These notes summarize the material in §2.6 of [1] covered in lecture.

9.1 Poincaré series

Let k be an integer and let $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ be a lattice with a character $\chi : \Gamma \to U(1)$ of finite order such that $\chi(-1) = (-1)^k$ if $-1 \in \Gamma$. Let $\Lambda \le \Gamma$ and let ϕ a meromorphic function on $\mathbf{H}$ such that

(i) $\phi|_k\lambda = \chi(\lambda)\phi$ for all $\lambda \in \Lambda$;

(ii) ϕ has only finitely many Λ-inequivalent poles $z_1, \ldots z_m$;

(iii) If $x_1, \ldots, x_r$ are the Γ-inequivalent cusps of Γ, then for any open neighborhoods $U_i \ni zi$ and $V_j \ni x_j$ we have

$$\int_{\Lambda\backslash\mathbf{H}'} |\phi(z)|\,\mathrm{Im}(z)^{k/2} d\nu(z) < \infty,$$

where $\mathbf{H}' = \mathbf{H} - \bigcup_{i=1}^m \bigcup_{\lambda\in\Lambda} \lambda U_i - \bigcup_{j=1}^r \bigcup_{\gamma\in\Gamma} \gamma V_j$ and $d\nu(z)$ is the hyperbolic measure.

We now define the Poincaré series

$$F(z) := F_k(z;\phi,\chi,\Lambda,\Gamma) = \sum_{\gamma\in\Lambda\backslash\Gamma} \overline{\chi(\gamma)}(\phi|_k\gamma)(z)$$

If the sum defining $F(z)$ converges, then by construction we have $F|_k\gamma = \chi(\gamma)F$ for all $\gamma \in \Gamma$ and $F \in \Omega_k(\Gamma,\chi)$ is an automorphic form of weight k for Γ with character χ.

Theorem 9.1. *Let $\phi, \chi, \Lambda, \Gamma, z_i, F$ be as above. Then F converges absolutely and uniformly on any compact subset of $\mathbf{H} - \{\gamma z_i | \gamma \in \Gamma, 1 \le i \le m\}$ and $F \in \Omega_k(\Gamma,\chi)$.*

Proof. See Theorem 2.6.6 in [1]. □

Now let $x \in \mathbb{P}^1(\mathbb{R})$ be a cusp of Γ, choose $\sigma \in \mathrm{SL}_2(\mathbb{R})$ so $\sigma x = \infty$, and suppose that *phi* also satisfies

(iv) if x is not a cusp of Λ then $|(\phi|_k\sigma^{-1})(z)| \le M|z|^{-1-\epsilon}$ on $\mathrm{Im}(z) > \ell$, for some $M, \ell, \epsilon > 0$.

(v) if x is a cusp of Λ then $|(\phi|_k\sigma^{-1})(z)| \le M|z|^{-\epsilon}$ on $\mathrm{Im}(z) > \ell$, for some $M, \ell > 0$ and $\epsilon \ge 0$.

Theorem 9.2. *Let $\Gamma, \Lambda, \phi, \chi, F$ be as above, let x_0 be a cusp of Γ such that (iv) and (v) hold for $x \in \Gamma x_0$. Then F is holomorphic at x_0, and if $\epsilon > 0$ in (v) then F vanishes at x_0.*

Proof. This is Theorem 2.6.7 in [1]. □

We now assume that Γ has at least one cusp x, with $\sigma x = \infty$ and that

$$\chi(\gamma) j(\sigma\gamma\sigma^{-1}, z)^k = 1 \text{ for all } \gamma \in \Gamma_x, \tag{1}$$

a condition that does not depend on the choice of σ and which implies that if χ is trivial, k is odd, and $-1 \notin \Gamma$ then x is a regular cusp. Recall that

$$\pm\sigma\Gamma_x\sigma^{-1} = \pm\left\langle\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)\right\rangle$$

for some $h > 0$. For each $m \in \mathbb{Z}_{\ge 0}$ we define

$$\phi_m(z) := \phi_m(z;x,\sigma) := j(\sigma,z)^{-k} e^{2\pi i m \sigma z/h}.$$

It is easy to check that for all $k \ge 3$ the functions ϕ_m satisfy conditions (i) through (v) for $\Gamma, \chi, \Lambda = \Gamma_x, x$, and we have the following theorem.

Theorem 9.3. *Assume $k \geq 3$ and let Γ, χ, ϕ_m, x be as above, satisfying conditions (i)–(v) and (1). Then*

- *If $m \geq 1$ then $F_k(z; \phi_m, \chi\Gamma_x, \Gamma) \in S_k(\Gamma, \chi)$ is a cusp form of weight k for Γ with character χ.*
- *If $m = 0$ then $F(z) := F_k(z; \phi_0, \chi\Gamma_x, \Gamma) \in M_k(\Gamma, \chi)$ is a modular form of weight k for Γ with character χ whose Fourier expansion at x has the form*

$$(F|_k\sigma^{-1})(z) = 1 + \sum_{n=1}^{\infty} a_n e^{2\pi i n z/h}$$

and which vanishes at all cusps that are Γ-inequivalent to x.

Definition 9.4. The Poincaré series $F_k(z; \phi_0, \chi, \Gamma_x, \Gamma)$ in Theorem 9.3 is an Eisenstein series.

Theorem 9.5. *Assume $k \geq 3$ and let Γ, χ, ϕ_m, x be as above, satisfying conditions (i)–(v) and (1). Let $\sigma x = \infty$ and for $m \in \mathbb{Z}_{\geq 0}$ define*

$$g_k^{(m)} := F_k(z; \phi_m, \chi, \Gamma_x, \Gamma).$$

Let $f \in S_k(\Gamma, \chi)$ have Fourier expansion

$$(f|_k\sigma^{-1})(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z/h}$$

at x, then

$$\left\langle f, g_k^{(m)} \right\rangle = \int_{\Gamma\backslash\mathbf{H}} f(z)\overline{g_k^{(m)}(z)}\,\mathrm{Im}(z)^k d\nu(z) = \begin{cases} 0 & \text{if } m = 0 \\ a_m(4\pi m)^{1-k} h^k (k-2)! & \text{if } m > 0 \end{cases}$$

proof sketch. Without loss of generality we may assume $x = \infty$ and $\sigma = 1$. As shown in the proof of [1, Theorem 2.6.10], the integral in the Petersson inner product converges uniformly, allowing us to swap the order of integration and the sum defining F_k. With $z = x + iy$ we have

$$\begin{aligned}
\int_{\Gamma\backslash\mathbf{H}} f(z)\overline{g_k^{(m)}(z)}\,\mathrm{Im}(z) d\nu(z) &= \sum_{\gamma \in \Gamma_x\backslash\Gamma} \chi(\gamma) \int_{\Gamma\backslash\mathbf{H}} f(z) e^{-2\mathfrak{p} i m y \bar{z}/h} j(\gamma, \bar{z})^{-k}\,\mathrm{Im}(z)^k d\nu(z) \\
&= \int_0^{\infty}\int_0^h f(z) e^{-2\pi i m \bar{z}/h} y^{k-2} dx\,dy \\
&= \sum_{n=1}^{\infty} a_n \int_0^{\infty} e^{-2\pi(m+n)y/h} y^{k-2} dy \int_0^h e^{2\pi i(n-m)x/h} dx \\
&= \begin{cases} 0 & \text{if } m = 0 \\ a_m(4\pi m)^{1-k} h^k (k-2)! & \text{if } m > 0 \end{cases}
\end{aligned}$$

where we have used the fact that $\int_0^1 e^{2\pi i n x} dx$ vanishes for all nonzero integers n. □

Corollary 9.6. *Under the hypotheses of Thoerem 9.5, the set $\{g_k^{(m)}(z) : m \geq 1\}$ generates $S_k(\Gamma, \chi)$.*

Proof. Let V be the subspace of $S_k(\Gamma,\chi)$ spanned by the $g_k^{(m)}$, let

$$V^\perp := \left\{ f : S_k(\Gamma,\chi) : \langle f, g_k^{(m)} \rangle = 0 \text{ for } m \geq 1 \right\}$$

be its orthogonal complement with respect to the Petersson inner product, and consider $f \in V$ with Fourier expansion $f(z) = \sum_{n\geq 1}^{\infty} a_n e^{2\pi i n z/h}$ at x. Theorem 9.5 implies that $a_m = 0$ for all $m \geq 1$, so $f = 0$. □

Let $\mathscr{E}_k(\Gamma,\chi)$ denote the orthogonal complement of $S_k(\Gamma,\chi)$ in $M_k(\Gamma,\chi)$ with respect to the Petersson inner product:

$$\mathscr{E}_k(\Gamma,\chi) := \{ g \in M_k(\Gamma,\chi) : \langle f, g \rangle = 0 \text{ for all } f \in S_k(\Gamma,\chi) \}.$$

Corollary 9.7. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a lattice and let $x_1,\ldots,x_r$ be the Γ-inequivalent cusps that satisfy (1). Assume $k \geq 3$ and let Γ, χ, ϕ_m be as above, satisfying conditions (i)–(v). Then the functions*

$$g_i(z) := F_k(z;\phi_0,\chi,\Gamma_{x_i},\Gamma) \in S_k(\Gamma,\chi) \qquad (1 \leq i \leq r)$$

are a basis for the $\mathbb{C}$-vector space $\mathscr{E}_k$.

Proof. Consider $f \in M_k(\Gamma,\chi)$. If x is a cusp of Γ that does not satisfy (1) then f vanishes at x. Otherwise, let $a_0^{(i)}$ be the constant term in the Fourier expansion of f at x_i. Theorem 9.3 implies that the function $f(z) - \sum_{i=1}^{r} a_0^{(i)} g_i(z)$ lies in $S_k(\Gamma,\chi)$, therefore $M_k(\Gamma,\chi) = S_k(\Gamma,\chi) \bigoplus V$, where V is the subspace of $M_k(\Gamma,\chi)$ spanned by the g_i, and Theorem 9.5 implies that V is contained in $\mathscr{E}_k(\Gamma,\chi) = S_k(\Gamma,\chi)^\perp$, so $E_k(\Gamma,\chi) = V$ is spanned by the g_i, and Theorem 9.3 implies that the g_i are linearly independent and thus form a basis. □

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.