18.786 Number Theory II **Spring 2024**
Problem Set #1 **Due: 2/16/2024**

Description

These problems are related to material covered in Lectures 1–3. Your solutions should be written up in latex (please do not submit handwritten solutions) and submitted as a pdf-file on Gradescope before midnight on the date due. You can use the latex source for this problem set as a template, and are welcome to use the latex environment of your choice (I currently use Overleaf and Texmaker, but there are many other options).

Collaboration is permitted/encouraged, but you must identify your collaborators or the name of your group on pset partners, as well any references you consulted that are not listed in the course syllabus. If there are none write "**Sources consulted: none**" at the top of your solution. Each student is expected to write their own solutions; it is fine to discuss the problems with others, but your work must be your own.

Instructions: Pick a combination of problems to solve that sum to 100 and write up your answers in latex, then complete the survey problem 4.

As in [1], we use $\mathbf{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ to denote the complex upper half plane.

Problem 1. Geodesic spaces (50 points)

Let (X,d) be a metric space. A path in X is a continuous function $\gamma\colon [a,b] \to X$ we may also denote $\gamma\colon x \to y$, where $x = \gamma(a)$ and $y = \gamma(b)$. The length of a path is

$$l(\gamma) := \sup_{a=t_0<\cdots<t_n=b} \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)).$$

where the supremum is taken over all finite subdivisions of $[a,b]$. A metric space is a length space if for all $x,y \in X$ we have $d(x,y) = \inf_{\gamma\colon x\to y} \ell(\gamma)$.

(a) For a piecewise smooth path $\gamma\colon [a,b] \to \mathbf{H}$ we define the hyperbolic length

$$\ell(\gamma) := \int_a^b \frac{|\gamma'(t)|}{\operatorname{Im}(\gamma(t))} dt.$$

Show that $d(x,y) := \inf_{\gamma\colon x\to y} \ell(\gamma)$ is a metric on $\mathbf{H}$ that makes $(\mathbf{H}, d)$ a length space, with $\ell = l$ (in other words, the length function ℓ induces the metric d, which induces the length function l). We call d the hyperbolic metric on $\mathbf{H}$.

Definition. A geodesic path $\gamma\colon [a,b] \to X$ is a distance preserving path; that is, we have $d(\gamma(s), \gamma(t)) = |s-t|$ for all $s,t \in [a,b]$. A geodesic line (resp. geodesic ray) is a distance preserving continuous map $\mathbb{R} \to X$ (resp. $\mathbb{R}_{\geq 0} \to X$).[1] By a geodesic path/line/ray in X we mean the image of a geodesic path/line/ray. A geodesic space is a metric space in which every pair of points can be connected by a geodesic path. A geodesic space is straight if every geodesic path lies in a geodesic line, and uniquely geodesic if every $x,y \in X$ are connected by a unique geodesic path. A subset of a geodesic space is convex if it contains all geodesic paths between points in the set.

[1]N.B.: This differs from the definition of a geodesic segment as a shortest path definition used in lecture, and from the two definitions most commonly used in Riemannian geometry; it is slightly stronger.

(b) Show that $\mathbf{H}$ is a geodesic space that is a uniquely geodesic space under both the Euclidean metric $|z_1 - z_2|$ and the hyperbolic metric defined above, but that it is only straight for the hyperbolic metric, and automorphisms of $\mathbf{H}$ (as a Riemann surface) are isometries only for the hyperbolic metric.

(c) Show that the topologies on $\mathbf{H}$ induced by the Euclidean metric and the hyperbolic metric coincide (in both cases the open balls $B_{<r}(x) := \{z \in \mathbf{H} : d(x,z) < r\}$ of radius $r > 0$ about some $x \in \mathbf{H}$ are a basis for the topology).

(d) Show that Euclidean and hyperbolic open balls are convex under both the Euclidean and hyperbolic metric, and give examples of subsets of $\mathbf{H}$ that are convex under one metric but not the other (in both directions).

(e) Show that under the hyperbolic metric (i) for any distinct $z_1, z_2 \in \mathbf{H}$ the set

$$C_{z_1,z_2} := \{z \in \mathbf{H} : d(z,z_1) = d(z,z_2)\}$$

is a geodesic line $\gamma\colon \mathbb{R} \to \mathbf{H}$, (ii) $\mathbf{H} - C_{z_1,z_2}$ is the union of two disjoint convex sets that can meaningfully be called left and right half-planes (relative to γ), (iii) every geodesic line arises in this way. Does this apply to all Riemann surfaces that are straight uniquely geodesic spaces for a given metric?

Problem 2. Fuchsian groups (50 points)

For any subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ we call $z \in \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ a limit point of Γ if there is a $w \in \mathbb{P}^1(\mathbb{C})$ and an infinite sequence of distinct $\gamma_n \in \Gamma$ such that $\lim \gamma_n w = z$, Let $\Lambda(\Gamma)$ to denote the set of of all limit points of Γ, and let $Z(\Gamma) := \Gamma \cap \{\pm 1\}$.

(a) Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{R})$ and let $\mathbf{H}$ be the upper half plane equipped with the hyperbolic metric defined in Problem 1. Show that the following are equivalent.

(1) Every $z \in \mathbf{H}$ lies in an open $U \subseteq \mathbf{H}$ for which $\{\gamma \in \Gamma : U \cap \gamma U \neq \emptyset\}$ is finite.
(2) Γ is discrete.
(3) Any sequence in Γ converging to 1 is eventually constant.
(4) For every real $B > 0$ the subset of Γ with matrix entries in $[-B, B]$ is finite.
(5) $\Lambda(\Gamma)$ is a closed subset of $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

Such a group is called a Fuchsian group.

(b) Let $\Gamma = \langle \gamma \rangle$ be a cyclic subgroup of $\mathrm{SL}_2(\mathbb{R})$ with $\gamma \neq \pm 1$ and consider the three cases where γ is elliptic, parabolic, or hyperbolic. Compute $\Lambda(\Gamma)$ in each case and determine whether (or under what conditions) Γ is Fuchsian.

(c) Show that if a Fuchsian group Γ is abelian then $\Gamma/Z(\Gamma)$ is cyclic.

(d) Show that the $\mathrm{SL}_2(\mathbb{R})$-normalizer of a nonabelian Fuchsian group is Fuchsian.

(e) Let Γ be a Fuchsian group. Show that the following are equivalent.

(1) $\#\Lambda(\Gamma) \leq 2$
(2) There is a finite Γ-orbit in $\mathbf{H} \cup \mathbb{RP}^1$.
(3) $\Gamma/Z(\Gamma)$ is abelian or conjugate to $\langle D, S \rangle$, where D is diagonal and $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Such a group is called an elementary Fuchsian group.

Problem 3. Dirichlet domains (100 points)

Throughout this problem $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group. Recall that given any $z_0 \in \mathbf{H}$ that is not an elliptic point we can construct a Dirichlet domain

$$F_{z_0} := \{z \in \mathbf{H} : d(z, z_0) \le d(z, \gamma z_0) \text{ for all } \gamma \in \Gamma\},$$

where d is the hyperbolic distance metric on $\mathbf{H}$. We proved in class that F_{z_0} is a convex locally finite fundamental domain for Γ whose boundary ∂F is a union of geodesic segments $L_\gamma := F_{z_0} \cap \gamma F_{z_0}$ called sides, whose intersections in $\mathbf{H}$ are vertices; points in $\partial\mathbf{H} = \mathbb{R} \cup \{\infty\}$ that are the common endpoint of two sides of F that are geodesic rays are ideal vertices and are not considered vertices for the purpose of this problem. Recall that sides are required to have nonzero (possibly infinite) length. Locally finite means that every compact $A \subseteq \mathbf{H}$ intersects only finitely many Γ-translates of F_{z_0}.

(a) Let F be a Dirichlet domain for Γ and let $T := \{\gamma \in \Gamma : F \cap \gamma F \neq \emptyset\}$. Show that T generates Γ and includes a conjugate of every elliptic element of Γ.

We now refine our notion of sides and vertices as follows. For sides L_γ with $\gamma^2 = \pm 1$ we have $\gamma L_\gamma = L_\gamma$ and γ thus fixes the midpoint of L_γ (and swaps its end points). In this case it will be convenient to view L_γ as two distinct sides that intersect at the midpoint of L_γ, which we now consider a vertex.

Each vertex of F lies in two sides of F, one of which we unambiguously distinguish as the clockwise side of v_1 (the one on the right if we look at v_1 from the interior of F).

Let S denote the set of sides of a Dirichlet domain F, and define

$$P := \{(\gamma, L, L') : L' = \gamma L\} \subseteq \Gamma \times (S \times S).$$

Let $G(P)$ be the projection of P to Γ modulo $Z(\Gamma)$ (keep only one of γ and $-\gamma$).

(b) Define a relation $\sim$ on S by letting $L \sim L'$ if and only if $(\gamma, L, L') \in P$ for some $\gamma \in \Gamma$. Show that this is an equivalence relation that partitions S into pairs of sides $\{L, L'\}$ with $L = F \cap \gamma^{-1}F$ and $L' = \gamma L = F \cap \gamma F$.

(c) Show that $G(P)$ generates $\Gamma/Z(\Gamma)$ and that if S is finite then $G(P)$ is finite.

(d) Let v_1 be a vertex of F. Show that there is an open neighborhood U of v_1 that contains no other vertices of F and intersects no sides of F that do not contain v_1, and a finite list of elements $1 = \delta_0, \delta_1, \ldots, \delta_n \in \Gamma$ such that the sets $\delta_i F$ are all distinct and $U \subseteq \cup_{i=0}^n \delta_i F$ with $v_1 \in \delta_i F$ for $0 \le i \le n$.

(e) Let v_1 be a vertex of F, let L_1 be the clockwise side of F, let $L_1' = \gamma_1 L_1$ be the unique side of F paired with L_1 via **(c)** (such a γ_1 exists, it need not be unique), and let $v_2 = \gamma_1 v_1$. Show that v_2 is a vertex of F, so we can similarly let L_2 be its clockwise side, let $L_2' = \gamma_2 L_2$ be the paired side, and define $v_3 = \gamma_2 v_2$. Continuing in this fashion, we get a sequence of vertices $v_1, v_2, \ldots$ and group elements $\gamma_1, \gamma_2, \ldots$ Show that after finitely many steps we return to $v_{m+1} = v_1$.

(f) With v_i and γ_i as in part **(e)**, let $\delta = (\gamma_m \cdots \gamma_1)^{-1}$. Show that the value of m in part **(e)** divides $n+1$, where n is as in part **(d)**, that $F \cap \Gamma v_1 = \{v_1, \ldots, v_{m-1}\}$ (with the v_i as in part **(e)**), and that δ fixes v_1 and has order $e := (n+1)/m$. We call $\delta^e = 1$ the vertex cycle relation for v_1. Show that if $v_1' = \gamma v_1$ is a vertex of F then the cycle relation for v_1' is $(\gamma^{-1}\delta\gamma)^e = \gamma^{-1}\delta^e\gamma = 1$. Let $R(P)$ be the set of vertex cycle relations for vertices in F.

(g) Show that $\Gamma/Z(\Gamma)$ is isomorphic to the free group on $G(P)$ with relations $R(P)$.

(h) For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, compute a Dirichlet domain using $z_0 = 2i$ and compute generators and relations for $\Gamma/\mathbb{Z}(\Gamma) = \mathrm{PSL}_2(\mathbb{Z})$ using the above method.

(i) Do the same for $\Gamma = \Gamma(2) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv 1 \bmod 2\}$.

Problem 4. Survey

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material to you (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
2/5	Automorphisms of **H**				
2/7	Fuchsian groups				
2/12	Dirichlet domains				
2/14	Quotients by Fuchsian groups				

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

References

[1] T. Miyake, *Modular forms*, Springer, 2006.