18.906: Universally closed maps

Definition. A continuous map $i : A \to B$ is *closed* if the image of every closed subset of A is closed in B. The continuous map i is *universally closed* if for every continuous map $Z \to B$, the pull-back $Z \times_B A \to Z$ is closed.

Universally closed maps are closed under pull-backs.

Lemma. A closed embedding is universally closed.

Proof. Let $i : A \to B$ be a closed embedding and let $f : Z \to B$ be continuous. The pull-back $Z \times_B A$ is topologized as a subspace of $Z \times A$, but I claim that this is the same as the subspace topology obtained from the composite monomorphism $Z \times_B A \to Z \times A \to Z$. The result then follows from the standard fact that preimages of closed sets under continuous maps are closed.

Let $g : W \to Z \times_B A$ and suppose that the composite to Z is continuous. Then the composite on to B is also continuous. That composite factors through A, and since A is topologized as a subspace of B the map $W \to A$ is continuous. Thus the map $W \to Z \times A$ is continuous, so $W \to Z \times_B A$ is continuous. □

Lemma. For A compact and any W, the projection map $\mathrm{pr}_1 : W \times A \to W$ is closed.

Proof. This is standard; e.g. Bredon, p. 22. □

Proposition. Any continuous map from a compact space to a Hausdorff space is universally closed.

Proof. Let A be compact, B be Hausdorff, and $i : A \to B$ be continuous, and let $f : Z \to B$ be continuous. The pull-back $Z \times_B A \to Z$ factors as the top row in

$$\begin{tikzcd}
Z \times_B A \arrow[r] \arrow[d] & Z \times A \arrow[r] \arrow[d] & Z \\
B \arrow[r, "\Delta"] & B \times B &
\end{tikzcd}$$

The second map along the top is closed by the second lemma above. The square is a pull-back square. The diagonal of B is a closed embedding since B is Hausdorff, so the pull-back is also a closed embedding. Composites of closed maps are closed so $Z \times_B A \to Z$ is closed. □

Lemma. Let A be a compact Hausdorff space and $i : A \to B$ a closed surjection. Then B is Hausdorff.

Proof. See Massey, p. 252. □

Corollary. A closed surjection from a compact Hausdorff space is universally closed. □

Lemma. Let B and C be compact Hausdorff spaces, A a closed subspace of C, and $f : A \to B$ a continuous map. Then the adjunction space $B \cup_f C$ is a compact Hausdorff space.

Proof. Spanier p. 56, or Bredon, p 43. □

Thus under these conditions the quotient map

$$B \coprod C \to B \cup_f C$$

is universally closed. In particular we can take $B = *$ and use the fact that $C \to * \coprod C$ is a closed inclusion to see that if A is a nonempty closed subspace of a compact Hausdorff space C then the quotient space C/A is Hausdorff and the quotient map $C \to C/A$ is universally closed.

A locally compact Hausdorff space A embeds as a subspace into its one-point compactification A^+. The one-point compactification is functorial on proper maps—continuous maps such that the preimage of any compact subspace is compact—and given a proper map $i : A \to B$ the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\
A^+ \arrow[r, "i^+"] & B^+
\end{tikzcd}$$

is a pull-back. So we have the

Corollary. Any proper map between locally compact Hausdorff spaces is universally closed. □

Another relevant fact:

Lemma. Let B be a locally compact Hausdorff space and let $X \to Y$ be any quotient map. Then $X \times B \to Y \times B$ is a quotient map.

Proof. Bredon p. 43.