphism, $g : U' \to V'$, is orientation preserving.

## 3.6 Techniques for computing the degree of a mapping

Let $U$ and $V$ be open subsets of $\mathbb{R}^n$ and $f : U \to V$ a proper $\mathcal{C}^\infty$ mapping. In this section we will show how to compute the degree of $f$ and, in particular, show that it is always an integer. From this fact we will be able to conclude that the degree of $f$ is a topological invariant of $f$: if we deform $f$ smoothly, its degree doesn't change.

**Definition 3.6.1.** *A point, $x \in U$, is a* critical point *of $f$ if the derivative*

$$Df(x) : \mathbb{R}^n \to \mathbb{R}^n$$

*fails to be bijective, i.e., if* $\det(Df(x)) = 0$.

We will denote the set of critical points of $f$ by $C_f$. It's clear from the definition that this set is a closed subset of $U$ and hence, by exercise 3 in §3.4, $f(C_f)$ is a closed subset of $V$. We will call this image the set of *critical values* of $f$ and the complement of this image the set of *regular values* of $f$. Notice that $V - f(U)$ is contained in $f - f(C_f)$, so if a point, $g \in V$ is not in the image of $f$, it's a regular value of $f$ "by default", i.e., it contains no points of $U$ in the pre-image and hence, a fortiori, contains no critical points in its pre-image. Notice also that $C_f$ can be quite large. For instance, if $c$ is a point in $V$ and $f : U \to V$ is the constant map which maps all of $U$ onto $c$, then $C_f = U$. However, in this example, $f(C_f) = \{c\}$, so the set of regular values of $f$ is $V - \{c\}$, and hence (in this example) is an open dense subset of $V$. We will show that this is true in general.

**Theorem 3.6.2.** *(Sard's theorem.)*

*If $U$ and $V$ are open subsets of $\mathbb{R}^n$ and $f : U \to V$ a proper $\mathcal{C}^\infty$ map, the set of regular values of $f$ is an open dense subset of $V$.*

We will defer the proof of this to Section 3.7 and, in this section, explore some of its implications. Picking a regular value, $q$, of $f$ we will prove:

**Theorem 3.6.3.** *The set, $f^{-1}(q)$ is a finite set. Moreover, if $f^{-1}(q) = \{p_1, \dots, p_n\}$ there exist connected open neighborhoods, $U_i$, of $p_i$ in $Y$ and an open neighborhood, $W$, of $q$ in $V$ such that:*

*i.* *for $i \neq j$ $U_i$ and $U_j$ are disjoint;*

*ii.* *$f^{-1}(W) = \bigcup U_i$,*

*iii.* *$f$ maps $U_i$ diffeomorphically onto $W$.*

*Proof.* If $p \in f^{-1}(q)$ then, since $q$ is a regular value, $p \notin C_f$; so

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^n$$

is bijective. Hence by the inverse function theorem, $f$ maps a neighborhood, $U_p$ of $p$ diffeomorphically onto a neighborhood of $q$. The open sets

$$\{U_p\,, \quad p \in f^{-1}(q)\}$$

are a covering of $f^{-1}(q)$; and, since $f$ is proper, $f^{-1}(q)$ is compact; so we can extract a finite subcovering

$$\{U_{p_i}\,, \quad i = 1, \dots, N\}$$

and since $p_i$ is the only point in $U_{p_i}$ which maps onto $q$, $f^{-1}(q) = \{p_1, \dots, p_N\}$.

Without loss of generality we can assume that the $U_{p_i}$'s are disjoint from each other; for, if not, we can replace them by smaller neighborhoods of the $p_i$'s which have this property. By Theorem 3.4.2 there exists a connected open neighborhood, $W$, of $q$ in $V$ for which

$$f^{-1}(W) \subset \bigcup U_{p_i}\,.$$

To conclude the proof let $U_i = f^{-1}(W) \cap U_{p_i}$. □

The main result of this section is a recipe for computing the degree of $f$ by counting the number of $p_i$'s above, keeping track of orientation.

**Theorem 3.6.4.** *For each $p_i \in f^{-1}(q)$ let $\sigma_{p_i} = +1$ if $f : U_i \to W$ is orientation preserving and $-1$ if $f : U_i \to W$ is orientation reversing. Then*

$$\deg(f) = \sum_{i=1}^{N} \sigma_{p_i}\,. \tag{3.6.1}$$

*Proof.* Let $\omega$ be a compactly supported $n$-form on $W$ whose integral is one. Then

$$\deg(f) = \int_U f^*\omega = \sum_{i=1}^{N} \int_{U_i} f^*\omega\,.$$

Since $f : U_i \to W$ is a diffeomorphism

$$\int_{U_i} f^*\omega = \pm \int_W \omega = +1 \text{ or } -1$$

depending on whether $f : U_i \to W$ is orientation preserving or not. Thus $\deg(f)$ is equal to the sum (3.6.1).

□

As we pointed out above, a point, $q \in V$ can qualify as a regular value of $f$ "by default", i.e., by not being in the image of $f$. In this case the recipe (3.6.1) for computing the degree gives "by default" the answer zero. Let's corroborate this directly.

**Theorem 3.6.5.** *If $f : U \to V$ isn't onto,* $\deg(f) = 0$.

*Proof.* By exercise 3 of §3.4, $V - f(U)$ is open; so if it is non-empty, there exists a compactly supported $n$-form, $\omega$, with support in $V - f(U)$ and with integral equal to one. Since $\omega = 0$ on the image of $f$, $f^*\omega = 0$; so

$$0 = \int_U f^*\omega = \deg(f) \int_V \omega = \deg(f)\,.$$

□

*Remark:* In applications the contrapositive of this theorem is much more useful than the theorem itself.

**Theorem 3.6.6.** *If* $\deg(f) \neq 0$ *$f$ maps $U$ onto $V$.*

In other words if $\deg(f) \neq 0$ the equation

$$f(x) = y \tag{3.6.2}$$

has a solution, $x \in U$ for *every* $y \in V$.

We will now show that the degree of $f$ is a topological invariant of $f$: if we deform $f$ by a "homotopy" we don't change its degree. To make this assertion precise, let's recall what we mean by a *homotopy* between a pair of $\mathcal{C}^\infty$ maps. Let $U$ be an open subset of $\mathbb{R}^m$, $V$ an open subset of $\mathbb{R}^n$, $A$ an open subinterval of $\mathbb{R}$ containing 0 and 1, and $f_i : U \to V$, $i = 0, 1$, $\mathcal{C}^\infty$ maps. Then a $\mathcal{C}^\infty$ map $F : U \times A \to V$ is a homotopy between $f_0$ and $f_1$ if $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. (See Definition **??**.) Suppose now that $f_0$ and $f_1$ are proper.

**Definition 3.6.7.** *$F$ is a* proper *homotopy between $f_0$ and $f_1$ if the map*

$$F^\sharp : U \times A \to V \times A \tag{3.6.3}$$

*mapping $(x, t)$ to $(F(x, t), t)$ is proper.*

Note that if $F$ is a proper homotopy between $f_0$ and $f_1$, then for every $t$ between 0 and 1, the map

$$f_t : U \to V\,, \quad f_t(x) = F_t(x)$$

is proper.

Now let $U$ and $V$ be open subsets of $\mathbb{R}^n$.

**Theorem 3.6.8.** *If $f_0$ and $f_1$ are properly homotopic, their degrees are the same.*

*Proof.* Let

$$\omega = \varphi(y)\, d\,y_1 \wedge \cdots \wedge d\,y_n$$

be a compactly supported $n$-form on $X$ whose integral over $V$ is 1. The the degree of $f_t$ is equal to

$$\int_U \varphi(F_1(x, t), \ldots, F_n(x, t)) \det D_x F(x, t)\, dx\,. \tag{3.6.4}$$

The integrand in (3.6.4) is continuous and for $0 \le t \le 1$ is supported on a compact subset of $U \times [0, 1]$, hence (3.6.4) is continuous as a function of $t$. However, as we've just proved, $\deg(f_t)$ is *integer* valued so this function is a constant. $\square$

(For an alternative proof of this result see exercise 9 below.) We'll conclude this account of degree theory by describing a couple applications.

*Application 1. The Brouwer fixed point theorem*

Let $B^n$ be the closed unit ball in $\mathbb{R}^n$:

$$\{x \in \mathbb{R}^n\,,\ \|x\| \leq 1\}\,.$$

**Theorem 3.6.9.** *If $f : B^n \to B^n$ is a continuous mapping then $f$ has a fixed point, i.e., maps some point, $x_0 \in B^n$ onto itself.*

The idea of the proof will be to assume that there isn't a fixed point and show that this leads to a contradiction. Suppose that for every point, $x \in B^n$ $f(x) \neq x$. Consider the ray through $f(x)$ in the direction of $x$:

$$f(x) + s(x - f(x))\,, \quad 0 \leq s < \infty\,.$$

This intersects the boundary, $S^{n-1}$, of $B^n$ in a unique point, $\gamma(x)$, (see figure 1 below); and one of the exercises at the end of this section will be to show that the mapping $\gamma : B^n \to S^{n-1}$, $x \to \gamma(x)$, is a continuous mapping. Also it is clear from figure 1 that $\gamma(x) = x$ if $x \in S^{n-1}$, so we can extend $\gamma$ to a continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$ by letting $\gamma$ be the identity for $\|x\| \geq 1$. Note that this extended mapping has the property

$$\|\gamma(x)\| \geq 1 \tag{3.6.5}$$

for all $x \in \mathbb{R}^n$ and

$$\gamma(x) = x \tag{3.6.6}$$

for all $\|x\| \geq 1$. To get a contradiction we'll show that $\gamma$ can be approximated by a $\mathcal{C}^\infty$ map which has similar properties. For this we will need the following corollary of Theorem 3.5.4.

**Lemma 3.6.10.** *Let $U$ be an open subset of $\mathbb{R}^n$, $C$ a compact subset of $U$ and $\varphi : U \to \mathbb{R}$ a continuous function which is $\mathcal{C}^\infty$ on the complement of $C$. Then for every $\epsilon > 0$, there exists a $\mathcal{C}^\infty$ function, $\psi : U \to \mathbb{R}$, such that $\varphi - \psi$ has compact support and $|\varphi - \psi| < \epsilon$.*

*Proof.* Let $\rho$ be a bump function which is in $\mathcal{C}_0^\infty(U)$ and is equal to 1 on a neighborhood of $C$. By Theorem 3.5.4 there exists a function, $\psi_0 \in \mathcal{C}_0^\infty(U)$ such that $|\rho\varphi - \psi_0| < \epsilon$. Let $\psi = (1-\rho)\varphi + \psi_0$, and note that

$$\begin{aligned} \varphi - \psi &= (1-\rho)\varphi + \rho\varphi - (1-\rho)\varphi - \psi_0 \\ &= \rho\varphi - \psi_0\,. \end{aligned}$$

By applying this lemma to each of the coordinates of the map, $\gamma$, one obtains a $\mathcal{C}^\infty$ map, $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\|g - \gamma\| < \epsilon < 1 \tag{3.6.7}$$

and such that $g = \gamma$ on the complement of a compact set. However, by (3.6.6), this means that $g$ is equal to the identity on the complement of a compact set and hence (see exercise 9) that $g$ is proper and has degree one. On the other hand by (3.6.8) and (3.6.6) $\|g(x)\| > 1-\epsilon$ for all $x \in \mathbb{R}^n$, so $0 \notin \operatorname{Im} g$ and hence by Theorem 3.6.4, $\deg(g) = 0$. Contradiction.

□

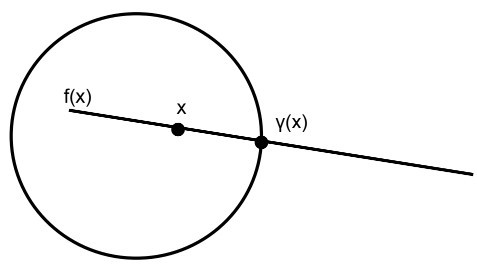



**Figure 3.6.1.**

*Application 2. The fundamental theorem of algebra*

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n$ with complex coefficients. If we identify the complex plane

$$\mathbb{C} = \{z = x + iy\,;\, x, y \in \mathbb{R}\}$$

with $\mathbb{R}^2$ via the map, $(x, y) \in \mathbb{R}^2 \to z = x + iy$, we can think of $p$ as defining a mapping

$$p : \mathbb{R}^2 \to \mathbb{R}^2\,,\ z \to p(z)\,.$$

We will prove

**Theorem 3.6.11.** *The mapping, $p$, is proper and* $\deg(p) = n$.

*Proof.* For $t \in \mathbb{R}$

$$\begin{aligned} p_t(z) &= (1-t)z^n + tp(z) \\ &= z^n + t\sum_{i=0}^{n-1} a_i z^i \,. \end{aligned}$$

We will show that the mapping

$$g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \,,\; z \to p_t(z)$$

is a proper homotopy. Let

$$C = \sup\{|a_i|\,,\; i = 0, \dots, n-1\}\,.$$

Then for $|z| \geq 1$

$$\begin{aligned} |a_0 + \cdots + a_{n-1}z^{n-1}| &\leq |a_0| + |a_1||z| + \cdots + |a_{n-1}|\,|z|^{n-1} \\ &\leq C|z|^{n-1}\,, \end{aligned}$$

and hence, for $|t| \leq a$ and $|z| \geq 2aC$,

$$\begin{aligned} |p_t(z)| &\geq |z|^n - aC|z|^{n-1} \\ &\geq aC|z|^{n-1}\,. \end{aligned}$$

If $A$ is a compact subset of $\mathbb{C}$ then for some $R > 0$, $A$ is contained in the disk, $|w| \leq R$ and hence the set

$$\{z \in \mathbb{C}\,,\; (p_t(z), t) \in A \times [-a, a]\}$$

is contained in the compact set

$$\{z \in \mathbb{C}\,,\; aC|z|^{n-1} \leq R\}\,,$$

and this shows that $g$ is a proper homotopy. Thus each of the mappings,

$$p_t : \mathbb{C} \to \mathbb{C}\,,$$

is proper and $\deg p_t = \deg p_1 = \deg p = \deg p_0$. However, $p_0 : \mathbb{C} \to \mathbb{C}$ is just the mapping, $z \to z^n$ and an elementary computation (see exercises 5 and 6 below) shows that the degree of this mapping is $n$. ∎

In particular for $n > 0$ the degree of $p$ is non-zero; so by Theorem 3.6.4 we conclude that $p : \mathbb{C} \to \mathbb{C}$ is surjective and hence has zero in its image.

**Theorem 3.6.12.** *(fundamental theorem of algebra)*
*Every polynomial,*

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 ,$$

*with complex coefficients has a complex root,* $p(z_0) = 0$*, for some* $z_0 \in \mathbb{C}$*.*

## Exercises for §3.6

1. Let $W$ be a subset of $\mathbb{R}^n$ and let $a(x)$, $b(x)$ and $c(x)$ be real-valued functions on $W$ of class $C^r$. Suppose that for every $x \in W$ the quadratic polynomial

$$(*) \qquad a(x)s^2 + b(x)s + c(x)$$

has two distinct real roots, $s_+(x)$ and $s_-(x)$, with $s_+(x) > s_-(x)$. Prove that $s_+$ and $s_-$ are functions of class $C^r$.

*Hint:* What *are* the roots of the quadratic polynomial: $as^2+bs+c$?

2. Show that the function, $\gamma(x)$, defined in figure 1 is a continuous mapping of $B^n$ onto $S^{2n-1}$. *Hint:* $\gamma(x)$ lies on the ray,

$$f(x) + s(x - f(x)) , \quad 0 \le s < \infty$$

and satisfies $\|\gamma(x)\| = 1$; so $\gamma(x)$ is equal to

$$f(x) + s_0(x - f(x))$$

where $s_0$ is a non-negative root of the quadratic polynomial

$$\|f(x) + s(x - f(x))\|^2 - 1 .$$

Argue from figure 1 that this polynomial has to have two distinct real roots.

3. Show that the Brouwer fixed point theorem isn't true if one replaces the closed unit ball by the open unit ball. *Hint:* Let $U$ be the open unit ball (i.e., the interior of $B^n$). Show that the map

$$h : U \to \mathbb{R}^n\,, \quad h(x) = \frac{x}{1 - \|x\|^2}$$

is a diffeomorphism of $U$ onto $\mathbb{R}^n$, and show that there are lots of mappings of $\mathbb{R}^n$ onto $\mathbb{R}^n$ which don't have fixed points.

4. Show that the fixed point in the Brouwer theorem doesn't have to be an interior point of $B^n$, i.e., show that it can lie on the boundary.

5. If we identify $\mathbb{C}$ with $\mathbb{R}^2$ via the mapping: $(x, y) \to z = x + iy$, we can think of a $\mathbb{C}$-linear mapping of $\mathbb{C}$ into itself, i.e., a mapping of the form

$$z \to cz\,, \quad c \in \mathbb{C}$$

as being an $\mathbb{R}$-linear mapping of $\mathbb{R}^2$ into itself. Show that the determinant of this mapping is $|c|^2$.

6. (a) Let $f : \mathbb{C} \to \mathbb{C}$ be the mapping, $f(z) = z^n$. Show that

$$Df(z) = nz^{n-1}\,.$$

*Hint:* Argue from first principles. Show that for $h \in \mathbb{C} = \mathbb{R}^2$

$$\frac{(z+h)^n - z^n - nz^{n-1}h}{|h|}$$

tends to zero as $|h| \to 0$.

(b) Conclude from the previous exercise that

$$\det Df(z) = n^2|z|^{2n-2}\,.$$

(c) Show that at every point $z \in \mathbb{C} - 0$, $f$ is orientation preserving.

(d) Show that every point, $w \in \mathbb{C} - 0$ is a regular value of $f$ and that

$$f^{-1}(w) = \{z_1, \ldots, z_n\}$$

with $\sigma_{z_i} = +1$.

(e) Conclude that the degree of $f$ is $n$.

7. Prove that the map, $f$, in exercise 6 has degree $n$ by deducing this directly from the definition of degree. *Some hints:*

(a) Show that in polar coordinates, $f$ is the map, $(r,\theta) \to (r^n, n\theta)$.

(b) Let $\omega$ be the two-form, $g(x^2+y^2)\,dx \wedge dy$, where $g(t)$ is a compactly supported $\mathcal{C}^\infty$ function of $t$. Show that in polar coordinates, $\omega = g(r^2) r\,dr \wedge d\theta$, and compute the degree of $f$ by computing the integrals of $\omega$ and $f^*\omega$, in polar coordinates and comparing them.

8. Let $U$ be an open subset of $\mathbb{R}^n$, $V$ an open subset of $\mathbb{R}^m$, $A$ an open subinterval of $\mathbb{R}$ containing 0 and 1, $f_i : U \to V$ $i = 0,1$, a pair of $\mathcal{C}^\infty$ mappings and $F : U \times A \to V$ a homotopy between $f_0$ and $f_1$.

(a) In §2.3, exercise 4 you proved that if $\mu$ is in $\Omega^k(V)$ and $d\mu = 0$, then

$$f_0^*\mu - f_1^*\mu = d\nu \tag{3.6.8}$$

where $\nu$ is the $(k-1)$-form, $Q\alpha$, in formula (**??**). Show (by careful inspection of the definition of $Q\alpha$) that if $F$ is a *proper* homotopy and $\mu \in \Omega_c^k(V)$ then $\nu \in \Omega_c^{k-1}(U)$.

(b) Suppose in particular that $U$ and $V$ are open subsets of $\mathbb{R}^n$ and $\mu$ is in $\Omega_c^n(V)$. Deduce from (3.6.8) that

$$\int f_0^*\mu = \int f_1^*\mu$$

and deduce directly from the definition of degree that degree is a proper homotopy invariant.

9. Let $U$ be an open connected subset of $\mathbb{R}^n$ and $f : U \to U$ a proper $\mathcal{C}^\infty$ map. Prove that if $f$ is equal to the identity on the complement of a compact set, $C$, then $f$ is proper and its degree is equal to 1. *Hints:*

(a) Show that for every subset, $A$, of $U$, $f^{-1}(A) \subseteq A \cup C$, and conclude from this that $f$ is proper.

(b) Let $C' = f(C)$. Use the recipe (1.6.1) to compute $\deg(f)$ with $q \in U - C'$.

10. Let $[a_{i,j}]$ be an $n \times n$ matrix and $A : \mathbb{R}^n \to \mathbb{R}^n$ the linear mapping associated with this matrix. Frobenius' theorem asserts: *If the $a_{i,j}$'s are non-negative then $A$ has a non-negative eigenvalue.* In

other words there exists a $v \in \mathbb{R}^n$ and a $\lambda \in \mathbb{R}$, $\lambda \geq 0$, such that $Av = \lambda v$. Deduce this linear algebra result from the Brouwer fixed point theorem. *Hints:*

(a) We can assume that $A$ is bijective, otherwise 0 is an eigenvalue. Let $S^{n-1}$ be the $(n-1)$-sphere, $|x| = 1$, and $f : S^{n-1} \to S^{n-1}$ the map,

$$f(x) = \frac{Ax}{\|Ax\|}\,.$$

Show that $f$ maps the set

$$Q = \{(x_1, \dots, x_n) \in S^{n-1}\,; \quad x_i \geq 0\}$$

into itself.

(b) It's easy to prove that $Q$ is homeomorphic to the unit ball $B^{n-1}$, i.e., that there exists a continuous map, $g : Q \to B^{n-1}$ which is invertible and has a continuous inverse. Without bothering to prove this fact deduce from it Frobenius' theorem.

## 3.7 Appendix: Sard's theorem

The version of Sard's theorem stated in §3.5 is a corollary of the following more general result.

**Theorem 3.7.1.** *Let $U$ be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ a $\mathcal{C}^\infty$ map. Then $\mathbb{R}^n - f(C_f)$ is dense in $\mathbb{R}^n$.*

Before undertaking to prove this we will make a few general comments about this result.

**Remark 3.7.2.** *If $\mathcal{O}_n$, $n = 1, 2,$ are open dense subsets of $\mathbb{R}^n$, the intersection*

$$\bigcap_n \mathcal{O}_n$$

*is dense in $\mathbb{R}^n$. (See [**?**], pg. 200 or exercise 4 below.)*

**Remark 3.7.3.** *If $A_n$, $n = 1, 2, \dots$ are a covering of $U$ by compact sets, $\mathcal{O}_n = \mathbb{R}^n - f(C_f \cap A_n)$ is open, so if we can prove that it's dense then by Remark 3.7.2 we will have proved Sard's theorem. Hence since we can always cover $U$ by a countable collection of closed cubes, it suffices to prove: for every closed cube, $A \subseteq U$, $\mathbb{R}^n - f(C_f \cap A)$ is dense in $\mathbb{R}^n$.*

**Remark 3.7.4.** *Let $g : W \to U$ be a diffeomorphism and let $h = f \circ g$. Then*

$$f(C_f) = h(C_h) \tag{3.7.1}$$

*so Sard's theorem for $g$ implies Sard's theorem for $f$.*

We will first prove Sard's theorem for the set of *super-critical* points of $f$, the set:

$$C_f^\sharp = \{p \in U\,, \quad Df(p) = 0\}\,. \tag{3.7.2}$$

**Proposition 3.7.5.** *Let $A \subseteq U$ be a closed cube. Then the open set $\mathbb{R}^n - f(A \cap C_f^\sharp)$ is a dense subset of $\mathbb{R}^n$.*

We'll deduce this from the lemma below.

**Lemma 3.7.6.** *Given $\epsilon > 0$ one can cover $f(A \cap C_f^\sharp)$ by a finite number of cubes of total volume less than $\epsilon$.*

*Proof.* Let the length of each of the sides of $A$ be $\ell$. Given $\delta > 0$ one can subdivide $A$ into $N^n$ cubes, each of volume, $\left(\frac{\ell}{N}\right)^n$, such that if $x$ and $y$ are points of any one of these subcubes

$$\left|\frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y)\right| < \delta\,. \tag{3.7.3}$$

Let $A_1, \dots, A_m$ be the cubes in this collection which intersect $C_f^\sharp$. Then for $z_0 \in A_i \cap C_f^\sharp$, $\frac{\partial f_i}{\partial x_j}(z_0) = 0$, so for $z \in A_i$

$$\left|\frac{\partial f_i}{\partial x_j}(z)\right| < \delta \tag{3.7.4}$$

by (3.7.3). If $x$ and $y$ are points of $A_i$ then by the mean value theorem there exists a point $z$ on the line segment joining $x$ to $y$ such that

$$f_i(x) - f_i(y) = \sum \frac{\partial f_i}{\partial x_j}(z)(x_j - y_j)$$

and hence by (3.7.4)

$$|f_i(x) - f_i(y)| \le \delta \sum |x_i - y_i| \le n\delta \frac{\ell}{N}\,. \tag{3.7.5}$$

Thus $f(C_f \cap A_i)$ is contained in a cube, $B_i$, of volume $\left(n\frac{\delta\ell}{N}\right)^n$, and $f(C_f \cap A)$ is contained in a union of cubes, $B_i$, of total volume less that

$$N^n n^n \frac{\delta^n \ell^n}{N^n} = n^n \delta^n \ell^n$$

so if w choose $\delta^n \ell^n < \epsilon$, we're done.

□

*Proof.* To prove Proposition 3.7.5 we have to show that for every point $p \in \mathbb{R}^n$ and neighborhood, $W$, of $p$, $W - f(C_f^\sharp \cap A)$ is non-empty. Suppose

$$W \subseteq f(C_f^\sharp \cap A)\,. \tag{3.7.6}$$

Without loss of generality we can assume $W$ is a cube of volume $\epsilon$, but the lemma tells us that $f(C_f^\sharp \cap A)$ can be covered by a finite number of cubes whose total volume is *less* than $\epsilon$, and hence by (3.7.6) $W$ can be covered by a finite number of cubes of total volume less than $\epsilon$, so its volume is less than $\epsilon$. This contradiction proves that the inclusion (3.7.6) can't hold.

□

To prove Theorem 3.7.1 let $U_{i,j}$ be the subset of $U$ where $\frac{\partial f_i}{\partial x_j} \neq 0$. Then

$$U = \bigcup U_{i,j} \cup C_f^\sharp\,,$$

so to prove the theorem it suffices to show that $\mathbb{R}^n - f(U_{i,j} \cap C_f)$ is dense in $\mathbb{R}^n$, i.e., it suffices to prove the theorem with $U$ replaced by $U_{i,j}$. Let $\sigma_i : \mathbb{R}^n \times \mathbb{R}^n$ be the involution which interchanges $x_1$ and $x_i$ and leaves the remaining $x_k$'s fixed. Letting $f_{\text{new}} = \sigma_i f_{\text{old}} \sigma_j$ and $U_{\text{new}} = \sigma_j U_{\text{old}}$, we have, for $f = f_{\text{new}}$ and $U = U_{\text{new}}$

$$\frac{\partial f_1}{\partial x_1}(p) \neq 0 \quad \text{for all } p \in U\} \tag{3.7.7}$$

so we're reduced to proving Theorem 3.7.1 for maps $f : U \to \mathbb{R}^n$ having the property (3.7.6). Let $g : U \to \mathbb{R}^n$ be defined by

$$g(x_1, \dots, x_n) = (f_1(x), x_2, \dots, x_n)\,. \tag{3.7.8}$$

Then

$$g^*x_1 = f^*x_1 \quad = \quad f_1(x_1, \dots, x_n) \tag{3.7.9}$$

and

$$\det(Dg) \quad = \quad \frac{\partial f_1}{\partial x_1} \neq 0\,. \tag{3.7.10}$$

Thus, by the inverse function theorem, $g$ is locally a diffeomorphism at every point, $p \in U$. This means that if $A$ is a compact subset of $U$ we can cover $A$ by a finite number of open subsets, $U_i \subset U$ such that $g$ maps $U_i$ diffeomorphically onto an open subset $W_i$ in $\mathbb{R}^n$. To conclude the proof of the theorem we'll show that $\mathbb{R}^n - f(C_f \cap U_i \cap A)$ is a dense subset of $\mathbb{R}^n$. Let $h : W_i \to \mathbb{R}^n$ be the map $h = f \circ g^{-1}$. To prove this assertion it suffices by Remark 3.7.4 to prove that the set

$$\mathbb{R}^n - h(C_h)$$

is dense in $\mathbb{R}^n$. This we will do by induction on $n$. First note that for $n = 1$, $C_f = C_f^\sharp$, so we've already proved Theorem 3.7.1 in dimension one. Now note that by (3.7.8), $h^*x_1 = x_1$, i.e., $h$ is a mapping of the form

$$h(x_1, \dots, x_n) = (x_1, h_2(x), \dots, h_n(x))\,. \tag{3.7.11}$$

Thus if we let $W_c$ be the set

$$\{(x_2, \dots, x_n) \in \mathbb{R}^{n-1}\,;\; (c, x_2, \dots, x_n) \in W_i\} \tag{3.7.12}$$

and let $h_c : W_c \to \mathbb{R}^{n-1}$ be the map

$$h_c(x_2, \dots, x_n) = (h_2(c, x_2, \dots, x_n), \dots, h_n(c, x_2, \dots, x_n))\,. \tag{3.7.13}$$

Then

$$\det(Dh_c)(x_2, \dots, x_n) = \det(Dh)(c, x_2, \dots, x_n) \tag{3.7.14}$$

and hence

$$(c, x) \in W_i \cap C_h \Leftrightarrow x \in C_{h_c}\,. \tag{3.7.15}$$

Now let $p_0 = (c, x_0)$ be a point in $\mathbb{R}^n$. We have to show that every neighborhood, $V$, of $p_0$ contains a point $p \in \mathbb{R}^n - h(C_h)$. Let $V_c \subseteq \mathbb{R}^{n-1}$ be the set of points, $x$, for which $(c, x) \in V$. By induction $V_c$ contains a point, $x \in \mathbb{R}^{n-1} - h_c(C_{h_c})$ and hence $p = (c, x)$ is in $V$ by definition and in $\mathbb{R}^n - h(C_n)$ by (3.7.15).

Q.E.D.

**Exercises for §3.7**

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be the map $f(x) = (x^2 - 1)^2$. What is the set of critical points of $f$? What is its image?

(b) Same questions for the map $f(x) = \sin x + x$.

(c) Same questions for the map

$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases}.$$

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map, i.e., a map of the form

$$f(x) = A(x) + x_0$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map. Prove Sard's theorem for $f$.

3. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function which is supported in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and has a maximum at the origin. Let $r_1, r_2, \ldots,$ be an enumeration of the rational numbers, and let $f : \mathbb{R} \to \mathbb{R}$ be the map

$$f(x) = \sum_{i=1}^{\infty} r_i \rho(x - i).$$

Show that $f$ is a $\mathcal{C}^\infty$ map and show that the image of $C_f$ is dense in $\mathbb{R}$. (The moral of this example: Sard's theorem says that the complement of $C_f$ is dense in $\mathbb{R}$, but $C_f$ can be dense as well.)

4. Prove the assertion made in Remark 3.7.2. *Hint:* You need to show that for every point $p \in \mathbb{R}^n$ and every neighborhood, $V$, of $p$, $\bigcap \mathcal{O}_n \cap V$ is non-empty. Construct, by induction, a family of closed balls, $B_k$, such that

(a) $B_k \subseteq V$

(b) $B_{k+1} \subseteq B_k$

(c) $B_k \subseteq \bigcap_{n \leq k} \mathcal{O}_n$

(d) radius $B_k < \frac{1}{k}$

and show that the intersection of the $B_k$'s is non-empty.

5. Verify (3.7.1).