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NOTE ON COFIBRATIONS

ARNE STRØM

In the first section of this note it is proved that cofibrations are homeomorphisms, and a characterization of closed cofibrations is given. The second section contains the proof of a homotopy lifting-extension theorem generalizing a result on relative CW-complexes.

All functions considered will be continuous.

1.

Definition. A (Hurewicz) *fibration* is a map $p:E \to B$ with the property that for any map $f:X \to E$ and any homotopy $F:X \times I \to B$ such that $F(x,0)=pf(x)$ for all $x \in X$, there exists a homotopy $\overline{F}:X \times I \to E$ such that $p\overline{F}=F$ and $\overline{F}(x,0)=f(x)$ for all $x \in X$.

A *cofibration* is a map $j:A \to X$ such that for any map $f:X \to Y$ and any homotopy $\overline{F}:A \times I \to Y$ such that $\overline{F}(a,0)=fj(a)$ for all $a \in A$, there exists a homotopy $F:X \times I \to Y$ such that $F(j \times 1_I)=\overline{F}$ and $F(x,0)=f(x)$ for all $x \in X$.

If A is a subspace of a space X such that the inclusion map $A \subset X$ is a cofibration, the pair (X,A) is called a *cofibered pair* or is said to possess the *absolute homotopy extension property* (AHEP). A necessary condition for (X,A) to be a cofibered pair is the existence of a retraction

$$r:\quad X \times I \to (X \times 0) \cup (A \times I)\,.$$

If A is closed, this condition is also sufficient.

The following theorem shows that the only cofibrations are cofibered pairs.

Theorem 1. *If $j:A \to X$ is a cofibration, then j is a homeomorphism $A \approx j(A)$.*

Proof. Let $j:A \to X$ be a cofibration and consider the mapping cylinder Z of j, that is, the quotient space of the topological sum $(X \times 0) \vee (A \times I)$ obtained by identifying $(a,0) \in A \times I$ with $(j(a),0) \in X \times 0$

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for each $a \in A$. Denote by q the quotient map $(X \times 0) \vee (A \times I) \to Z$. There is a continuous map $i: Z \to X \times I$ defined by

$$\begin{aligned} iq(x,0) &= (x,0), \qquad && x \in X\,, \\ iq(a,t) &= (j(a),t), && a \in A,\ t \in I\,. \end{aligned}$$

Define maps $f: X \to Z$ and $\bar{F}: A \times I \to Z$ by

$$f(x) = q(x,0), \qquad \bar{F}(a,t) = q(a,t)\,.$$

Because j is a cofibration there exists a map $F: X \times I \to Z$ such that $F(j(a),t) = q(a,t)$ and $F(x,0) = q(x,0)$ for all $a \in A$, $t \in I$, and $x \in X$. Then $Fi = 1_Z$, and i is, therefore, a homeomorphism of Z onto $i(Z) = (X \times 0) \cup (j(A) \times I)$. Also, $q | A \times 1$ is a homeomorphism of $A \times 1$ onto $q(A \times 1)$, and consequently $iq | A \times 1$ is a homeomorphism of $A \times 1$ onto $iq(A \times 1) = j(A) \times 1$.

Next we shall prove a theorem which generalizes 3.1 of [1].

THEOREM 2. *Let A be a closed subspace of a topological space X. Then (X,A) is a cofibered pair if and only if there exist*

(i) *a neighborhood U of A which is deformable in X to A* rel A (*that is, there exists a homotopy $H: U \times I \to X$ such that $H(x,0) = x$, $H(a,t) = a$, and $H(x,1) \in A$ for all $x \in U$, $a \in A$, $t \in I$*), *and*

(ii) *a continuous function $\varphi: X \to I$ such that $A = \varphi^{-1}(0)$ and $\varphi(x) = 1$ for all $x \in X - U$.*

PROOF. Suppose that (X,A) is a cofibered pair. Then there exists a retraction

$$r:\ X \times I \to (X \times 0) \cup (A \times I)\,,$$

and U, H and φ may be chosen as follows:

$$\begin{aligned} U &= \{x \in X \mid pr_1 r(x,1) \in A\}\,, \\ H &= pr_1 r \mid U \times I\,, \\ \varphi(x) &= \sup_{t \in I} |t - pr_2 r(x,t)|\,, \end{aligned}$$

pr_1 and pr_2 denoting projections on X and I, respectively.

Conversely, suppose that U, H and φ are given and satisfy the conditions of the theorem. Since A is closed it suffices to prove the existence of a retraction

$$r:\ X \times I \to (X \times 0) \cup (A \times I)\,.$$

The required retraction may be constructed as follows.

(i) If $\varphi(x) = 1$, let $r(x,t) = (x,0)$.

(ii) If $\frac{1}{2} \leqq \varphi(x) < 1$, let $r(x,t) = \big(H\big(x, 2(1-\varphi(x))t\big), 0\big)$.

(iii) If $0 < \varphi(x) \leqq \frac{1}{2}$ and $0 \leqq t \leqq 2\varphi(x)$, let
$$r(x,t) = \big(H\big(x, t/(2\varphi(x))\big), 0\big)\ .$$

(iv) If $0 < \varphi(x) \leqq \frac{1}{2}$ and $2\varphi(x) \leqq t \leqq 1$, let
$$r(x,t) = \big(H(x,1),\, t - 2\varphi(x)\big)\ .$$

(v) If $\varphi(x) = 0$, let $r(x,t) = (x,t)$.

(This construction is that of [2].) The proof of continuity is straightforward and will be omitted.

2.

It was remarked in Section 1 that if (X,A) is a cofibered pair, then $(X \times 0) \cup (A \times I)$ is a retract of $X \times I$. In fact, we have the following stronger result.

Lemma. *If (X,A) is a cofibered pair, then $(X \times 0) \cup (A \times I)$ is a strong deformation retract of $X \times I$.*

Proof. Let $i\colon (X \times 0) \cup (A \times I) \subset X \times I$ be the inclusion map, and let
$$r\colon\ X \times I \to (X \times 0) \cup (A \times I)$$
be a retraction. A homotopy
$$D\colon\ ir \simeq 1_{X \times I}\ \ \mathrm{rel}(X \times 0) \cup (A \times I)$$
is given by
$$D(x,t,t') = \big(pr_1 r(x, (1-t')t),\, (1-t')pr_2 r(x,t) + t't\big)\ .$$

Theorem 3. *Suppose that $p\colon E \to B$ is a fibration, that A is a strong deformation retract of X, and that there exists a map $\varphi\colon X \to I$ such that $A = \varphi^{-1}(0)$. Then any commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f''"] \arrow[d, phantom, "i\,\cap"] & E \arrow[d, "p"] \\ X \arrow[r, "f'"'] & B \end{tikzcd}$$

may be filled in with a map $f\colon X \to E$ such that $pf = f'$ and $fi = f''$. f is unique up to homotopy rel$A$.

Proof. By hypothesis there exists a retraction $r\colon X \to A$ and a homotopy
$$D\colon\ ir \simeq 1_X\ \ \mathrm{rel}A\ .$$

If $f\colon X \to E$ is such that $fi = f''$, then $f \simeq fir = f''r \operatorname{rel} A$, which proves the last assertion of the theorem. Define $\bar{D}\colon X \times I \to X$ by

$$\bar{D}(x,t) = \begin{cases} D\big(x, t/(\varphi(x))\big), & t < \varphi(x)\,, \\ D(x,1), & t \geqq \varphi(x)\,. \end{cases}$$

D is easily shown to be continuous. Because p is a fibration there exists a homotopy $\bar{F}\colon X \times I \to E$ such that $p\bar{F} = f'\bar{D}$ and $\bar{F}(x,0) = f''r(x)$ for each $x \in X$. f is given by $f(x) = \bar{F}(x, \varphi(x))$.

We are now in a position to prove

THEOREM 4. *Suppose that $p\colon E \to B$ is a fibration, that (X,A) is a cofibered pair, and that A is closed. Then any commutative diagram*

$$\begin{tikzcd} (X \times 0) \cup (A \times I) \arrow[r, "f"] \arrow[d, phantom, "\cap"] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"'] & B \end{tikzcd}$$

may be filled in with a homotopy $\bar{F}\colon X \times I \to E$ such that $p\bar{F} = F$ and $\bar{F} \mid (X \times 0) \cup (A \times I) = f$.

PROOF. According to the Lemma $(X \times 0) \cup (A \times I)$ is a strong deformation retract of $X \times I$, and by Theorem 2 there exists a function $\psi\colon X \to I$ such that $A = \psi^{-1}(0)$. Define $\varphi\colon X \times I \to I$ by $\varphi(x,t) = t\psi(x)$. Then $(X \times 0) \cup (A \times I) = \varphi^{-1}(0)$, and the theorem follows from Theorem 3.

The condition that A be closed is not very restrictive. For instance, A will always be closed if X is Hausdorff. Not all cofibrations are closed, however. The most trivial example of a non-closed cofibration is the pair (X,a) where X is the two-point space $\{a,b\}$ with the trivial topology.

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