# Appendix A

## Bump Functions and Partitions of Unity

We will discuss in this section a number of "global to local" techniques in multi-variable calculus: techniques which enable one to reduce global problems on manifolds and large open subsets of $\mathbb{R}^n$ to local problems on small open subsets of these sets. Our starting point will be the function, $\rho$ on $\mathbb{R}$ defined by

$$\text{(A1)} \quad \rho(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0 \quad , & x \leq 0 \end{cases}$$

which is positive for $x$ positive, negative for $x$ negative and everywhere $C^\infty$. (We will sketch a proof of this last assertion at the end of this appendix). From $\rho$ one can construct a number of other interesting $C^\infty$ functions.

**Example 1** For $a > 0$ the function $\rho(x) + \rho(a - x)$ is positive for all $x$ so the quotient, $\rho_a(x)$, of $\rho(x)$ by this function is a well-defined $C^\infty$ function with the properties,

$$\text{(A2)} \quad \begin{aligned} &\rho_a(x) = 0 && \text{for } x \leq 0 \\ &0 \leq \rho_a(x) \leq 1 \\ &\rho_a(x) = 1 && \text{for } x \geq a \end{aligned}$$

**Example 2** Let $I$ be the open interval, $a < x < b$, and $\rho_I$ the function, $\rho_I(x) = \rho(x - a)$ $\rho(b - x)$. Then $\rho_I(x)$ is positive for $x \in I$ and zero on the complement of $I$.

**Example 3** More generally let $I_1, \ldots, I_n$ be open intervals and let $Q = I_1 \times \ldots \times I_n$ be the open rectangle in $\mathbb{R}^n$ having these intervals as sides. Then the function

$$\text{(A3)} \quad \rho_Q(x) = \rho_{I_1(x_1)} \cdots \rho_{I_n(x_n)}$$

is a $C^\infty$ function on $\mathbb{R}^n$ that's positive on $Q$ and zero on the complement of $Q$.

Using these functions we will prove

**Lemma *A*1** Let $C$ be a compact subset of $\mathbb{R}^n$ and $U$ an open set containing $C$: Then there exists a $C^\infty$ function $\phi$ on $\mathbb{R}^n$ such that

$$\text{(A4)} \quad \begin{aligned} &\phi(x) \geq 0 \text{ for all } x \in \mathbb{R}^n, \\ &\phi(x) > 0 \text{ on } C \\ &\phi \in C_0^\infty(U) \end{aligned}$$

**Proof** For each $p \in C$ let $Q_p$ be an open rectangle with $p \in Q_p$ and $\overline{Q}_p \subseteq U$. The $Q_p$'s cover $C_i$ so, by Heine-Basel there exists a finite subcover, $Q_i = Q_{p_i}$, $u = 1, \dots, N$. Now let $\phi = \sum p_{Qi}$

This result can be slightly strengthened: namely we claim

**Theorem** $A2$ There exists a function, $\psi \in C_0^\infty(U)$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on $C$.

**Proof** Let $\phi$ be as in lemma A.1 and let $a > 0$ be the greatest lower bound of the restriction of $\phi$ to $C$. Then if $\rho_a$ is the function in example 1 the function, $\rho_a \cdot \phi$ has the properties indicated above.

**Remark**: The function, $\psi$, in this theorem is an example of a "bump function". If one wants to study the behavior of a vector field, $v$, or a $k$-form, $w$, on the set, $C$, then by multiplying $v$ (or $w$) by $\psi$ one can, without loss or generality assume that $v$ (or $w$) is compactly supported on a small neighborhood of $C$.

Bump functions are one of the standard tools in calculus for converting global problems to local problems. Another such tool is *partitions of unity*:

Let $U$ be an open subset of $\mathbb{R}^n$ and $\mathbb{U} = \{U_\alpha,\ \alpha \in I\}$ a covering of $U$ by open subsets (indexed by the elements of the "index set", $I$). Then the partition of unity theorem asserts:

**Theorem** $A3$ There exists a sequence of functions, $\rho_i \in C_0^\infty(U)$ such that

(a) $\rho_i \geq 0$

(b) For every $i$ there is an $\alpha \in I$ with $\rho_i \in C_0^\infty(U_\alpha)$

(c) For every $p \in U$ there exists a neighborhood, $U_p$, of $p$ in $U$ and an $N_p > 0$ such that $\rho_i | U_p = 0$ for $u > N_p$.

(d) $\sum \rho_i = 1$

**Remark** Because of item (c) the sum in item (d) is well defined.

We will derive this result from a somewhat simpler set theoretical result:

**Theorem** $A4$ There exists a countable covering of $U$ by open rectangles, $Q_i$, such that

(a) $\overline{Q}_i \subseteq U$

(b) For each $i$ there is an $\alpha \in I$ with $\overline{Q}_i \subseteq U_\alpha$

(c) For every $p \in U$ there exists a neighborhood, $U_p$ of $p$ in $U$ and $N_p > 0$ such that $Q_i \cap U_p$ is empty for $i > N_p$

We first note that this theorem implies the preceding theorem. (To see this note that the functions $\rho_{Q_i}$ in example 3 above have all the properties indicated in theorem A3 except for property (d). Moreover since the $Q_i$'s are a covering of $U$ the sum

$$\sum \rho_{Q_i}$$

is everywhere positive. Thus we get a sequence of $\rho_i$'s satisfying (a) – (d) by taking $\rho_i$ to be the quotient of $\rho_{Q_i}$ by this sum.)

To prove theorem A4 let $d(x, U^c)$ be the distance of a point $x \in U$ to the complement, $U^c$, of $U$ in $\mathbb{R}^n$ and let $A_r$ be the compact subset of $U$ consisting of points, $x \in U$, satisfying $d(x, U^c) \geq 1/r$ and $|x| \leq r$. By Heine-Basel we can find, for each $r$, a collection of open rectangles, $Q_i$, $r$, $i = 1, \ldots, N_r$, such that $\overline{Q}_i, r$ is contained in Int $A_{r+1} - A_{r-2}$ and in some $U_\alpha$ and such that the $Q_i, r$'s are a covering of $A_r$ − Int $A_{r-1}$. Thus the $Q_i, r$'s have the properties listed in theorem A.4, and by relabelling: i.e. setting $Q_i = Q_{i,1}$ for $1 \leq i \leq N_i$, $Q_i = Q_{i-N_1}$, for $N_1 + 1 \leq i \leq N_1 + N_2$ etc. we get a sequence, $Q_1, Q_2, \ldots$ with the desired properties. We will next describe a couple of applications of theorem A4.

## Application 1 Improper integrals

Let $f: U \to \mathbb{R}$ be a continuous function. We will say that $f$ is integrable over $U$ if the infinite sum

$$\text{(A5)} \qquad \sum \int_U \rho_i \, |f| \, dx$$

converges and if so we will define the *improper integral* of $f$ over $U$ to be the sum

$$\text{(A6)} \qquad \sum \int_U \rho_i f dx$$

(Notice that each of the summands in this series is the integral of a compactly supported continuous function over $\mathbb{R}^n$ so the individual summands are well-defined. Also it's clear that if $f$ itself is a compactly supported function on $\mathbb{R}^n$, $(A5)$ is bounded by $\int_{\mathbb{R}^n} |f| \, dx$, so for every $f \in C_0(\mathbb{R}^n)$ the improper integral of $f$ over $U$ is well-defined.)

## Application 2 An extension theorem for $C^\infty$ maps

Let $X$ be a subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}^m$ a continuous map. We will say that $f$ is $C^\infty$ if, for every $p \in X$, there exists an open neighborhood, $U_p$, of $p$ in $\mathbb{R}^n$ and a $C^\infty$ map, $g_p : U_p \to \mathbb{R}^m$ such that $g_p = f$ on $U_p \cap X$.

## The extension theorem

If $f : X \to \mathbb{R}^m$ is $C^\infty$ there exists an open neighborhood, $U$, of $X$ in $\mathbb{R}^n$ and a $C^\infty$ map, $f : U \to \mathbb{R}^m$, such that $g = f$ on $X$.

**Proof** Let $U = \cup U_p$ and $\rho_i$, $i = 1, \ldots,$ a partition of unity with respect to the covering, $\mathcal{U} = \{U_p, p \in X\}$, of $U$. Then, for each $i$, there exists a $p$ such that the support of $\rho_i$ is contained in $U_p$. Let

$$g_i = \begin{cases} \rho_i g_p & \text{on} \quad U_p \\ 0 & \text{on} \quad U_p^c \end{cases}$$

Then $g = \sum g_i$ is well defined by item (c) of theorem A3 and the restriction of $g$ to $X$ :

$$\sum \rho_u f$$

is equal to $f$. □

# Exercises

**Exercise 1** Show that the function A1 is $C^\infty$. Hints

(a) From the Taylor series expansion

$$e^x = \sum \frac{x^k}{k!}$$

conclude that for $x > 0$

$$e^x \geq \frac{x^{k+n}}{(k+n)!}$$

(b) Replacing $x$ by $1/x$ conclude that for $x > 0$

$$e^{1/x} \geq \frac{1}{(n+k)!} \, \frac{1}{x^{n+k}}$$

(c) From this inequality conclude that for $x > 0$

$$e^{-\frac{1}{x}} \leq (n+k)! \; x^{n+k}$$

(d) Let $f_n(x)$ be the function

$$f_n(x) = \begin{cases} e^{-1/x} x^{-n}, & x > 0 \\ 0 \quad , & x \leq 0 \end{cases}$$

Conclude from (c) that for $x > 0$

$$f_n(x) \leq (n+k)! \, x^k \quad \text{for all } k$$

(e) Conclude that $f_n$ is $C^1$ differentiable

(f) Show that

(A7) $$\frac{d}{dx} f_n = f_{n+2} - n f_{n+1}$$

(g) Deduce by induction that the $f_n$'s are $C^1$ differentiable for all $r$ and $n$.

**Exercise 2** Show that the improper integral (A6) is well-defined independent of the choice of partition of unity. Hint Let $\rho'_j$, $j = 1, 2, \ldots$, be another partition of unity. Show that (A6) is equal to

$$\text{(A8)} \qquad \sum \int_U \rho_i \, \rho'_j \, f \, dx$$

APPENDIX B

# THE IMPLICIT FUNCTION THEOREM

Let $U$ be an open neighborhood of the origin in $\mathbb{R}^n$ and $f_i$, $i = 1, \dots, k$, $C^\infty$ functions on $U$ with the property, $f_1(0) = \cdots = f_k(0) = 0$. Our goal in this appendix is to prove the following "implicit function theorem".

**Theorem B.1.** *Suppose the matrix*

$$\left[\frac{\partial f_i}{\partial x_j}(0)\right], \quad 1 \le i, j \le k \tag{B1}$$

*is non-singular. Then there exists a neighborhood, $U_0$, of $0$ in $\mathbb{R}^n$ and a diffeomorphism, $g : (U_0, 0) \hookrightarrow (U, 0)$, of $U_0$ onto an open neighborhood of $0$ in $U$ such that*

$$g^* f_i = x_i\,, \quad i = 1, \dots, k \tag{B2}$$

*and*

$$g^* x_j = x_j\,, \quad j = k+1, \dots, n\,. \tag{B3}$$

**Remarks.**

1. Let $g(x) = (g_1(x), \dots, g_n(x))$. Then the second set of equations just say that

$$g_j(x_1, \dots, x_n) = x_j \tag{B4}$$

for $j = k+1, \dots, n$, so the first set of equations can be written more concretely in the form

$$f_i(g_1(x_1, \dots, x_n), \dots, g_k(x_1, \dots, x_n), x_{k+1}, \dots, x_n) = x_i \tag{B5}$$

for $i = 1, \dots, k$ .

2. Letting $y_i = g_i(x_1, \dots, x_n)$ the equations (B5) become

$$f_i(y_1, \dots, y_k\,, x_{k+1}, \dots, x_n) = x_i\,. \tag{B5$'$}$$

Hence what the implicit function theorem is saying is that, *modulo the assumption (B1) the system of equations (B5$'$) can be solved for the $y_i$'s in terms of the $x_i$'s.*

3. By a linear change of coordinates:

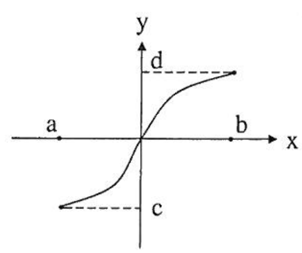
$$x_i \to \sum_{r=1}^{k} a_{i,r} x_r \,, \quad i = 1, \dots, k$$

we can arrange without loss of generality for the matrix (B1) to be the identity matrix, i.e.,

$$\frac{\partial f_i}{\partial x_j}(0) = \delta_{ij} \,, \quad 1 \le i, j \le k \,. \tag{B6}$$

Our proof of this theorem will be by induction on $k$.

First, let's suppose that $k = 1$ and, for the moment, also suppose that $n = 1$. Then the theorem above is just the inverse function theorem of freshman calculus. Namely if $f(0) = 0$ and $df/dx(0) = 1$, there exists an interval, $a \le x \le b$, about the origin on which $df/dx$ is greater than $1/2$, so $f$ is strictly increasing on this interval and its graph looks like the curve in the figure below




with $c = f(a) \le -\frac{1}{2}a$ and $d = f(b) \ge \frac{1}{2}b$.

The graph of the inverse function, $g : [c, d] \to [a, b]$ is obtained from this graph by just rotating it through ninety degrees, i.e., making the $y$-axis the horizontal axis and the $x$-axis the vertical axis. (From the picture it's clear that $y = f(x) \Leftrightarrow x = g(y) \Leftrightarrow f(g(y)) = y$.)

Most elementary text books regard this intuitive argument as being an adequate proof of the inverse function theorem; however, a slightly beefed-up version of this proof (which is completely rigorous) can be found in Spivak, *Calculus*, Chapter 12. Moreover, as Spivak points out, if the slope of the curve in the figure above at the point $(x, y)$ is equal to $\lambda$ the slope of the rotated curve at $(y, x)$ is $1/\lambda$, so from this proof one concludes that if $y = f(x)$

$$\frac{dg}{dy}(y) = \left(\frac{df}{dx}(x)\right)^{-1} = \left(\frac{df}{dx}(g(y))\right)^{-1} \,. \tag{B7}$$

Since $f$ is a continuous function, its graph is a continuous curve and, therefore, since the graph of $g$ is the same curve rotated by ninety degrees, $g$ is also a continuous functions. Hence by (B7), $g$ is also a $C^1$ function and hence by (B7), $g$ is a $C^2$ function and hence .... In other words $g$ is in $\mathcal{C}^\infty([c, d])$.

Let's now prove that the implicit function theorem with $k = 1$ and $n$ arbitrary. This amounts to showing that if the function, $f$, in the discussion above depends on the parameters, $x_2, \dots x_n$ in a $\mathcal{C}^\infty$ fashion, then so does its inverse, $g$. More explicitly let's suppose $f = (x_1, \dots, x_n)$ is a $\mathcal{C}^\infty$ function on a subset of $\mathbb{R}^n$ of the form $[a, b] \times V$ where $V$ is a compact, convex neighborhood of the origin in $\mathbb{R}^n$ and satisfies $\partial f / \partial x_1 \geq \frac{1}{2}$ on this set. Then by the argument above there exists a function, $g = g(y, x_2, \dots, x_n)$ defined on the set, $[a/2, b/2] \times V$, and having the property

(B8) $$f(x_1, x_2, \dots, x_n) = y \Leftrightarrow g(y, x_2, \dots, x_n) = x_1 \,.$$

Moreover by (B7) $g$ is a $\mathcal{C}^\infty$ function of $y$ and

(B9) $$\frac{\partial g}{\partial y}(y, x_2, \dots, x_n) = \frac{\partial f}{\partial x_1}(x_1, x_2, \dots, x_n)^{-1}$$

at $x_1 = g(y)$. In particular, since $\frac{\partial f}{\partial x_1} \geq \frac{1}{2}$

(B10) $$0 < \frac{\partial g}{\partial y} < 2$$

and hence

(B11) $$|g(y', x_2, \dots, x_n) - g(y, x_2, \dots, x_n)| < 2|y' - y|$$

for points, $y$ and $y'$, or the interval, $[a/2, b/2]$.

The $k = 1$ case of Theorem B1 is almost implied by (B8) and (B9) except that we must still show that $g$ is a $\mathcal{C}^\infty$ function, not just of $y$, but of all the variables, $y, x_2, \dots, x_n$, and this we'll do by quoting another theorem from freshman calculus (this time a theorem from the second semester of freshman calculus).

*The mean value theorem in n-variables.* Let $U$ be a convex open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a $\mathcal{C}^\infty$ function. Then for $a, b \in U$ there exists a point, $c$, on the line interval joining $a$ to $b$ such that

$$f(b) - f(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(c)(b_i - a_i) \,.$$

*Proof.* Apply the $1-D$ mean value theorem to the function, $h(t) = f((1-t)a+tb)$.

□

Let's now show that the function, $g$, in (B8) is a $\mathcal{C}^\infty$ function of the variables, $y$ and $x_2$. To simplify our notation we'll suppress the dependence of $f$ on $x_3, \dots, x_n$ and write $f$ as $f(x_1, x_2)$ and $g$ as $g(y, x_2)$. For $h \in (-\epsilon, \epsilon)$, $\epsilon$ small, we have

$$y = f(g(y, x_2+h), x_2+h) = f(g(y, x_2), x_2),$$

and, hence, setting $x_1' = g(y, x_2+h)$, $x_1 = g(y, x_2)$ and $x_2' = x_2 + h$, we get from the mean value theorem

$$\begin{aligned} 0 &= f(x_1', x_2') - f(x_1, x_2) \\ &= \frac{\partial f}{\partial x_1}(c)(x_1' - x) + \frac{\partial f}{\partial x_2}(c)(x_2' - x_2) \end{aligned}$$

and therefore

$$\text{(B12)} \qquad g(y, x_2+h) - g(y, x_2) = \left(-\frac{\partial f}{\partial x_1}(c)\right)^{-1} \frac{\partial f}{\partial x_2}(c)\, h$$

for some $c$ on the line segment joining $(x_1, x_2)$ to $(x_1', x_2')$. Letting $h$ tend to zero we can draw from this identity a number of conclusions:

I. Since $f$ is a $\mathcal{C}^\infty$ function on the compact set, $[a, b] \times V$, its derivatives are bounded on this set, so the right hand side of B12 tends to zero as $h$ tends to zero, i.e., for fixed $y$, $g$ is continuous as a function of $x_2$.

II. Now divide the right hand side of (B12) by $h$ and let $h$ tend to zero. Since $g$ is continuous in $x_2$, the quotient of (B12) by $h$ tends to its value at $(x_1, x_2)$. Hence for fixed $y$ $f$ is differentiable as a function of $x_2$ and

$$\text{(B13)} \qquad \frac{\partial g}{\partial x_2}(y, x_2) = -\left(\frac{\partial f}{\partial x_1}\right)^{-1} (x_1, x_2)\, \frac{\partial f}{\partial x_2}(x_1, x_2)$$

where $x_1 = g(y, x_2)$.

III. Moreover, by the inequality (B11) and the triangle inequality

$$\begin{aligned}|g(y',x_2') - g(y,x_2)| &\le |g(y',x_2') - g(y,x_2')| + |g(y,x_2') - g(y,x_2)| \\ &\le 2|y'-y| + |g(y,x_2') - g(y,x_2)|\,,\end{aligned}$$

hence $g$ is continuous as a function of $y$ *and* $x_2$.

IV. Hence by (B9) and (B13), $g$ is a $C^1$ function of $y$ and $x_2$, and hence ....

In other words $g$ is a $\mathcal{C}^\infty$ function of $y$ and $x_2$. This argument works, more or less verbatim, for more than two $x_i$'s and proves that $g$ is a $\mathcal{C}^\infty$ function of $y, x_2, \ldots, x_n$. Thus with $f = f_1$ and $g = g_1$ Theorem B.1 is proved in the special case $k = 1$.

We'll now prove Theorem B.1 for arbitrary $k$ by induction on $k$. By induction we can assume that there exists a neighborhood $U_0'$, of the origin in $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ diffeomorphism $\varphi : (U_0', 0) \hookrightarrow (U, 0)$ such that

$$\varphi^* f_i = x_i \tag{B14}$$

for $2 \le i \le k$ and

$$\varphi^* x_j = x_j \tag{B15}$$

for $j = 1$ and $k + 1 \le j \le n$. Moreover, by (B6)

$$\left(\frac{\partial}{\partial x_1}\varphi^* f_1\right)(0) = \sum_{i=1}^{k} \frac{\partial f_i}{\partial x_1}(0)\frac{\partial}{\partial x_i}\varphi_i(0) = \frac{\partial \varphi_1}{\partial x_1} = 1$$

since $\varphi_1 = \varphi^* x_1 = x_1$. Therefore we can apply Theorem B.1, with $k = 1$, to the function, $\varphi^* f_1$, to conclude that there exists a neighborhood, $U_0$, of the origin in $\mathbb{R}^n$ and a diffeomorphism, $\psi : (U_0, 0) \to (U_0', 0)$ such that $\psi^*\varphi^* f_1 = x_1$ and $\psi^*\varphi^* x_i = x_i$ for $1 \le i \le n$. Thus by (B14) $\psi^*\varphi^* f_i = \psi^* x_i = x_i$ for $2 \le i \le k$, and by (B15) $\psi^*\varphi^* x_j = \psi^* x_j = x_j$ for $k + 1 \le j \le n$. Hence if we let $g = \varphi \circ \psi$ we see that:

$$g^* f_i = (\varphi \circ \psi)^* f_i = \psi^*\varphi^* f_i = x_i$$

for $i \le i \le k$ and

$$g^* x_j = (\varphi \circ \psi)^* x_j = \psi^*\varphi^* x_j = x_j$$

for $k+1 \leq j \leq n$. □

We'll derive a number of subsidiary results from Theorem B.1. The first of these is the $n$-dimensional version of the inverse function theorem:

**Theorem B.2.** *Let $U$ and $V$ be open subsets of $\mathbb{R}^n$ and $\varphi : (U,p) \to (V,q)$ a $C^\infty$ map. Suppose that the derivative of $\varphi$ at $p$*

$$D\varphi(p) : \mathbb{R}^n \to \mathbb{R}^n$$

*is bijective. Then $\varphi$ maps a neighborhood of $p$ in $U$ diffeomorphically onto a neighborhood of $q$ in $V$.*

*Proof.* Composing fore and aft by translations we can assume that $p = q = 0$. Let $\varphi = (f_1, \dots, f_n)$. Then the condition that $D\varphi(0)$ be bijective is the condition that the matrix (B.1) be non-singular. Hence, by Theorem B.1, there exists a neighborhood, $V_0$ of 0 in $V$, a neighborhood, $U_0$, of 0 in $U_0$ and a diffeomorphism, $g : V_0 \to U_0$, such that

$$g^* f_i = x_i$$

for $i = 1, \dots, n$. However, these equations simply say that $g$ is the inverse of $\varphi$, and hence that $\varphi$ is a diffeomorphism.

□

A second result which we'll extract from Theorem B.1 is the *canonical submersion theorem.*

**Theorem B.3.** *Let $U$ be an open subset of $\mathbb{R}^n$ and $\varphi : (U,p) \to (\mathbb{R}^k, 0)$ a $C^\infty$ map. Suppose $\varphi$ is a* submersion *at $p$, i.e., suppose its derivative*

$$D\varphi(p) : \mathbb{R}^n \to \mathbb{R}^k$$

*is onto. Then there exists a neighborhood, $\mathcal{O}$, of $p$ in $U$, a neighborhood, $U_0$, of the origin in $\mathbb{R}^n$ and a diffeomorphism, $g : (U_0, 0) \to (\mathcal{O}, p)$ such that the map $\varphi \circ g : (U_0, 0) \to (\mathbb{R}^n, 0)$ is the restriction to $U_0$ of the* canonical submersion*:*

(B16) $$\pi : \mathbb{R}^n \to \mathbb{R}^k\,,\ \pi(x_1, \dots, x_n) = (x_1, \dots, x_k)\,.$$

*Proof.* Let $\varphi = (f_1, \dots, f_k)$. Composing $\varphi$ with a translation we can assume $p = 0$ and by a permutation of the variables $x_1, \dots, x_n$ we can assume that the matrix (B1) is non-singular. By Theorem B.1

we conclude that there exists a diffeomorphism. $g : (U_0, 0) \hookrightarrow (U, p)$ with the properties $g^* f_i = x_i$, $i = 1, \ldots, k$, and hence,

$$\varphi \circ g(x_1, \ldots, x_n) = (x_1, \ldots, x_k)\,.$$

□

As a third application of Theorem B.1 we'll prove a theorem which is similar in spirit to Theorem B.3, the *canonical immersion theorem.*

**Theorem B.4.** *Let $U$ be an open neighborhood of the origin in $\mathbb{R}^k$ and $\varphi : (U, 0) \to (\mathbb{R}^n, p)$ a $C^\infty$ map. Suppose that the derivative of $\varphi$ at 0*

$$D\varphi(0) : \mathbb{R}^k \to \mathbb{R}^n$$

*is injective. Then there exists a neighborhood, $U_0$, of 0 in $U$, a neighborhood , $V$, of $p$ in $\mathbb{R}^n$ and a diffeomorphism.*

$$\psi : V \to U_0 \times \mathbb{R}^{n-k}$$

*such that the map, $\psi \circ \varphi : U_0 \to U_0 \times \mathbb{R}^{n-k}$ is the restriction to $U_0$ of the canonical immersion*

(B17) $\iota : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^{n-k}, \quad \iota(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)\,.$

*Proof.* Let $\varphi = (f_1, \ldots, f_n)$. By a permutation of the $f_i$'s we can arrange that the matrix

$$\left[\frac{\partial f_i}{\partial x_j}(0)\right], \quad 1 \le i, j \le k$$

is non-singular and by composing $\varphi$ with a translation we can arrange that $p = 0$. Hence by Theorem B.2 the map

$$\chi : (U, 0) \to (\mathbb{R}^k, 0) \quad x \to (f_1(x), \ldots, f_k(x))$$

maps a neighborhood, $U_0$, of 0 diffeomorphically onto a neighborhood, $V_0$, of 0 in $\mathbb{R}^k$. Let $\psi : (V_0, 0) \to (U_0, 0)$ be its inverse and let

$$\gamma : V_0 \times \mathbb{R}^{n-k} \to U_0 \times \mathbb{R}^{n-k}$$

be the map

$$\gamma(x_1, \ldots, x_n) = (\psi(x_1, \ldots, x_k), x_{k+1}, \ldots, x_n)\,.$$

Then

$$\text{(B18)}\qquad \begin{aligned}\gamma\circ\varphi(x_1,\dots,x_k) &= \gamma(\chi(x_1,\dots,x_k), f_{k+1}(x),\dots,f_n(x))\\ &= (x_1,\dots,x_k, f_{k+1}(x),\dots,f_n(x))\,.\end{aligned}$$

Now note that the map

$$h : U_0\times\mathbb{R}^{n-k}\to U_0\times\mathbb{R}^{n-k}$$

defined by

$$h(x_1,\dots,x_n) = (x_1,\dots,x_k\,,\; x_{k+1}-f_{k+1}(x),\dots,x_n-f_n(x))$$

is a diffeomorphism (Why? What is its inverse?) and, by (B18),

$$h\circ\gamma\circ\varphi(x_1,\dots,x_k) = (x_1,\dots,x_k,0,\dots,0)\,,$$

i.e., $(h\circ\gamma)\circ\varphi = i$. Thus we can take the $V$ in Theorem B.4 to be $V_0\times\mathbb{R}^{n-k}$ and the $\psi$ to be $h\circ\gamma$.

□

**Remarks.**

The canonical submersion and immersion theorems can be succinctly summarized as saying that every submersion "looks locally like the canonical submersion" and every immersion "looks locally like the canonical immersion".