18.906: Problem Set II

Due Monday, March 16, in class.

5. Show that the class of fibrations enjoys the following closure properties.

(a) If $p : E \to B$ is a fibration and $f : B' \to B$ is any continuous map, then the pullback $B' \times_B E \to B'$ is a fibration.

(b) If $p_i : E_i \to B_i$ is a fibration for each i in a set I, then $\prod p_i : \prod E_i \to \prod B_i$ is a fibration.

(c) If Y is a locally compact Hausdorff space and $p : E \to B$ is a fibration, then $p^Y : E^Y \to B^Y$ is a fibration.

(d) If $p : E \to C$ and $q : C \to B$ are both fibrations, then $qp : E \to B$ is a fibration.

(e) If $p : E \to B$ is a fibration then so is any retract of p. That is: suppose that we have a commutative diagram

$$\begin{array}{ccccc} E' & \xrightarrow{i} & E & \xrightarrow{r} & E' \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B' & \xrightarrow{j} & B & \xrightarrow{s} & B' \end{array}$$

in which $ri = 1_{E'}$ and $sj = 1_{B'}$. Then p' is again a fibration.

(f) If $p_i : E_i \to B_i$ is a fibration for each i in a set I, is the coproduct $\coprod p_i : \coprod E_i \to \coprod B_i$ necessarily a fibration?

6. Suppose A and B are pointed compact Hausdorff spaces and let X be any pointed space. Show that the natural identification of $(X \wedge A) \wedge B$ and $X \wedge (A \wedge B)$ as quotients of $X \times A \times B$ is a homeomorphism; that $A \wedge B$ is a compact Hausdorff space; and that the evident bijection between $(X_*^B)_*^A$ and $X_*^{B \wedge A}$ is a pointed homeomorphism.

7. Let $f : X \to Y \leftarrow Z : g$. Write $\Delta : Y \to Y \times Y$ for the diagonal map.

Recall that the homotopy pullback was defined as the pullback in

$$\begin{array}{ccc} X \times_Y^h Z & \longrightarrow & P_g \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Repair the asymmetry of this definition by showing that there is also a pullback diagram of the form

$$\begin{array}{ccc} X \times_Y^h Z & \longrightarrow & P_\Delta \\ \downarrow & & \downarrow \\ X \times Z & \xrightarrow{f \times g} & Y \times Y \end{array}$$

8. Let $p : E \to B$ be a fibration. Construct a contravariant functor

$$\Pi_1(B) \to \mathrm{Ho}(\mathbf{Top})$$

from the fundamental groupoid of B to the homotopy category of spaces, which assigns to $a \in B$ the fiber $E_a = p^{-1}(a)$ (and verify that it is a functor).

9. Let $p : E \to B$ be a fibration, pick $* \in E$, and write $*$ also for its image in B. Show that if A has a nondegenerate basepoint (also denoted by $*$!) then any diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & E \\ {\scriptstyle i_0}\downarrow & \overset{\overline{G}}{\nearrow} & \downarrow{\scriptstyle p} \\ A \times I & \xrightarrow{G} & B \end{array}$$

in which $G(*, t) = *$ for all t and $g(*) = *$ completes via a map $\overline{G}$ such that $\overline{G}(*, t) = *$ for all t.

4 Mar 2009