§5.8 Čech cohomology

We pointed out at the end of § 5.3 that if a manifold, X, admits a finite good cover:

$$\mathbb{U} = \{U_i, i = 1, \cdots, d\}$$

then in principle, its cohomology can be read off from the intersection properties of the U_i's. In this section we'll explain in more detail how to do this. More explicitly we will show how to construct from the intersection properties of the U_i's a sequence of cohomology groups $H^k(\mathbb{U}, \mathbb{R})$, and we will prove that these "Čech cohomology groups" are isomorphic to the DeRham cohomology groups of X. (However we will leave for the reader a key ingredient in this proof: a diagram chasing argument that's similar to the diagram chasing arguments that were used to prove the Mayer-Vietoris theorem in §5.2 and the five lemma in §5.4.)

The details We will denote by N^k the set of all multi-indices

$$I = (i_o, \ldots, i_k), 1 \leq i_r \leq d \tag{5.8.1}$$

with the property that the intersection

$$U_I = U_{i_o} \cap \ldots \cap U_{i_k} \tag{5.8.2}$$

is non empty. (For example for $1 \leq i \leq d$ the multi-index, $I = (i_0, \ldots, i_k)$, $i_r = i$, is in N^k since for this I, U_I is just U_i). The disjoint union, N, of the N^k's is called the nerve of the cover, U, and N^k the k-skeleton of this nerve. Note that if we delete from the multi-index, (5.8.1) the index, i_r, i.e. replace I by

$$I_r = (i_0, \ldots, i_{r-1}, i_{r+1}, \ldots, i_k) \tag{5.8.3}$$

then I_r is in N^{k-1}.

We will now associate with N a cochain complex by defining $C^k(\mathbb{U}, \mathbb{R})$ to be the finite dimensional vector space consisting of all maps, $c : N^k \to \mathbb{R}$ and defining

$$\delta : C^{k-1}(\mathbb{U}, \mathbb{R}) \to C^k(\mathbb{U}, \mathbb{R})$$

to be the coboundary operator

$$\delta c(I) = \sum_{r=0}^{k} (-1)^r c(I_r) \tag{5.8.4}$$

We claim that this operator is a coboundary operator, i.e. $\delta(\delta C) = 0$. To see this we note that for $c \in C^{k-1}(\mathbb{U}, \mathbb{R})$ and $I \in N^k \delta(\delta c)(I)$ is equal by (5.8.4) to the sum, $\sum_{r=0}^{k+1}(-1)^r \delta c(I_r)$, and hence equal to the sum

$$\sum_{r=0}^{k+1}(-1)^r \left(\sum_{s<r}(-1)^s c(I_{r,s}) + \sum_{s>r}(-1)^{s-1} c(I_{r,s}) \right)$$

Thus the term $c(I_{r,s})$ occurs twice in this sum but occurs with opposite signs, and hence $\delta(\delta c) = 0$.
The complex

$$0 \to C^0(\mathbb{U}, \mathbb{R}) \xrightarrow{\delta} \cdots \to C^k(\mathbb{U}, \mathbb{R}) \to \tag{5.8.5}$$

is the Čech cochain complex of the cover $\mathbb{U}$ and the Čech cohomology groups that we mentioned above are the cohomology groups

$$H^k(\mathbb{U}, \mathbb{R}) = \frac{\text{Ker } \delta : C^k \to C^{k+1}}{\text{Im } \delta : C^{k-1} \to C^k} \tag{5.8.6}$$

of this complex.
The rest of this section will be devoted to proving

Theorem 5.8.1. *For $\mathbb{U}$ a finite good cover of X*

$$H^k(\mathbb{U}, \mathbb{R}) = H^k(X) \tag{5.8.7}$$

Remark

1. The definition of $H^k(\mathbb{U}, \mathbb{R})$ only involves the nerve N of this cover so this result is in effect a proof of the claim we made above: that the cohomology of X is in principle determined by the intersection properties of the U_i's

2. This result gives us another proof of an assertion we proved earlier: if X admits a finite good cover its cohomology groups are finite dimensional.

The proof of theorem 5.8.1 will involve an interesting DeRham theoretic generalization of the Čech cochain complex (5.8.5). Namely for k and l non-negative integers we will define a *Čech cochain of degree k with values in Ω^l* to be a map, c, which assigns to each $I \in N^k$ an l-form, $c(I)$, in $\Omega^l(U_I)$.

The set of these cochains forms an infinite dimensional vector space: $C^k(\mathbb{U}, \Omega^l)$, and we will show how to define for these cochains a coboundary operator

$$\delta : C^{k-1}(\mathbb{U}, \Omega^l) \to C^k(\mathbb{U}, \Omega^l) \tag{5.8.8}$$

similar to (5.8.4). To define this operator let I be an element of N^k and for $0 \le r \le k$ let $\gamma_r : \Omega^l(I_r) \to \Omega^l(I)$ be the restriction map.

$$\omega \in \Omega^l(U_{I_r}) \to \omega | U_I$$

(Note that the right hand side is well-defined since U_I is an open subset of U_{I_r}.) Thus given a Čech cochain $c \in C^{k-1}(\mathbb{U}, \Omega^l)$ we can, mimicking (5.8.4), define a Čech cochain, $\delta c \in C^k(\mathbb{U}, \Omega^l)$ by setting

$$\delta c(I) = \sum (-1)^r \gamma_r \, c(I_r) \tag{5.8.9}$$

In other words except for the γ_r's the definition of this cochain is formally identical with the definition (5.8.4). We will leave as an exercise the proof that

$$\delta : C^{k-1}(\mathbb{U}, \Omega^l) \to C^k(\mathbb{U}, \Omega^l)$$

is, like the δ defined by (5.8.4), a coboundary operator, i.e. $\delta(\delta c) = 0$.

Hint Except for keeping track of the γ_r's the proof of this is identical with the proof of this assertion for the coboundary operator (5.8.4). Thus to summarize we get for every l a Čech cochain complex

$$0 \to C^0(\mathbb{U}, \Omega^l) \xrightarrow{\delta} \cdots \to C^k(\mathbb{U}, \Omega^l) \to \tag{5.8.10}$$

and our next task in this section will be to compute the cohomology of this complex. We'll begin with the cohomology in the first slot, the kernel of the operator

$$\delta : C^0(\mathbb{U}, \Omega^l) \to C^\perp(\mathbb{U}, \Omega^l)$$

An element, c, of $C^0(\mathbb{U}, \Omega^l)$ is by definition a map which assigns to each $i \in \{1, \dots, d\}$ an element $c(u) = \omega_i$ of $\Omega^l(U_i)$.

Now let $I = (i_0, i_1)$ be an element of N^1. Then, by definition, $U_{i_o} \cap U_{i_1}$ is non-empty and

$$\delta c(I) = \gamma_{i_0} \omega_{i_1} - \gamma_{i_1} \omega_{i_0}$$

Thus $\delta c = 0$ iff

$$\omega_{i_0} | U_{i_0} \cap U_{i_1} = \omega_{i_1} | U_{i_0} \cap U_{i_1}$$

for all $I \in N^1$; and this is true if and only if the l-form, ω_i, is the restriction to U_i of a globally defined l-form, ω in $\Omega^l(X)$. Thus the kernel of the operator

$$\delta : C^0(\mathbb{U}, \Omega^l) \to C^{\perp}(\mathbb{U}, \Omega^l)$$

is $\Omega^l(X)$. Inserting this into the sequence above we get a new sequence:

$$0 \to \Omega^l(X) \to C^0(\mathbb{U}, \Omega^l) \to \cdots \to C^k(\mathbb{U}, \Omega^l) \to \tag{5.8.11}$$

and we will prove

Theorem 5.8.2. *The sequence (5.8.11) is exact.*

Proof. We've just proved that it is exact in its first slot. To prove that it's exact in its k-th slot we will construct a chain chomotopy operator:

$$Q : C^k(\mathbb{U}, \Omega^l) \to C^{k+1}(\mathbb{U}, \Omega^l)$$

with the property that

$$\delta Q c + Q \delta c = c \tag{5.8.12}$$

for all $c \in C^k(\mathbb{U}, \Omega^l)$. To define this operator we'll need a slight generalization of theorem 5.2.7.

Theorem 5.8.3. *There exist functions, $\phi_\alpha \in C^\infty(X), \alpha = 1, \cdots, N$ such that the support of ϕ_α is contained in U_α and $\sum \phi_\alpha = 1$.*

Proof. Let $\rho_i \in C_0^\infty(X)$, $i = 1, 2, 3, \ldots$, be a partition of unity subordinate to the cover $\mathbb{U}$ and let ϕ_1 be the sum of the ρ_i's with support in U_1. Deleting these ρ_i's from the sequence $\rho_1, \rho_2, \rho_3, \ldots$, we get a new sequence, $\rho'_1, \rho'_2, \rho'_3, \ldots$ Let ϕ_2 be the sum of the ρ'_i's in this sequence with support in U_2. Now delete these from the sequence $\rho'_1, \rho'_2, \rho'_3, \ldots$ and continue. □

To define the operator (5.8.12) let c be an element of $C^k(\mathbb{U}, \Omega^l)$ and I be an element of N^k. Then we define $Qc \in C^{k-1}(\mathbb{U}, \Omega^l)$ by defining its value at $I \in N^{k-1}$ to be:

$$Qc(I) = \sum_{u=1} \phi_i c(i, i_0, \ldots, i_{k-1})$$

where $I = (i_0, \ldots, i_{k-1})$ and the ith summand on the left is defined to be equal to zero when $U_i \cap U_I$ is empty and defined to be the product of the function ϕ_i and the l-form, $c(i, i_0, \ldots, i_{k-1})$, when $U_i \cap U_I$ is not empty. (Note that in this case the multi-index, $(i, i_0, \ldots, i_{k-1})$, is in N^k and so by definition, $c(i, i_0, \ldots, i_{k-1})$ is an element of $\Omega^l(U_i \cap U_I)$. However, since ϕ_i is supported on U_i we can extend the l-form, $\phi_i c(i, i_0, \ldots, i_{k-1})$, to U_I by setting it equal to zero outside U_i.)

To prove (5.8.12) we note that for $c \in C^k(\mathbb{U}, \Omega^l)$

$$Q\delta c(i_0, \ldots, i_k) = \sum \phi_i \delta c(i, i_0, \ldots, i_k),$$

so by (5.8.9) the right hand side is equal to

$$\sum_i \phi_i \gamma_i c(i_0, \ldots, i_k) + \sum_{i,s} (-1)^{s+1} \phi_i \gamma_{i_s} c(i, \ldots, \hat{i_s}, \ldots, i_k) \tag{5.8.13}$$

where $(i, \ldots, \hat{i_s}, \ldots, i_k)$ is the multi index $(i, i_0, \ldots, i_k)$ with i_s deleted. Similarly

$$\begin{aligned} \delta Q c(i_0, \ldots, i_k) &= \sum (-1)^s \gamma_{i_s} Q c(i_0, \ldots, \hat{i_s}, \ldots, i_k) \\ &= \sum_{i,s} (-1)^s \gamma_{i_s} \phi_i c(i, i_0, \ldots, \hat{i_s}, \ldots, i_k) \end{aligned}$$

However, this is the negative of the second summand in (5.8.13); so, by adding these two summands we get

$$(Q\delta c + \delta Q c)(i_o, \ldots, i_k) = \sum \phi_i \gamma_i c(i_0, \ldots, i_k)$$

and since $\sum \phi i = 1$, this identity reduces to

$$(Q\delta c + \delta Q c)(I) = c(I)$$

or, since I is an arbitrary element of N^k, $Q\delta c + \delta Q c = c$ □

Finally note that theorem 5.8.2 is an immediate consequence of the existence of this chain homotopy operator. Namely if $c \in C^k(\mathbb{U}, \Omega^l)$ is in the kernel of δ then

$$c = Q\delta c + \delta Q c = \delta Q c$$

so c is in the image of the previous δ.

To prove theorem 5.8.1, we let $C^{k,l} = C^k(\mathbb{U}, \Omega^l)$, and note that in addition to the exact sequence

$$0 \to \Omega^l \to C^{0,l} \xrightarrow{\delta} C^{1,l} \xrightarrow{\delta} \cdots \tag{5.8.14}$$

given by (5.8.11) one has another exact sequence

$$0 \to C^k(\mathbb{U}, \mathbb{R}) \to C^{k,0} \xrightarrow{d} C^{k,1} \xrightarrow{d} \cdots \tag{5.8.15}$$

where the d's are defined as follows: Given $c \in C^{k,l}$ and $I \in N^k$, $c(I)$ is an element of $\Omega^l(U_I)$ and $dc(I)$ an element of $\Omega^{l+1}(U_I)$, and we define $dc \in C^{k,l+1}$ to be the map

$$I \in N^k \to dc(I) \in \Omega^{l+1}(U_I)$$

It is clear from this definition that $d(dc) = 0$ so the sequence

$$C^{k,0} \xrightarrow{d} C^{k,1} \xrightarrow{d} \cdots \to C^{k,l} \xrightarrow{d} \cdots \tag{5.8.16}$$

is a complex and the exactness of this sequence follows from the fact that, since $\mathbb{U}$ is a good cover, the sequence

$$\Omega^0(U_I) \xrightarrow{d} \Omega^1(U_I) \to \cdots \to \Omega^l(U_I) \to \cdots$$

is exact. Moreover if c is in $C^{k,0}$ and $dc = 0$, then for every $I \in N^k$, $dc(I) = 0$ i.e. the zero form, $c(I)$, in $\Omega^0(U_I)$ is a constant. Hence the map, $I \in N^k \to c(I)$, is just a Čech cochain of the type we defined earlier, i.e. a map from N^k to $\mathbb{R}$. Therefore the kernel of the map, $d : C^{k,0} \to C^{k,1}$, is $C^0(\mathbb{U}, \mathbb{R})$; and , adjoining this term to the sequence (5.8.16) we get the exact sequence (5.8.15).

Lastly we note that the two operations we've defined above, the d operation, $d : C^{k,l} \to C^{k,l+1}$, and the δ : operation, $\delta : C^{k,l} \to C^{k+1,l}$, commute i.e. for $c \in C^{k,l}$

$$\delta dc = d\delta c \tag{5.8.17}$$

Proof. : For $I \in N^{k+1}$

$$\delta c(I) = \sum (-1)^r \gamma_r c(I_r)$$

so if $c(I_r) = \omega_{I_r} \in \Omega^l(U_{I_r})$

$$\delta c(I) = \sum (-1)^r \omega_{I_r} | U_I$$

and

$$\begin{aligned} d\delta(I) &= \sum (-1)^r d\omega_{I_r} | U_I \\ &= \delta dc(I) \end{aligned}$$

□

Putting these results together we get the commutative diagram in figure one (the $\check{C}^i$'s in the bottom row being the spaces of Čech cochains, $C^i(\mathbb{U}, \mathbb{R})$.

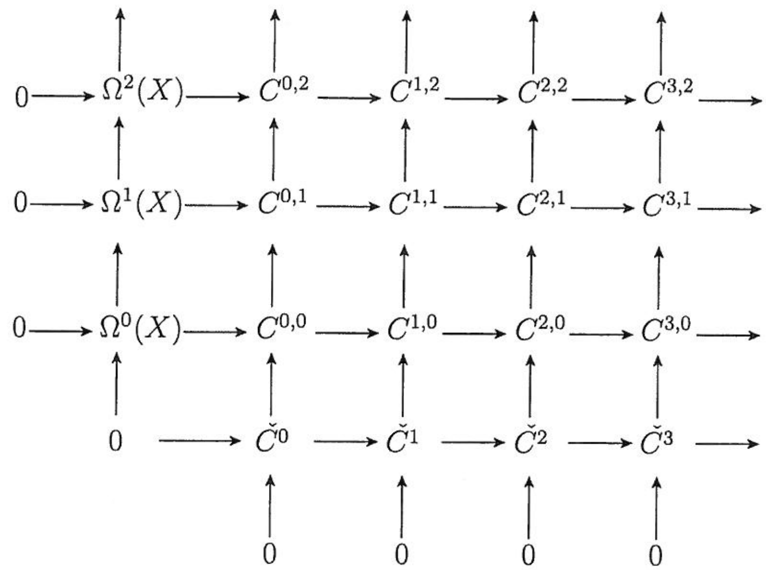
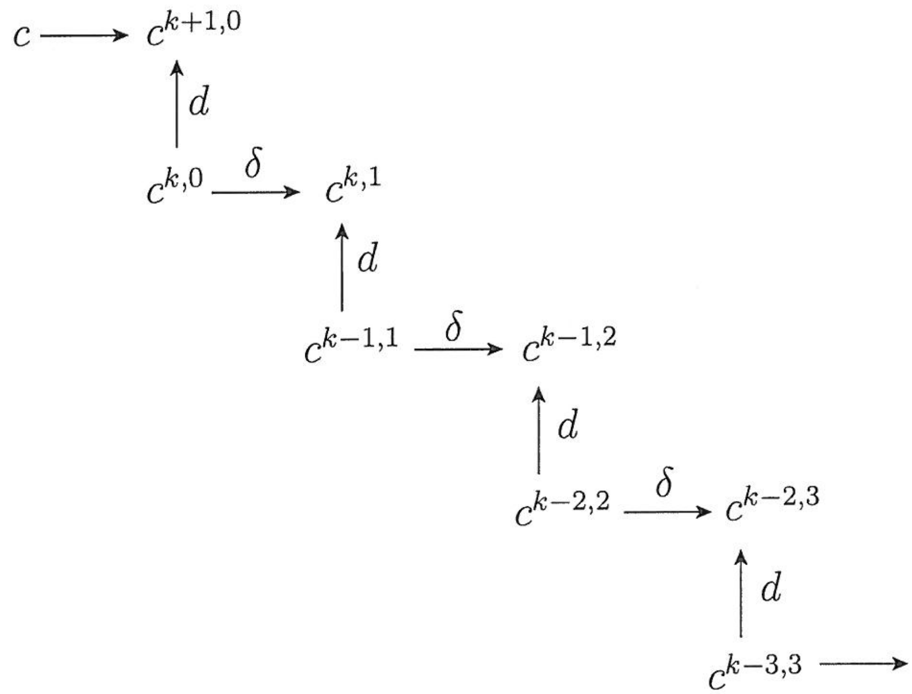


Figure 5.1:

In this diagram all columns are exact except for the extreme left hand column, which is the usual DeRham complex, and all rows are exact except for the bottom row which is the usual Čech cochain complex.

Exercise : Deduce from this diagram that $H^k(\mathbb{U}, \mathbb{R}) = H^k(X)$.
Hint : Let c be an element of $\Omega^{k+1}(X)$ satisfying $dc = 0$. Show that one can, by a sequence of chess moves (of which the first few stages are illustrated in figure 2) convert c into an element $\check{c}$ of $\check{C}^{k+1}$ satisfying $\delta\check{c} = 0$.



C
$C^{k+1,0}$
d
$C^{k,0}$
δ
$C^{k,1}$
d
$C^{k-1,1}$
δ
$C^{k-1,2}$
d
$C^{k-2,2}$
δ
$C^{k-2,3}$
d
$C^{k-3,3}$

Figure 5.2: