

### 18.906: On the Serre spectral sequence

Recall the construction, due to Andreas Dress, Zur Spectralsequenz von Faserungen, *Inv. math.* 3 (1967) 172–178: Let  $p : E \rightarrow B$  be any map of spaces, and define the bisimplicial set  $\text{Sin}(p)$  with

$$\text{Sin}_{s,t}(p) = \left\{ \begin{array}{ccc} \Delta^s \times \Delta^t & \cdots \cdots \rightarrow & E \\ \downarrow \text{pr}_1 & & \downarrow p \\ \Delta^s & \cdots \cdots \rightarrow & B \end{array} \right\}$$

Let  $R$  be a commutative ring, and write  $S(p; R)$  or just  $S(p)$  for the double complex associated to  $R\text{Sin}(p)$ .

We first study the spectral sequence associated to the transpose of this double complex—that is, we compute the homology with respect to the differential in the  $s$  direction first. We will need to extend the functoriality a bit.

Let  $p' : E' \rightarrow B'$  be another map, and define a simplicial set  $\text{Sin}(p', p)$  with  $t$ -simplices

$$\text{Sin}_s(p', p) = \left\{ \begin{array}{ccc} \Delta^s \times E' & \cdots \cdots \rightarrow & E \\ \downarrow p' & & \downarrow p \\ \Delta^s \times B' & \cdots \cdots \rightarrow & B \end{array} \right\}$$

One should think of this as the simplicial set of maps from  $p'$  to  $p$ . Thus on the one hand  $\text{Sin}(* \downarrow *, p) = \text{Sin}(E)$ , the usual singular simplicial set of  $E$ ; while on the other hand  $\text{Sin}_{\cdot,t}(p) = \text{Sin}(\Delta^t \downarrow *, p)$ .

Let  $S(p', p; R)$  denote the chain complex associated with  $R\Delta(p', p)$ .

Let  $\Delta_{ss}^s$  denote the simplicial  $s$ -simplex—the simplicial set represented by  $[s] \in \mathbf{\Delta}$ . There is a natural map of simplicial sets

$$\text{Sin}(p' \times 1_{\Delta^1}, p) \times \Delta_{ss}^1 \rightarrow \text{Sin}(p', p)$$

which sends  $(f : \Delta^s \times E \times \Delta^1 \rightarrow E), (\sigma : \Delta_{ss}^s \rightarrow \Delta_{ss}^1)$  to

$$(v, e) \mapsto f(v, e, \sigma(v))$$

From it one can construct a homotopy from  $\text{in}_0^* : S(p' \times \Delta^1, p) \rightarrow S(p', p)$  to  $\text{in}_1^* : S(p' \times \Delta^1, p) \rightarrow S(p', p)$ . This serves as a universal example to show that if  $f_0 : p'' \rightarrow p'$  and  $f_1 : p'' \rightarrow p'$  are homotopic then they induce homotopic maps  $f_i^* : S(p', p) \rightarrow S(p'', p)$ . Thus if  $f : p'' \rightarrow p'$  is a homotopy equivalence then it induces a chain equivalence  $f^* : S(p', p) \rightarrow S(p'', p)$ .

For example, the homotopy equivalences

$$\begin{array}{ccc} \Delta^s & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

induce chain equivalences (for each  $t$ )

$$S(E) = S(* \downarrow *, p) \rightarrow S(\Delta^t \downarrow *, p) = S_{.,t}(p)$$

Taking homology, then, we find that  $H_{s,t}(S(p); d_h) = H_s(E)$  for each  $t$ . The next differential is the alternating sum of the face maps, all of which are the identity, so the differential is alternately 0 and 1, and  $E_{s,t}^2 = H_s(E)$  for  $t = 0$  and 0 for  $t \neq 0$ . This proves:

**Proposition.** The homology of the total complex of  $\text{Sin}(p)$  is  $H(E)$ .

Now we look at the other spectral sequence.

It's not too hard to see that the vertical homology in bidegree  $(s, t)$  is

$$E_{s,t}^1 = C_s(B; R_t^p)$$

Here  $R_t^p : (\Delta/B)^{\text{op}} \rightarrow R - \text{mod}$  by

$$R_t^p(\sigma : \Delta^s \rightarrow B) = H_t S^p(\sigma)$$

where, given  $p : E \rightarrow B \leftarrow X : q$ ,  $S^p(q)$  is the chain complex of  $R$  modules generated by the simplicial set  $\text{Sin}^p(q)$  with  $t$ -simplices given by

$$\text{Sin}^p(q)_t = \left\{ \begin{array}{ccc} X \times \Delta^t & \cdots \cdots \rightarrow & E \\ \downarrow \text{pr}_1 & & \downarrow p \\ X & \xrightarrow{q} & B \end{array} \right\}$$

It's also not hard to see that  $d^1$  in the spectral sequence, which is induced by the horizontal differential, is the differential in the chain complex  $C(B; R_t^p)$  defining the homology of  $B$  with generalized coefficients in  $R_t^p$ . If  $T : (\Delta/B)^{\text{op}} \rightarrow R - \text{mod}$  is such a generalized coefficient system,  $C(B; T)$  is defined as the chain complex associated to the simplicial  $R$  module with  $s$ -simplicies given by

$$C_s(B; T) = \bigoplus_{\sigma: \Delta^s \rightarrow B} T(\sigma)$$

A simplicial operator  $\phi : [m] \rightarrow [n]$  induces  $\phi^* : C_n(B; T) \rightarrow C_m(B; T)$  by requiring that for every  $\sigma : \Delta^n \rightarrow B$ , the diagram

$$\begin{array}{ccc} C_n(B; T) & \xleftarrow{\text{in}_\sigma} & T(\sigma) \\ \vdots \downarrow \phi^* & & \downarrow T(\phi) \\ C_m(B; T) & \xleftarrow{\text{in}_{\sigma \circ \phi}} & T(\sigma \circ \phi) \end{array}$$

commutes.

**Proposition.** If  $p$  is a fibration, then  $R_t^p$  is the generalized coefficient system associated to the local coefficient system  $\Pi_1(B)^{\text{op}} \rightarrow R - \text{mod}$  sending  $b \in B$  to  $H_t(E_b)$  and a path  $\omega : I \rightarrow B$  with  $\omega(1) = a$  and  $\omega(0) = b$  to the map  $H_t(E_a) \rightarrow H_t(E_b)$  it induces by lifting in the diagram

$$\begin{array}{ccc} E_a \times 0 & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ E_a \times I & \xrightarrow{\omega \circ \text{pr}_2} & B \end{array}$$

A local coefficient system  $M : \Pi_1(B)^{\text{op}} \rightarrow R - \text{mod}$  induces a generalized coefficient system  $T_M$  by defining

$$T_M(\sigma) = M_{\sigma(0)}$$

and for  $\phi : [m] \rightarrow [n]$ , defining

$$T_M(\sigma \circ \phi) \rightarrow T_M(\sigma)$$

i.e.

$$M_{\sigma(\phi(0))} \rightarrow M_{\sigma(0)}$$

as the map induced by the path  $\sigma \circ e_{0,\phi(0)}$ , where  $e_{i,j}$  is the affine path in the standard simplex joining the  $i$ th vertex to the  $j$ th vertex. Functoriality follows from the fact that if  $i \leq j \leq k$  then  $e_{i,j} * e_{j,k}$  and  $e_{i,k}$  are homotopic rel endpoints as paths in the simplex, and from the fact that  $M$  sends path-homotopic paths to the same map.

The proposition follows from:

**Proposition.** Let  $p : E \rightarrow B$  be a fibration and  $A$  a CW complex. Let  $q : A \times I \rightarrow B$  and let  $i_0 : A \rightarrow A \times I$  be the inclusion  $a \mapsto (a, 0)$ . The induced map  $S^p(q) \rightarrow S^p(q \circ i_0)$  is then a chain homotopy equivalence.

This is proven inductively, using the homotopy lifting property valid for  $p$ .