18.786 Number Theory II **Spring 2024**
Problem Set #2 **Due: 2/25/2024**

Description

These problems are related to Lectures 3-5. Your solutions should be written up in latex and submitted as a pdf-file on Gradescope before midnight on the date due.

Collaboration is permitted/encouraged, but you must identify your collaborators or the name of your group on pset partners, and any references you consulted. If there are none write "**Sources consulted: none**" at the top of your solution.

Instructions: In the problems below we reference terms and definitions from earlier problems, so be sure to read through the problems in order. You may use results proved in an earlier problem in any later problem (even if you do not choose to solve the earlier problem). Pick any combination of problems to solve that sum to 100 and write up your answers in latex (be sure to include Problem 4, which is a survey).

Notation: $\mathbf{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, $\mathbf{D} := \{z \in \mathbb{C} : |z| < 1\}$, $Z(\Gamma) := \Gamma \cap \{\pm 1\}$.

Problem 1. Classification of Fuchsian groups (50 points)

As in Problem 2 of Problem Set 1, we call $z \in \mathbb{P}^1(\mathbb{C}) := \mathbb{C} \cup \{\infty\}$ a limit point of a group $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ if there is a $w \in \mathbb{P}^1(\mathbb{C})$ and an infinite sequence of distinct $\gamma_n \in \Gamma$ such that $\lim \gamma_n w = z$, and we use $\Lambda(\Gamma)$ to denote the set of of all limit points of Γ.

Recall that a Fuchsian group is a subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ that satisfies any of the equivalent properties listed in part (a) of Problem 2 on Problem Set 1; in particular, Γ is discrete and its limit set $\Lambda(\Gamma)$ is a closed subset of $\mathbb{P}^1(\mathbb{R}) := \mathbb{R} \cup \{\infty\}$. Elliptic points are elements of $\mathbf{H}$ fixed by an elliptic $\gamma \in \Gamma$, cusps are elements of $\mathbb{P}^1(\mathbb{R})$ fixed by a parabolic $\gamma \in \Gamma$, and hyperbolic points are elements of $\mathbb{P}^1(\mathbb{R})$ fixed by a hyperbolic $\gamma \in \Gamma$.

Recall that in a topological space X a point $x \in X$ is a limit point of a set $S \subseteq X$ if every punctured neighborhood of x contains a point in S. A set $S \subseteq X$ is perfect if it is equal to its set of limit points and nowhere dense if its interior is empty, equivalently, every punctured neighborhood of $x \in S$ contains a point not in S. Let Γ be a Fuchsian group.

(a) Show that every $x \in \mathbb{P}^1(\mathbb{R})$ fixed by an element of $\Gamma - Z(\Gamma)$ lies in $\Lambda(\Gamma)$; in other words, $\Lambda(\Gamma)$ contains all cusps and hyperbolic points of Γ.

(b) Show that Γ is countable and $\mathbf{H}$ contains uncountably many non-elliptic points.

(c) Show that for any non-elliptic point $w \in \mathbf{H}$ the set $\Lambda(\Gamma)$ is equal to the set of limit points of the Γ-orbit of w, and in particular is Γ-invariant.

(d) Show that if $x \in \Lambda(\Gamma)$ is fixed by a hyperbolic $\gamma \in \Gamma$ and $y \in \Lambda(\Gamma)$ is not fixed by γ then either $\lim_{n\to\infty} \gamma^n y = x$ or $\lim_{n\to\infty} \gamma^{-n} y = x$, so x is a limit point of $\Lambda(\Gamma)$.

(e) Show that if $x \in \Lambda(\Gamma)$ is not a hyperbolic point and Γ contains a hyperbolic γ, then x is a limit of hyperbolic points (hint: apply **(c)** to a point fixed by γ)

(f) Show that if $\#\Lambda(\Gamma) > 2$ then Γ contains a hyperbolic element γ and $\Lambda(\Gamma)$ is perfect, and therefore uncountable.

(g) Show that if $\Lambda(\Gamma) \neq \mathbb{P}^1(\mathbb{R})$ is perfect then $\Lambda(\Gamma)$ is nowhere dense. Use this to show that $\Lambda(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{P}^1(\mathbb{R})$.

It follows from **(f)** and **(g)** that there are three kinds of Fuchsian groups:

- elementary: $\#\Lambda(\Gamma) \leq 2$.
- the first kind: $\Lambda(\Gamma) = \mathbb{P}^1(\mathbb{R})$.
- the second kind: $\Lambda(\Gamma)$ is perfect and nowhere dense.

Problem 2. Finitely generated Fuchsian groups (50 points)

Let Γ be a Fuchsian group, $w \in \mathbf{H}$ a non-elliptic point, and F_w the corresponding Dirichlet domain. Recall that the boundary of F_w is a union of geodesic segments of the form $L_\gamma := F_w \cap \gamma F_w$ called sides; we require geodesic segments to have nonzero (possibly infinite) length (so if L_γ is a point it is not a side).

If F_w has only finitely many sides then Γ is called geometrically finite (this does not depend on the choice of w). Problem 3 of Problem Set 1 proves that if Γ is geometrically finite then it is finitely generated. The goal of this problem is to prove the converse.

As in Problem 3 of Problem Set 1, we split sides of the form L_γ with $\gamma^2 = \pm 1$ into two sides that intersect at the midpoint of L_γ and group sides into pairs that are Γ-translates (and not Γ-equivalent to any other sides). Let $G(\Gamma)$ be the subset of Γ consisting of elements γ that appear in a pair $(L, \gamma L)$. As proved in Problem Set 1, the set $G(\Gamma)$ generates Γ.

Now suppose that Γ is finitely generated. Out goal is to show F_w has finitely many sides.

(a) Show that if Γ is finitely generated then any generating set for Γ contains a finite subset that generates Γ. Let $\gamma_1, \ldots \gamma_n$ be a minimal subset of $G(\Gamma)$ generating Γ and let $L_1, \ldots, L_n$ be corresponding sides so that $(L_i, \gamma_i L_i)$ is a side pair for $1 \leq i \leq n$.

(b) Show that there exists a hyperbolic ball B_w about w whose interior contains arcs of positive length on all of the sides $L_1, \ldots, L_n, \gamma_1 L_1, \ldots \gamma_n L_n$, whose boundary contains no vertices of F_w and is not tangent to any side of F_w. Then show that only finitely many sides of F_w intersect B_w and that ΓB_w is connected.

(c) Let $s_1, \ldots, s_m$ be the arcs that make up $\partial B_w \cap F_w$. Pick an arc s_i and let e be one of its endpoints. Show that for some $\gamma \in G(\Gamma)$ the point γe is an endpoint of some s_j, so that $s_i \cup \gamma^{-1} s_j$ is a connected path. Show that by continuing in this fashion in both directions one can construct a connected path C_i composed of Γ-translates of the s_j that separates $\mathbf{H}$ into two disjoint sets, and that $\omega_i C_i = C_i$ for some $\omega_i \in \Gamma - Z(\Gamma)$. We thus obtain C_i and ω_i for each $1 \leq i \leq m$.

(d) Let $E_1, \ldots, E_m$ be the components of $F_w - B_w$ adjacent to $s_1, \ldots, s_m$. Show that every side of F_w disjoint from B_w intersects one of the E_i, so that it suffices to show that each E_i intersects only finitely many sides of F_w. Then show that if no element of $\Lambda(\Gamma)$ is a limit point of E_i then E_i can intersect only finitely many sides of F_w.

(e) Show that each path C_i of part **(c)** separates the interior of E_i from Γw, so that any limit point of E_i in $\Lambda(\Gamma)$ is an endpoint of C_i, and show that C_i has 0, 1, or 2 endpoints, depending on whether ω_i is elliptic, parabolic, or hyperbolic, respectively.

(f) Show that if ω_i is hyperbolic then E_i has no limit points in $\Lambda(\Gamma)$.

(g) Show that if ω_i is parabolic and Γ is not elementary then E_i lies inside C_i and intersects F_w in only finite many sides. Then show that if Γ is elementary then F_w has finitely many sides, and conclude that in every case F_w has finitely many sides.

Problem 3. Fuchsian groups of the first kind (50 points)

For a Fuchsian group Γ we use X_Γ to denote the Riemann surface $\Gamma\backslash\mathbf{H}_\Gamma^*$, where $\mathbf{H}_\Gamma^*$ is the topological space obtained by adjoining the cusps of Γ to $\mathbf{H}$ (and giving them open neighborhoods), and X_Γ is equipped with the complex structure defined in [2, §1.8].

Recall that under the measure μ induced by the hyperbolic metric every fundamental domain F_Γ for Γ has the same area, and we say that Γ is cofinite if this $\mu(F_\Gamma)$ is finite. Siegel's theorem [2, §1.9.1] implies that Γ is cofinite if and only if X_Γ is compact.

In Problem 1 (and many texts), a Fuchsian group Γ of the first kind is one that satisfies $\Lambda(\Gamma) = \mathbb{P}^1(\mathbb{R})$. Miyake [2, p. 28] uses this term to indicate that X_Γ is compact. The goal of this problem is to understand how these two notions are related.

Let Γ be a Fuchsian group.

(a) Show that if $\Gamma' \leq \Gamma$ then $\mu(F'_\gamma) = [\Gamma\colon \Gamma']\mu(F_\Gamma)$ and conclude that a subgroup of a cofinite Fuchsian group is cofinite if and only if it is of finite index.

(b) Show that if Γ is cofinite then it is geometrically finite, hence finitely generated.

(c) Show that if Γ is cofinite then $\Lambda(\Gamma) = \mathbb{P}^1(\mathbb{R})$.

(d) Show that elementary Fuchsian groups are geometrically finite but not cofinite.

(e) Show that for geometrically finite Fuchsian groups, if $\Lambda(\Gamma) = \mathbb{P}^1(\mathbb{R})$ then Γ is cofinite.

(f) Assume Γ is not elementary, and let $H(\Gamma)$ be the (hyperbolic) convex hull of $\Lambda(\Gamma)$ in $\mathbf{H}$, and let $D(\Gamma)$ be a Dirichlet domain. Show that $H(\Gamma) \cap D(\Gamma)$ has finite area if and only if Γ is geometrically finite, and $H(\Gamma) = \mathbf{H}$ if and only if $\Lambda(\Gamma) = \mathbb{P}^1(\mathbb{R})$.

(g) Prove that for finitely generated Fuchsian groups Γ, the Riemann surface X_Γ is compact if and only if $\Lambda(\Gamma) = \mathbb{P}^1(\mathbb{R})$.

Remark. In Poincaré's original exposition, Fuchsian groups were assumed to be finitely generated. Some authors still make this assumption (possibly implicitly), in which case the two definitions of first kind we have been considering are equivalent. However, there are infinitely generated non-cofinite Fuchsian groups for which $\Lambda(\Gamma) = \mathbb{P}^1(\mathbb{R})$; see [1].

Problem 4. Survey

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material to you (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
2/20	Arithmetic groups				
2/21	Automorphic forms				

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

References

[1] A. Basmajian, *Hyperbolic structures for surfaces of finite type*, Trans. Amer. Math. Soc. **336** (1993), 421–444.

[2] T. Miyake, *Modular forms*, Springer, 2006.