18.906: Problem Set I

Due Monday, February 23, in class.

1. (a) Let $\mathrm{Gr}_k(\mathbf{R}^n)$ be the Grassmann manifold of k-planes in $\mathbf{R}^n$, and let E_n^k be the tautologous vector bundle over it, $E_n^k = \{(V, x) : x \in V\} \subseteq \mathrm{Gr}_k(\mathbf{R}^n) \times \mathbf{R}^n$. Show that this is indeed a vector bundle over $\mathrm{Gr}_k(\mathbf{R}^n)$.

(b) Let $\mathrm{Aff}_k(\mathbf{R}^n)$ be the space of all k-dimensional affine subspaces of $\mathbf{R}^n$—that is, all translates of k-dimensional vector subspaces of $\mathbf{R}^n$. Put a topology on this space, and identify it with a space related to those you considered in **(a)**.

(c) Find an expression relating the tangent bundle of the Grassmannian $\mathrm{Gr}_k(\mathbf{R}^n)$ to the tautologous bundle E_n^k and its orthogonal complement.

Note. It may (or may not) be useful to regard $\mathrm{Gr}_k(\mathbf{R}^n)$ as the subspace of the vector space of symmetric $n \times n$ matrices which are idempotent and of rank k.

2. (a) I asserted that if k-plane bundle $E \downarrow B$ has a basis of sections then it is trivial. It's true that there is a continuous fiberwise linear bijection $B \times \mathbf{R}^n \to E$ over B, but why should this map be a homeomorphism?

(b) Let E and F be vector bundles over B and suppose that $f : F \to E$ is a continuous map covering the identity map of B which is a fiber-wise linear embedding. Is $f(F)$ a sub-bundle of E? (That is, is there an open cover of B such that for each open U in it the pair $(E, F)|_U$ is fiberwise-linearly homeomorphic to $U \times (\mathbf{R}^{k(U)}, \mathbf{R}^{l(U)})$ over U?)

3. Let B be a compact Hausdorff space, and let $C(B)$ be the commutative ring of continuous real valued functions on B. The set $\Gamma(E)$ of sections of a vector bundle $E \downarrow B$ forms a module over $C(B)$ in an evident way. (How?) Show that it is a finitely generated projective module. Conversely, given a finitely generated projective module over $C(B)$, construct a vector bundle having it as its module of sections.

4. Show that the infinite Stiefel variety $V_k(\mathbf{R}^\infty)$ is contractible. (For example the infinite-dimensional sphere S^∞ is contractible.) Here are some tools you might want to build in order to prove this.
(a) There is a family of orthogonal operators on $\mathbf{R}^\infty$, say α_t, such that α_0 is the identity operator and $\alpha_1(v_1, v_2, \ldots) = (0, v_1, 0, v_2, \ldots)$.
(b) For any pair of unit vectors $v, w \in \mathbf{R}^\infty$ which don't sum to 0, there is a family of orthogonal operators $\beta_t^{v,w}$ such that $\beta_0^{v,w} = \mathrm{id}$, $\beta_1^{v,w}(v) = w$, and $\beta_t^{v,w} u = u$ whenever u is orthogonal to both v and w.

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