18.906: Problem Set III

Due Monday, April 13, in class.

10. Show that the derived couple of an exact couple is indeed exact.

11. Perhaps I should have termed the convergence condition I discussed in class on March 13 "first quadrant" rather than "regular." Here's another such condition, with just as much claim to the term "regular." Maybe I'll call it "finite width."

Let s_0 be a nonnegative integer. A filtration $\{F_sC\}$ on a chain complex is "of width s_0" if $H(F_sC) = 0$ for all $s < 0$ and $H(F_sC) \to H(C)$ is an isomorphism for all $s \geq s_0$.

For example it might be that $F_{-1}C = 0$ and $F_{s_0}C = C$.

In **(a)**–**(c)** assume that $\{F_sC\}$ is of width s_0.

(a) Show that $A^r_{s,t} = 0$ for $s < 0$ for all $r \geq 1$.

(b) Show that the natural epimorphism $A^r_{s,t} \to F_sH_{s+t}(C)$ is an isomrophism for $r \geq s_0 - s$.

(c) Conclude that for $r > s$ there is a natural map $E^r_{s,t} \to E^0_sH_{s+t}(C)$ which is an isomorphism if also $r > s_0 - s$. Observe that there is an epimorphism $E^r_{s,t} \to E^{r+1}_{s,t}$ if $r > s$, and a monomorphism $E^{r+1}_{s,t} \to E^r_{s,t}$ if $r > s_0 - s$, which are inverse isomorphims if both conditions hold.

(d) Let $C' \to C \to C''$ be a short exact sequence of chain complexes. Define a filtration on C by $F_sC = 0$ for $s < 0$, $F_0C = C'$ (or the image of C' in C), and $F_sC = C$ for $s \geq 1$. Describe the exact couple and spectral sequence associated to this filtration, and relate it to the homology long exact sequence associated to the original short exact sequence of chain complexes.

12. Let p be a prime number and let C be a chain complex such that for each n the group C_n has no p-torsion. The *p-adic filtration* is defined by $F_sC = C$ for $s \geq 0$, $F_sC = p^{-s}C$ for $s \leq 0$.

(a) Establish natural isomorphisms

$$E^0_{s,t} \cong \begin{cases} C_{s+t}/pC_{s+t} & \text{if} \quad s \leq 0 \\ 0 & \text{if} \quad s > 0 \end{cases}$$

and $A^1_{s,t} \cong H_{s+t}(C)$. What is E^1?

(b) Let k be a non-negative integer, and consider the very simple chain complex C with $C_0 = \mathbb{Z} = C_1$, $C_n = 0$ otherwise, and $d = p^k : C_1 \to C_0$. What is $H(C)$? Describe the exact couples (A^r, E^r) for $r \geq 0$, and draw a picture of (E^r, d^r) when $d_r \neq 0$. Check that in this case there is an isomorphism $E^r_{s,t} \cong E^0_{s,t} H(C)$ for r large enough.

(c) Determine $A^r_{s,t}$ and $i : A^r_{s,t} \to A^r_{s+1,t-1}$ in general, in terms of $H(C)$. What is the filtration on $H(C)$ associated to this exact couple?

(d) Give an example of a chain complex C in which C_n is a free abelian group for each n, $E^1 = 0$, but $H(C) \neq 0$. You can even take $C_n = 0$ for $n \neq 0, 1$.

13. This problem concerns the Serre spectral sequence of a fibration $p : E \to B$, which is endowed with a natural isomorphism

$$E^2_{s,t} \cong H_s(B; \mathcal{F}_t)$$

where $\mathcal{F}_t$ is the local coefficient system given by the functor $\Pi_1(B) \to (R-$mod) sending b to $H_t(p^{-1}(b))$ and sending a path class from a to b in B to the effect in homology of the homotopy equivalence $p^{-1}(a) \to p^{-1}(b)$ it induces.

(a) Assume that F is path connected. Show that the composition

$$H_s(B) \cong E^2_{s,0} \supseteq E^3_{s,0} \supseteq \cdots \supseteq E^\infty_{s,t} \cong H_s(E)/F_{s-1}H_s(E) \twoheadleftarrow H_s(E)$$

is the map induced by $p : E \to B$.

(b) Assume that B is path connected and $\mathcal{F}_t$ is trivial. Pick $b \in B$ and let $i : F = p^{-1}(b) \hookrightarrow E$ be the inclusion of the fiber over b. Show that the composition

$$H_t(F) \cong E^2_{0,t} \subseteq E^3_{0,t} \subseteq \cdots \subseteq E^\infty_{0,t} \cong F_0 H_t(E) \rightarrowtail H_t(E)$$

is the map induced by $i : F \to E$.

14. Verify that the claim made in class on April 1 is no joke: in the Leray-Serre-Dress spectral sequence of a map $p : E \to B$, there is a natural isomorphism

$$E^2_{s,t} \cong H_s(B; R^p_t)$$

1 April 2009