Description

These problems are related to the material covered in Lectures 6-9. Your solutions are to be written up in latex and submitted as a pdf-file with a filename of the form SurnamePset2.pdf via e-mail to drew@math.mit.edu by noon on the date due. Collaboration is permitted/encouraged, but you must identify your collaborators, and any references consulted. If there are none, write “Sources consulted: none” at the top of your problem set. The first person to spot each non-trivial typo/error in any of the problem sets or lecture notes will receive 1-5 points of extra credit.

Instructions: In some of the problems below we reference terms and definitions from earlier problems, so be sure to read through the whole problem set first. You may use results proved in an earlier problem in any later problem (even if you don’t choose to solve the earlier problem). Pick a combination of problems to solve that sum to 100 and write up your answers in latex, then complete the survey problem 5.

As in [1], we use $H := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ to denote the complex upper half plane.

Problem 1. Geodesic spaces (50 points)

Let $(X,d)$ be a metric space. Recall that a path $\gamma : [a,b] \to X$ is a continuous function whose domain is a real interval, which we may also denote $\gamma : x \to y$, where $x = \gamma(a)$ and $y = \gamma(b)$. The length of a path is

$$l(\gamma) := \sup_{a=t_0 < \cdots < t_n = b} \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)).$$

where the supremum is taken over all finite subdivisions of $[a,b]$. A metric space is a length space if for all $x,y \in X$ we have $d(x,y) = \inf_{\gamma : x \to y} l(\gamma)$.

(a) For a path $\gamma : [a,b] \to H$ we define the hyperbolic length

$$\ell(\gamma) := \int_a^b \frac{|\gamma'(t)|}{\text{Im}(\gamma(t))} dt.$$

Show that $d(x,y) := \inf_{\gamma : x \to y} \ell(\gamma)$ is a metric on $H$ that makes $(H,d)$ a length space with $l(\gamma) = \ell(\gamma)$. We call $d$ the hyperbolic metric on $H$.

**Definition.** A geodesic path $\gamma : [a,b] \to X$ is a distance preserving path; that is, we have $d(\gamma(s), \gamma(t)) = |s-t|$ for all $s,t \in [a,b]$. A geodesic line (resp. geodesic ray) is a distance preserving continuous map $\mathbb{R} \to X$ (resp. $\mathbb{R}_{\geq 0} \to X$).\(^1\) By a geodesic path/line/ray in $X$ we mean the image of a geodesic path/line/ray. A geodesic space is a metric space in which every pair of points can be connected by a geodesic path. A geodesic space is straight if every geodesic path lies in a geodesic line, and uniquely geodesic if every $x,y \in X$ are connected by a unique geodesic path. A subset of a geodesic space is convex if it contains all geodesic paths between points in the set.

\(^1\)This definition differs from the shortest path definition used in class, and from the two definitions most commonly used in Riemannian geometry (it is slightly stronger).
(b) Show that $H$ is a geodesic space that is a uniquely geodesic space under both the Euclidean metric $|z_1 - z_2|$ and the hyperbolic metric defined above, but that it is only straight for the hyperbolic metric, and automorphisms of $H$ (as a Riemann surface) are isometries only for the hyperbolic metric.

(c) Show that the topologies on $H$ induced by the Euclidean metric and the hyperbolic metric coincide.

(d) Show that Euclidean and hyperbolic open balls are convex under both the Euclidean and hyperbolic metric, and give examples of subsets of $H$ that are convex under one metric but not the other (in both directions).

(e) Show that under the hyperbolic metric (i) for any distinct $z_1, z_2 \in H$ the set $C_{z_1, z_2} := \{z \in H : d(z, z_1) = d(z, z_2)\}$ is a geodesic line $\gamma : \mathbb{R} \to H$, (ii) $H - C_{z_1, z_2}$ is the union of two disjoint convex sets that can meaningfully be called left and right half-planes (relative to $\gamma$), (iii) every geodesic line arises in this way. Does this apply to all Riemann surfaces that are straight uniquely geodesic spaces for a given metric?

Problem 2. Topological group actions (50 points)

Recall that locally compact groups are (by definition) Hausdorff and that whenever a topological group acts on a topological space the action is understood to be continuous, that is, the map $(g, x) \mapsto gx$ is a morphism of topological spaces $G \times X \to X$. A group acting on a metric space $(X, d)$ is said to act via isometries if $x \mapsto gx$ is an isometry for every $g \in G$, in other words, $d(x, y) = d(gx, gy)$ for all $x, y \in X$ and $g \in G$.

A morphism of topological spaces is proper if it is a closed map and inverse images of compact sets are compact. A topological group $G$ acts properly on a topological space $X$ if the action map $(g, x) \mapsto (x, gx)$ defines a proper morphism $G \times X \to X \times X$.

(a) Let $G$ be a locally compact group acting on a locally compact metric space $(X, d)$ via isometries. Prove that the following are equivalent

1. All $x, y \in X$ have open neighborhoods $U, V$ with $\{g \in G : gU \cap V \neq \emptyset\}$ finite.
2. Every $x \in X$ has an open neighborhood $U$ with $\{g \in G : gU \cap U \neq \emptyset\}$ finite.
3. For all compact $A, B \subseteq X$ the set $\{g \in G : gA \cap B \neq \emptyset\}$ is finite.
4. For all compact $A \subseteq X$ the set $\{g \in G : gA \cap A \neq \emptyset\}$ is finite.
5. $G$ is discrete and acts properly on $X$.
6. For every $x \in X$ the orbit $Gx$ is discrete and the stabilizer $G_x$ is finite.

(b) Suppose we drop the assumption in (a) that $X$ is a metric space (so we cannot speak of isometries) and instead only assume $X$ (and $G$) is locally compact Hausdorff. Which of the six properties listed in (a) are equivalent to each other?

(c) Let $G$ be a Hausdorff group acting on a Hausdorff space $X$. Let $\pi : X \to G\backslash X$ be the quotient map. Show that the following are equivalent:

1. $G\backslash X$ is Hausdorff.
(2) All \( x, y \in X \) with \( \pi(x) \neq \pi(y) \) have open neighborhoods \( U, V \) with \( gU \cap V = \emptyset \) for all \( g \in G \).

(3) The image of the action map \( G \times X \to X \times X \) is closed.

(d) Let \( G \) be a Hausdorff group that acts properly on a Hausdorff space \( X \). Show that the following hold:

(1) \( G \setminus X \) is Hausdorff.
(2) Every orbit \( Gx \) is closed.
(3) Every stabilizer \( G_x \) is compact.
(4) For all \( x \in X \) the map \( G/G_x \to Gx \) defined by \( g \mapsto gx \) is a homeomorphism.

(e) Let the discrete group \( G = \mathbb{Z} \) act on \( \mathbb{R}^2 - \{(0, 0)\} \) via \( n \cdot (x, y) := (2^n x, y/2^n) \), where \( \mathbb{R}^2 - \{(0, 0)\} \) is equipped with the Euclidean metric. Determine which of the 6 properties listed in (a) and the 4 properties listed in (d) are enjoyed by \( G \) and \( X \). You should find that \( G \) and \( X \) satisfy some but not all of the properties of both (a) and (d), implying that they do not satisfy the hypotheses of either (a) or (d). Which hypotheses fail?

Problem 3. Fuchsian groups (50 points)

For any subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{R}) \) we call \( z \in \mathbb{CP}^1 := \mathbb{C} \cup \{\infty\} \) a limit point of \( \Gamma \) if there is a \( w \in \mathbb{CP}^1 \) and an infinite sequence of distinct \( \gamma_n \in \Gamma \) such that \( \lim \gamma_n w = z \). Let \( \Lambda(\Gamma) \) to denote the set of of all limit points of \( \Gamma \).

(a) Let \( \Gamma \) be a subgroup of \( \text{SL}_2(\mathbb{R}) \) and let \( \mathcal{H} \) be the upper half plane equipped with the hyperbolic metric defined in Problem 1. Show that the following are equivalent.

(1) \( \Gamma = \Gamma \) and \( X = \mathcal{H} \) satisfy property (1) listed in part (a) of problem 2.
(2) \( \Gamma \) is discrete.
(3) Any sequence in \( \Gamma \) converging to 1 is eventually constant.
(4) For every real \( B > 0 \) the subset of \( \Gamma \) with matrix entries in \([-B, B]\) is finite.
(5) \( \Lambda(\Gamma) \) is a closed subset of \( \mathbb{RP}^1 := \mathbb{R} \cup \{\infty\} \).

Such a group is called a Fuchsian group.

(b) Let \( \Gamma = \langle \gamma \rangle \) be a cyclic subgroup of \( \text{SL}_2(\mathbb{R}) \) with \( \gamma \neq \pm 1 \) and consider the three cases where \( \gamma \) is elliptic, parabolic, or hyperbolic. Compute \( \Lambda(\Gamma) \) in each case and determine whether (or under what conditions) \( \Gamma \) is Fuchsian.

(c) Show that if a Fuchsian group \( \Gamma \) is abelian then \( \Gamma/Z(\Gamma) \) is cyclic.

(d) Show that the \( \text{SL}_2(\mathbb{R}) \)-normalizer of a nonabelian Fuchsian group is Fuchsian.

(e) Let \( \Gamma \) be a Fuchsian group. Show that the following are equivalent.

(1) \( \# \Lambda(\Gamma) \leq 2 \)
(2) There is a finite \( \Gamma \)-orbit in \( \mathcal{H} \cup \mathbb{RP}^1 \).
(3) \( \Gamma/Z(\Gamma) \) is abelian or conjugate to \( \langle D, S \rangle \), where \( D \) is diagonal and \( S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

Such a group is called an elementary Fuchsian group.
Problem 4. Dirichlet domains (100 points)

Throughout this problem Γ ⊆ SL₂(ℝ) is a Fuchsian group. Recall that given any z₀ ∈ H that is not an elliptic point we can construct a Dirichlet domain

\[ F_{z₀} := \{ z ∈ H : d(z, z₀) ≤ d(z, γz₀) \text{ for all } γ ∈ Γ \}, \]

where \( d \) is the hyperbolic distance metric on H. We proved in class that \( F_{z₀} \) is a convex locally finite fundamental domain for Γ whose boundary \( ∂F \) is a union of geodesic segments \( L_γ := F_{z₀} ∩ γF_{z₀} \) called sides, whose intersections are vertices; we include vertices in \( ℝ \cup \{ ∞ \} \) that are the common endpoint of two geodesic rays. Recall that sidelines are required to have nonzero (possibly infinite) length. Locally finite means that every compact \( A ⊆ H \) intersects only finitely many Γ-translates of \( F_{z₀} \).

(a) Let \( F \) be a Dirichlet domain for Γ and let \( T := \{ γ ∈ Γ : F ∩ γF ≠ ∅ \} \). Show that \( T \) generates Γ and includes all elliptic elements of Γ.

We now refine our notion of sides and vertices as follows. For sides \( L_γ \) with \( γ² = ±1 \) we have \( ∇L_γ = L_γ \) and \( γ \) thus fixes the midpoint of \( L_γ \) (and swaps its end points). In this case it will be convenient to view \( L_γ \) as two distinct sides that intersect at the midpoint of \( L_γ \), which we now consider a vertex.

Each vertex of \( F \) lies in two sides of \( F \), one of which we unambiguously distinguish as the clockwise side of \( v_1 \) (the one on the right if we look at \( v_1 \) from the interior of \( F \)).

Let \( S \) denote the set of sides of a Dirichlet domain \( F \), and define

\[ P := \{ (γ, L, L') : L' = γL \} ⊆ Γ × (S × S). \]

Let \( G(𝑃) \) be the projection of \( P \) to Γ modulo \( Z(Γ) \) (keep only one of \( γ \) and \( −γ \)).

(b) Define a relation \( ∼ \) on \( S \) by letting \( L ∼ L' \) if and only if \( (γ, L, L') ∈ P \) for some \( γ ∈ Γ \). Show that this is an equivalence relation that partitions \( S \) into pairs of the form \( \{ (γ, L, L'), (γ⁻¹, L', L) \} \) with \( L = F ∩ γ⁻¹F \) and \( L' = F ∩ γF \).

(c) Show that \( G(P) \) generates \( Γ/Z(Γ) \) and that if \( S \) is finite then \( G(P) \) is finite.

(d) Let \( v_1 \) be a vertex of \( F \). Show that there is an open neighborhood \( U \) of \( v_1 \) that contains no other vertices of \( F \) and intersects no sides of \( F \) that do not contain \( v_1 \), and a finite list of elements \( 1 = δ₀, δ₁, ..., δₙ ∈ Γ \) such that the sets \( δ_i F \) are all distinct and \( U ⊆ \bigcup_{i=0}^{n} δ_i F \) with \( v_1 ∈ δ_i F \) for \( 0 ≤ i ≤ n \).

(e) Let \( v_1 \) be a vertex of \( F \), let \( L_1 \) be the clockwise side of \( F \), let \( L'_1 = γ₁L₁ \) be the unique side of \( F \) paired with \( L_1 \) via (e) (such a \( γ₁ \) exists, it need not be unique), and let \( v_2 = γ₁v_1 \). Show that \( v_2 \) is a vertex of \( F \), so we can similarly let \( L_2 \) be its clockwise side, let \( L'_2 = γ₂L₂ \) be the paired side, and define \( v₃ = γ₂v₂ \). Continuing in this fashion, we get a sequence of vertices \( v₁, v₂, ..., \) and group elements \( γ₁, γ₂, ... \). Show that after finitely many steps we return to \( v_m = v₁ \).

(f) With \( v₁ \) and \( γ₁ \) as in part (e), let \( δ = (γₘ ... γ₁)⁻¹ \). Show that the value of \( m \) in part (e) divides \( n + 1 \), where \( n \) is as in part (d), that \( F ∩ Γv₁ = \{ v₁, ..., v_{m−1} \} \) (with the \( v₁ \) as in part (e)), and that \( δ \) fixes \( v₁ \) and has order \( e := (n + 1)/m \). We call \( δ^e = 1 \) the vertex cycle relation for \( v₁ \). Show that if \( v'_1 = γ₁v₁ \) is a vertex of \( F \) then the cycle relation for \( v'_1 \) is \( (γ⁻¹δγ)^e = γ⁻¹δ^eγ = 1 \). Let \( R(P) \) be the set of vertex cycle relations for vertices in \( F \).
(g) Show that $\Gamma/Z(\Gamma)$ is isomorphic to the free group on $G(P)$ with relations $R(P)$.

(h) For $\Gamma = \text{SL}_2(\mathbb{Z})$, compute a Dirichlet domain using $z_0 = 2i$ and compute generators and relations for $\Gamma/Z(\Gamma) = \text{PSL}_2(\mathbb{Z})$ using the above method.

(i) Do the same for $\Gamma = \Gamma(2) := \{\gamma \in \text{SL}_2(\mathbb{Z}) : \gamma \equiv 1_{\text{SL}_2(\mathbb{Z})} \mod 2\}$.

**Problem 5. Survey**

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = “mind-numbing,” 10 = “mind-blowing”), and how difficult you found it (1 = “trivial,” 10 = “brutal”). Also estimate the amount of time you spent on each problem to the nearest half hour.

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Please rate each of the following lectures that you attended, according to the quality of the material (1=“useless”, 10=“fascinating”), the quality of the presentation (1=“epic fail”, 10=“perfection”), the pace (1=“way too slow”, 10=“way too fast”, 5=“just right”) and the novelty of the material to you (1=“old hat”, 10=“all new”).

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Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

**References**