These notes summarize the material in §2.3–2.5 of [1] covered in lecture.

8.1 Divisors of automorphic forms

For a compact Riemann surface X we define

$$\mathrm{Div}(X)_{\mathbb{Q}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and call elements of $\mathrm{Div}(X)_{\mathbb{Q}}$ divisors with rational coefficients.

Let $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ be a lattice, so that $X_\Gamma := \Gamma\backslash\mathbf{H}^*_\Gamma$ is a compact Riemann surface, and let $\pi_\Gamma\colon \mathbf{H}^*_\Gamma \to X$ be the projection map. Let $f \in A_{2n}(\Gamma)$ be a nonzero automorphic form of weight $2n$, and let $P \in X_\Gamma(\mathbb{C})$ and $z_P \in \pi_\Gamma^{-1}(P)$. In the previous lecture we defined

$$\mathrm{ord}_P(f) = \begin{cases} \frac{1}{e}\mathrm{ord}_{z_P}(f) & \text{if } P \text{ is an elliptic point of order } e,\\ \mathrm{ord}_\infty(f|_{2n}\sigma^{-1}) & \text{if } P \text{ is a cusp and } \sigma z_P = \infty,\\ \mathrm{ord}_{z_P}(f). & \text{otherwise}\end{cases}$$

Here $\mathrm{ord}_\infty(f|_{2n}\sigma^{-1})$ is the index m of the first nonzero coefficient a_m in the Fourier expansion $(f|_{2n}\sigma^{-1})(z) = \sum e^{2\pi i m z/h}$ with $\pm\sigma\Gamma_{z_P}\sigma^{-1} = \pm\left\langle\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)\right\rangle$. We also recall that $e = \#\Gamma_{z_P}/Z(\Gamma)$ is equal to the ramification index of the map π_Γ at e (the cardinality of $\pi_\Gamma^{-1}(Q)$ for every $Q \ne P$ in any sufficiently small neighborhood of P).

The lattice Γ has only finitely many inequivalent elliptic points and cusps (it is cofinite, hence geometrically finite), so we may define

$$\mathrm{div}(f) := \sum_{P\in X_\Gamma(\mathbb{C})} \mathrm{ord}_P(f)P \in \mathrm{Div}(X_\Gamma)_{\mathbb{Q}}.$$

In the previous lecture we associated to f a meromorphic differential $\omega_f \in \Omega^n(X_\Gamma)$ with

$$\mathrm{ord}_P(\omega_f) = \begin{cases} \mathrm{ord}_P(f) - n\left(1-\frac{1}{e}\right) & \text{if } P \text{ is an elliptic point of order } e,\\ \mathrm{ord}_P(f) - n & \text{if } P \text{ is a cusp,}\\ \mathrm{ord}_P(f) & \text{otherwise.}\end{cases}$$

Note that unlike $\mathrm{div}(f)$, the divisor $\mathrm{div}(\omega_f) \in \mathrm{Div}(X_\Gamma)$ has integer coefficients (it is the divisor of a differential form), and the Riemann–Roch theorem implies

$$\deg(\mathrm{div}(\omega_f)) = 2n(g-1), \tag{1}$$

where $g := g(X_\Gamma)$ is the genus of X_Γ, as is true for any meromorphic differential of degree n on a compact Riemann surface of genus g.

The following theorem follows immediately from the formulas above.

Theorem 8.1. *For $k \in 2\mathbb{Z}$ and nonzero $f \in A_k(\Gamma)$ we have*

$$\mathrm{div}(f) = \mathrm{div}(\omega_f) + \frac{k}{2}\sum_{P\in X_\Gamma(\mathbb{C})}\left(1-\frac{1}{e_P}\right)P$$
$$\deg\mathrm{div}(f) = k(g-1) + \frac{k}{2}\sum_{P\in X_\Gamma(\mathbb{C})}\left(1-\frac{1}{e_P}\right)$$

where e_P is the ramification index of P and $g = g(X_\Gamma)$, with $1/e_P = 0$ when P is a cusp.

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Corollary 8.2 (Sturm bound). *Let $k \in 2\mathbb{Z}_{>0}$, let $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ be a lattice with $r \ge 1$ inequivalent cusps and $s \ge 0$ inequivalent elliptic points, and let g be the genus of X_Γ. Let x be a cusp of Γ and choose $\sigma \in \mathrm{SL}_2(\mathbb{R})$ so that $\sigma x = \infty$. Suppose $f_1, f_2 \in M_k(\Gamma)$ have Fourier expansions*

$$(f_1|_k\sigma^{-1} = \sum_{n\ge 0} a_n e^{2\pi i n z/h} \qquad \textit{and} \qquad (f_2|_k\sigma^{-1})(z) = \sum_{n\ge 0} b_n e^{2\pi i n z/h}$$

at x, with $a_n = b_n$ for $n \le k(g-1+(r+s)/2)$. Then $f_1 = f_2$.

Proof. Suppose $f_1 \ne f_1$. Then $f_1 - f_2$ and $\omega := \omega_{f_1-f_2}$ are nonzero. Let $P = \pi_\Gamma(x)$, and let $n > k(g-1) + k(r+s)/2$ be the least positive integer for which $a_n \ne b_n$. Then

$$\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f-g) - \tfrac{k}{2} = n - \tfrac{k}{2} > k(g-1+(r+s-1)/2).$$

Now $f_1 - f_2$ is holomorphic, so $\mathrm{ord}_Q(\omega) \ge -k/2$ for every elliptic point or cusp $Q \ne P$, and $\mathrm{ord}_R(\omega) \ge 0$ for every ordinary point R. It follows that

$$\deg(\mathrm{div}(\omega)) > k(g-1+(r+s-1)/2) - (r+s-1)k/2 = k(g-1)$$

but this contradicts the degree equality implied by Riemann–Roch (1), so $f_1 = f_2$. □

Remark 8.3. This corollary has some remarkable implications. If $k(g-1) + k(r+s)/2 < 0$ the hypothesis always holds, implying that every $f \in M_k(\Gamma)$ is zero. It also implies that every $f \in M_k(\Gamma)$ is uniquely determined by a finite prefix of its Fourier expansion at any cusp. This yields an effective algorithm for testing whether two modular forms are equal or not, and if we fix a basis for the finite dimensional $\mathbb{C}$-vector space $M_k(\Gamma)$ it allows us to explicitly compute the representation of any $f \in M_k(\Gamma)$ as a linear combination of basis elements.

8.2 Odd weight

For odd k we may assume $-1 \notin \Gamma$, since $f|_k{-1} = -f$ is satisfied only when $f = 0$. Let $f \in A_k(\Gamma)$ be nonzero. Then $f^2 \in A_{2k}(\Gamma)$, and we define

$$\mathrm{ord}_P(f) := \mathrm{ord}_P(f^2)/2$$

and

$$\mathrm{div}(f) := \sum_{P\in X_\Gamma(\mathbb{C})} \mathrm{ord}_p(f)P$$

Theorem 8.4. *Let k be an odd integer and assume $-1 \notin \Gamma$. For each nonzero $f \in A_k(\Gamma)$ we have*

$$\mathrm{div}(f) = \tfrac{1}{2}\mathrm{div}(\omega_{f^2}) + \tfrac{k}{2}\sum_P \big(1 - \tfrac{1}{e_P}\big)P$$

$$\deg\mathrm{div}(f) = k(g-1) + \tfrac{k}{2}\sum_P \big(1 - \tfrac{1}{e_P}\big)$$

$$\mathrm{ord}_P(f) \in \begin{cases} \tfrac{1}{2} + \mathbb{Z} & \textit{if } P \textit{ is an irregular cusp} \\ \tfrac{1}{e_P} + \mathbb{Z} & \textit{if } P \textit{ is an elliptic point} \\ 0 & \textit{otherwise} \end{cases}$$

$$\mathrm{ord}_P(\omega_{f^2}) \in \begin{cases} 1 + 2\mathbb{Z} & \textit{if } P \textit{ is a regular cusp} \\ 2\mathbb{Z} & \textit{otherwise} \end{cases}$$

where e_P is the ramification index at P with $1/e_P = 0$ if P is a cusp.

8.3 Computing the measure of X_Γ

As above, let $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ be a lattice and let $X_\Gamma := \Gamma\backslash\mathbf{H}^*_\Gamma$.

Lemma 8.5. *Let m be the* lcm *of the orders of all elliptic $P \in X_\Gamma(\mathbb{C})$. There is a nonzero $f \in A_{2m}(\Gamma)$ which has no zeros or poles at any cusp or any elliptic point in $X_\Gamma(\mathbb{C})$.*

Proof. Let $f \in A_{2m}(\Gamma)$ be nonzero. Then $\mathrm{div}(f) = \mathrm{div}(\omega_f) + \frac{k}{2}\sum_P\big(1-\frac{1}{e_P}\big)P \in \mathrm{Div}(X_\Gamma)$, since $k/2 = m \in \mathbb{Z}$ is divisible by every $e_P \in \mathbb{Z}$. Let S be the (finite) set of elliptic points and cusps in $X_\Gamma(\mathbb{C})$, let $g := g(X_\Gamma)$, and choose $n \in \mathbb{Z}$ so that

$$-\deg(\mathrm{div}(g)) - 1 + n > 2g - 2.$$

Fix $Q \notin S$ and consider the divisors $D_Q := -\mathrm{div}(g) + nQ$ and $D_P := -\mathrm{div}(g) - P + nQ$ for each $P \in S$. We have $D_Q \ge D_P$, so $L(D_Q) \supsetneq L(D_P)$, and $\deg(D_Q) > \deg(D_P) > 2g-2$, so

$$\dim L(D_Q) - \dim L(D_P) = \deg(D_Q) - \deg(D_P) = 1,$$

for all $P \in S$ by Riemann–Roch, since $\ell(D) = \deg(D) - g + 1$ whenever $\deg(D) > 2g-2$.

It follows that the Riemann–Roch space

$$L(D_Q) = \{h \in \mathbb{C}(X_\Gamma) : h = 0 \text{ or } \mathrm{div}(h) + D_Q \ge 0\}$$

contains some $h \in \mathbb{C}(X_\Gamma) \simeq A_0(\Gamma)$ that does not lie in any of the $L(D_P)$, with $\mathrm{ord}_P(f/h) = 0$ for $P \in S$, since $\mathrm{ord}_P(h) = \mathrm{ord}_P(f)$ for $P \in S$. Then $f/h \in A_{2m}(\Gamma)$ has not zeros or poles on any cusp or elliptic point in $X_\Gamma(\mathbb{C})$. □

Recall that $\nu(X_\Gamma)$ is defined as the measure of any (measurable) fundamental domain for Γ under the hyperbolic measure $d\nu(z) := \frac{dxdx}{y^2}$ on $\mathbf{H}$, where $z = x + iy$, which is induced by the Haar measure of $\mathrm{SL}_2(\mathbb{R})$.

Theorem 8.6. *For any lattice $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ we have*

$$\tfrac{1}{2\pi}\nu(X_\Gamma) = 2g - 2 + \sum_{P \in X_\Gamma(\mathbb{C})}\big(1 - \tfrac{1}{e_P}\big),$$

where e_P is the ramification index of P, with $1/e_P = 0$ when P is a cusp, and g is the genus of X_Γ.

Proof sketch. Pick m and nonzero $f \in A_{2m}(\Gamma)$ via Lemma 8.5, and a measurable fundamental domain F for Γ such that f has no zeros or poles on ∂F. Let $v_1, \ldots, v_r$ be the vertices of F that are cusps, and pick curves $C_1, \ldots, C_r$ with C_i in a neighborhood of v_i such that $\pi_\Gamma(C_i)$ is a circle around $\pi(v_i)$ oriented counterclockwise. Let M be the compact set bounded by F and the curves $C_1, \ldots, C_r$. Then

$$\nu(X_\Gamma) = \nu(F) = \lim_{C_i \to x_i} \int_M \frac{dx \wedge dy}{y^2},$$

where $C_i \to x_i$ means we are shrinking the neighborhoods of x_i in which the C_i lie so that the measure of the part of F outside M tends to zero. Applying Stokes Theorem yields

$$\int_M \frac{dx \wedge dy}{y^2} = \int_{\partial M} \frac{dz}{y} = \int_{\partial M}\left(\frac{dz}{y} + \frac{i}{m}d(\log f)\right) - \frac{i}{m}\int_{\partial M} d(\log f),$$

where ∂M is oriented counterclockwise. One can show that the integrand in the first term on the RHS is Γ-invariant, and since Γ acts on pairs of sides of F with opposite orientation, the integral tends to zero as the $C_i \to x_i$, and we have

$$\nu(X_\Gamma) = 2\pi \lim_{C_i \to x_i} -\frac{i}{m} \int_{\partial M} d(\log f) = \tfrac{2\pi}{m} \deg(\operatorname{div}(f))$$

by Cauchy's residue formula, since we can assume f has no zeros or poles on ∂M and its zeros and poles in F all lie inside M. Applying Theorem 8.1 yields

$$\tfrac{1}{2\pi}\nu(X_\Gamma) = \tfrac{1}{m}\left(2m(g-1) + m\sum_P \left(1 - \tfrac{1}{e_P}\right)\right) = 2g - 2 + \sum_{P \in \mathbb{C}(X_\Gamma)} \left(1 - \tfrac{1}{e_p}\right). \qquad \square$$

8.4 Dimension formulas

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a lattice. For $D \in \operatorname{Div}(X_\Gamma)_\mathbb{Q}$ we define

$$\lfloor D \rfloor := \sum_P \lfloor r_P \rfloor P$$

Lemma 8.7. *Let $\Gamma \in \mathrm{SL}_2(\mathbb{R})$ be a lattice with regular cusps $P_1, \ldots, P_s$ and irregular cusps $P_{s+1}, \ldots, P_t$. Let $f \in A_k(\Gamma)$ be nonzero. We have the following isomorphisms of $\mathbb{C}$-vector-spaces*

- $M_k(\Gamma) \simeq L(\lfloor \operatorname{div}(f) \rfloor)$;
- $S_k(\Gamma) \simeq \begin{cases} L(\lfloor \operatorname{div}(f) - (P_1 + \cdots + P_t) \rfloor) & k \text{ even}, \\ L(\lfloor \operatorname{div}(f) - (P_1 + \cdots + P_s + \frac{1}{2}P_{s+1} + \cdots + \frac{1}{2}P_t) \rfloor) & k \text{ odd}. \end{cases}$

Proof. We have $A_k(\Gamma) = \{fh | h \in A_0(\Gamma)\}$, and it follows that

$$\begin{aligned} M_k &= \{fh : h \in A_0(\Gamma) : h = 0 \text{ or } \operatorname{div}(fh) \geq 0\} \\ &= \{h \in \mathbb{C}(X_\Gamma) : h = 0 \text{ or } \operatorname{div}(h) + \operatorname{div}(f) \geq 0\} \\ &= L(\lfloor (\operatorname{div}(f) \rfloor). \end{aligned}$$

since $\operatorname{div}(h) + \operatorname{div}(f) \geq 0$ if and only if $\operatorname{div}(h) + \lfloor \operatorname{div}(f) \rfloor \geq 0$.

Now consider $fh \in M_k(\Gamma)$ with $h \in A_0(\Gamma) \simeq \mathbb{C}(X)$. Then

$$fh \in S_k(\Gamma) \quad \Longleftrightarrow \quad \operatorname{ord}_{P_i}(ff_0) > 0 \text{ for } 1 \leq i \leq t$$

If k is even then

$$\begin{aligned} fh \in S_k(\Gamma) \quad &\Longleftrightarrow \quad \operatorname{ord}_{P_i}(fh) \geq 1 \text{ for } 1 \leq i \leq t \\ &\Longleftrightarrow \quad \operatorname{ord}_{P_i}(f) + \operatorname{ord}_{P_i}(h) - 1 \geq 0 \text{ for } 1 \leq i \leq t \\ &\Longleftrightarrow \quad h \in L(\lfloor \operatorname{div}(f) - (P_1 + \cdots + P_t) \rfloor) \end{aligned}$$

which implies $S_k(\Gamma) \simeq L(\lfloor \operatorname{div}(f) - (P_1 + \cdots + P_t) \rfloor)$. For k odd a similar argument works, except we use the bound $\operatorname{ord}_{P_i}(fh) \geq 1/2$ for $s < i \leq t$. $\square$

Let $e_1,\ldots,e_r$ be the orders of the elliptic points and t the number of cusps in $X_\Gamma(\mathbb{C})$. Define

$$d := 2g-2+\sum_{i=1}^r\left(1-\tfrac{1}{e_i}\right)+t,$$

let $g := g(X_\Gamma)$, and fix a nonzero $f\in A_k(\Gamma)$. Theorem 8.1 implies that

$$\deg(\operatorname{div}(f)) = \tfrac{k}{2}d \qquad\text{and}\qquad \deg(\lfloor \operatorname{div}(f)\rfloor) = k(g-1)+\sum_{i=1}^r\left\lfloor \tfrac{k}{2}\left(1-\tfrac{1}{e_i}\right)\right\rfloor+\lfloor\tfrac{k}{2}\rfloor t$$

and we have $d = v(X_\Gamma)/(2\pi)>0$, by Theorem 8.6. We now consider $k\in 2\mathbb{Z}$ as follows:

- $k<0$: we have $\deg(\lfloor\operatorname{div}(f)\rfloor)\le\deg(\operatorname{div}(f)) = kd/2<0$ so $\dim M_k(\Gamma)=\dim S_k(\Gamma)=0$.
- $k=0$: we have $f\in\mathbb{C}(X_\Gamma)^\times$ and $\dim M_0(\Gamma)=\ell(\operatorname{div}(f))=1$ and

$$\dim S_0(\Gamma)=\ell(\operatorname{div}(f)-(P_1+\cdots+P_t))=\begin{cases}1 & \text{if } t=0\\ 0 & \text{if } t>0.\end{cases}$$

- $k=2$: we have $S_2(\Gamma)\simeq\Omega^1_0(X_\Gamma)$, as shown in Lecture 7, so $\dim S_2(\Gamma)=\ell(\operatorname{div}(\omega_f))=g$, and $M_k(\Gamma)\simeq S_k(\Gamma)$ if $t=0$, otherwise Riemann–Roch implies

$$\dim M_k(\Gamma)=\ell(\lfloor\operatorname{div}(f)\rfloor)=\deg(\lfloor\operatorname{div}(f)\rfloor)-g+1+\ell(\operatorname{div}(\omega_1)-\lfloor\operatorname{div}(f)\rfloor)=g-1+t.$$

- For $k>2$. Fix any nonzero $f\in A_k(\Gamma)$ and let $D=\operatorname{div}(f)-(P_1+\cdots+P_t)$, where $P_1,\ldots,P_t$ are the cusps in $X_\Gamma(\mathbb{C})$. Then $\deg(D)=\deg(\operatorname{div}(f))-t$ and

$$\deg(\lfloor D\rfloor)=k(g-1)+\sum_{i=1}^r\left\lfloor\tfrac{k}{2}\left(1-\tfrac{1}{e_i}\right)\right\rfloor+\left(\tfrac{k}{2}-1\right)t \;\ge\; \tfrac{k-2}{2}d+(2g-2)>2g-2.$$

  Lemma 8.7 and Riemann–Roch imply $\dim S_k(\Gamma)=\ell(\lfloor D\rfloor)=\deg(\lfloor D\rfloor)-g+1$, and we also have $\dim M_k(\Gamma)=\dim S_k(\Gamma)+\ell(\lfloor\operatorname{div}(f)\rfloor)-\ell(\lfloor D\rfloor)=\dim S_k(\Gamma)+t$.

This yields the following theorem.

Theorem 8.8. *Let $\Gamma\in\mathrm{SL}_2(\mathbb{R})$ be a lattice, let $e_1,\ldots,e_r$ be the orders of the elliptic points in $X_\Gamma(\mathbb{C})$, let t be the number of cusps in $X_\Gamma(\mathbb{C})$, let g be the genus of X_Γ and let $k\in 2\mathbb{Z}$. Then*

$$\dim S_k(\Gamma)=\begin{cases}0 & k<0 \text{ or } k=0, t>0\\ 1 & k=t=0\\ g & k=2\\ (k-1)(g-1)+\sum_{i=1}^r\left\lfloor\tfrac{k}{2}\left(1-\tfrac{1}{e_i}\right)\right\rfloor+\left(\tfrac{k}{2}-1\right)t & k>2\end{cases}$$

and

$$\dim M_k(\Gamma)=\begin{cases}0 & k<0\\ 1 & k=0\\ g & k=2, t=0\\ g-1+t & k=2, t>0\\ \dim S_k(\Gamma)+t & k>2\end{cases}$$

For odd $k \neq 1$ a similar series of calculations yields the following result.

Theorem 8.9. *Let $\Gamma \in \mathrm{SL}_2(\mathbb{R})$ be a lattice with $-1 \notin \Gamma$, let $e_1, \ldots, e_r$ be the orders of the elliptic points in $X_\Gamma(\mathbb{C})$, let u and v be the number of regular and irregular cusps in $X_\Gamma(\mathbb{C})$, let g be the genus of X_Γ and let $k \in 1 + 2\mathbb{Z}$. Then*

$$dim S_k(\Gamma = \begin{cases} 0 & k < 0 \\ (k-1)(g-1) + \sum_{i=1}^{r} \left\lfloor \frac{k}{2}\left(1 - \frac{1}{e_i}\right) \right\rfloor + \frac{k-2}{2} u + \frac{k-1}{2} v & k \geq 3 \end{cases}$$

and

$$\dim M_k(\Gamma) = \begin{cases} 0 & k < 0 \\ \dim S_1(\Gamma) + \frac{u}{2} & k = 1 \\ \dim S_k(\Gamma) + \frac{u}{2} & k \geq 3 \end{cases}$$

Proof. See [1, Theorem 2.5.3] □

Remark 8.10. No general formula is known for $\dim S_1(\Gamma)$, even for congruence subgroups.

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.