18.906: Cofibration sequences

Proposition. If $f : A \to B$ is a cofibration, then the collapse map $C_f \to B/A$ is a homotopy equivalence.

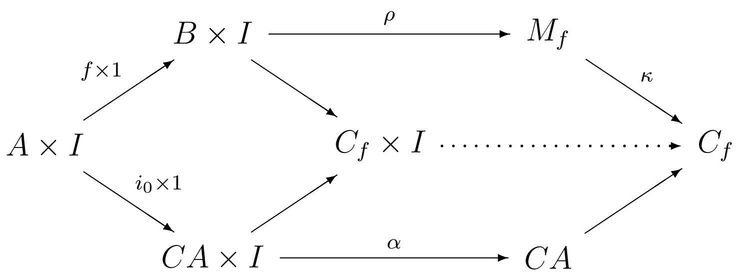
Proof. Let $\rho : B \times I \to M_f = B \cup_A A \times I$ be a retraction, so that if $i : M_f \to B \times I$ is the inclusion then $\rho \circ i = 1_{M_f}$. I am including A into $A \times I$ by $a \mapsto (a, 0)$.

Let $j : B \to B \times I$ by $b \mapsto (b, 1)$ and let $\kappa : M_f \to C_f$ be the map collapsing CA to a point.

Then $\kappa \circ \rho \circ j$ collapses A to a point, and so factors through a map $\overline{\rho} : B/A \to C_f$. This will be our homotopy inverse.

The homotopy $\pi \circ \overline{\rho} \sim 1_{B/A}$ is the factorization of $B \times I \xrightarrow{\kappa\rho} C_f \xrightarrow{\pi} B/A$ through $(B \times I)/(A \times I) = (B/A) \times I$ (here using the fact that I is a compact Hausdorff space).

The homotopy $\overline{\rho} \circ \pi \sim 1_{C_f}$ is obtained by gluing in the following diagram.



where $\alpha([a, s], t) = [a, \max\{s, t\}]$. $\square$