These notes summarize the material in §2.2–2.3 of [1] presented in lecture.

7.1 Meromorphic differentials on compact Riemann surfaces

Let X be a (connected) Riemann surface with holomorphic atlas $\{\psi_i\colon U_i \xrightarrow{\sim} V_i \subseteq \mathbb{C}\}$ and transition maps $\psi_{ij} := \psi_j \circ \psi_i^{-1}\colon V_{ij} \to V_j$ defined for all $U_i \cap U_j \neq \emptyset$, where $V_{ij} := \psi_i(U_i \cap U_j)$.

Definition 7.1. Let n be an integer. A meromorphic differential ω of degree n on X is (the equivalence class of) a collection of meromorphic functions $\{\omega_i\colon V_i \to \mathbb{C}\}$ such that

$$\omega_i = (\omega_j \circ \psi_{ij})(d\psi_i/d\psi_j)^n$$

on V_{ij} whenever $U_i \cap U_j \neq \emptyset$. Meromorphic differentials $\{\omega_i\}$ and $\{\theta_i\}$ of degree n are equivalent if $\{\omega_i\} \cup \{\theta_i\}$ is also a meromorphic differential of degree n.

Remark 7.2. A meromorphic differential of degree 0 is a meromorphic function $X \to \mathbb{C}$, equivalently, a holomorphic map $X \to \mathbb{P}^1(\mathbb{C})$ that is not identically ∞. Nonconstant meromorphic functions $X \to \mathbb{C}$ are the same thing as nonconstant holomorphic maps $X \to \mathbb{P}^1(\mathbb{C})$.

We note the following, all of which are compatible with equivalence:

- Taking all $\omega_i = 0$ yields the zero differential, which is a meromorphic differential of degree n for all $n \in \mathbb{Z}$.
- If $\omega = \{\omega_i\}$ is a nonzero differential of degree n then $\omega^{-1} := \{1/\omega_i\}$ is a nonzero differential of degree $-n$.
- If ω is a meromorphic differential of degree n then so is $c\omega := \{c\omega_i\}$ for any $c \in \mathbb{C}$.
- If ω and ω' are meromorphic differentials of degree n then so is $\omega + \omega' := \{\omega_i + \omega_i'\}$.
- If ω and ω' are meromorphic differentials of degrees m and n then their product $\omega\omega' := \{\omega_i\omega_i'\}$ is a meromorphic differential of degree $m+n$.

It follows that the set of meromorphic differentials of degree n on X forms a $\mathbb{C}$-vector space $\Omega^n(X)$, and we have a graded algebra of meromorphic differentials

$$\Omega(X) := \bigoplus_{n\in\mathbb{Z}} \Omega^n(X),$$

in which $\Omega^0(X) \simeq \mathbb{C}(X)$ is the field of meromorphic functions on X. In particular, each $\mathbb{C}$-vector space $\Omega^n(X)$ is also an $\Omega^0(X)$-vector space.

From now on we shall assume that X is compact. This implies that X is a smooth projective curve over $\mathbb{C}$ with function field $\mathbb{C}(X) \simeq \Omega^0(X)$ Recall that the categories of Riemann surfaces (with holomorphic maps), smooth projective curves over $\mathbb{C}$ (with dominant morphisms), and function fields of transcendence degree 1 over $\mathbb{C}$ (with field homomorphisms) are equivalent categories. The functor from Riemann surfaces to curves is covariant, while the functor from curves to function fields is contravariant.

For each $f \in \mathbb{C}(X)$ we define a differential $df := \{df_i\}$, where $f_i := f \circ \psi_i^{-1}$, which is zero if and only if f is constant, and $(df)^n \in \Omega^n(X)$ for all $n \in \mathbb{Z}$. Note that $\mathbb{C}(X) \neq \mathbb{C}$ contains a nonconstant f, thus there are nonzero differentials of every degree and $\dim_{\mathbb{C}(X)} \Omega^n(X) > 0$. On the other hand, if we pick a nonzero $\omega \in \Omega^n$, multiplication by ω^{-1} defines an isomorphism $\Omega^n(X) \xrightarrow{\sim} \Omega^0(X)$ of $\mathbb{C}(X)$-vector spaces. Thus for every $n \in \mathbb{Z}$ we have

$$\dim_{\mathbb{C}(X)} \Omega^n(X) = 1.$$

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7.2 Divisors

The divisor group $\operatorname{Div}(X) := \bigoplus_{P\in X(\mathbb{C})}\mathbb{Z}$ of X is the free abelian group generated by $X(\mathbb{C})$:

$$\operatorname{Div}(X) := \left\{ \sum_{P\in X(\mathbb{C})} n_P P : n_P \in \mathbb{Z} \text{ with } n_P = 0 \text{ for all but finitely many } P \right\}.$$

For $D = \sum n_P P \in \operatorname{Div}(X)$, the degree of the divisor D is $\deg(D) := \sum_P n_P$, and the map

$$\begin{aligned} \deg\colon \operatorname{Div}(X) &\to \mathbb{Z} \\ D &\mapsto \deg(D) \end{aligned}$$

is a group homomorphism. For each $f \in \mathbb{C}(X)^\times$ we have a divisor

$$\operatorname{div}(f) := \sum_{P\in X(\mathbb{C})} \operatorname{ord}_P(f)P,$$

where $\operatorname{ord}_P(f)$ is the order of vanishing of f at P. Divisors arising in this way are called principal.

For any $f, g \in \mathbb{C}(X)^\times$ we have $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$ and $\deg(\operatorname{div}(f)) = 0$. The set of principal divisors form a subgroup $\operatorname{Prin}(X) \le \operatorname{Div}^0(X) \le \operatorname{Div}(X)$, where $\operatorname{Div}^0(X) := \ker(\deg)$. We now define the divisor class group

$$\operatorname{Pic}(X) := \operatorname{Div}(X)/\operatorname{Prin}(X).$$

We also define $\operatorname{Pic}^0(X) := \ker(\deg)/\operatorname{Prin}(X)$.

For each $\omega \in \Omega^n(X)$ and $P \in X(\mathbb{C})$ we define the order of vanishing of ω at $P \in U_i$ to be

$$\operatorname{ord}_P(\omega) = \operatorname{ord}_P(\omega_i \circ \psi_i).$$

The transition maps ψ_{ij} are holomorphic and nonzero, so $\operatorname{ord}_P(\omega_i) = \operatorname{ord}_P(\omega_j)$ for $U_i \cap U_j \neq \emptyset$. This implies that $\operatorname{ord}_P(\omega)$ does not depend on the choice of $U_i \ni P$. We now define the divisor

$$\operatorname{div}(\omega) = \sum_{P\in X(\mathbb{C})} \operatorname{ord}_P(\omega)P,$$

which for $\omega \in \Omega^0(X)$ coincides with our previous definition. For any $\omega \in \Omega^m(X)$ and $\omega' \in \Omega^n(X)$ with $\omega, \omega' \neq 0$ we have

$$\operatorname{div}(\omega\omega') = \operatorname{div}(\omega) + \operatorname{div}(\omega').$$

Recall that $\dim_{\mathbb{C}(X)} \Omega^n = 1$. This implies that $\{\operatorname{div}(\omega) : 0 \neq \omega \in \Omega^n(X)\}$ is a divisor class.

We call a divisor $D = \sum_{P\in X} n_P P \in \operatorname{Div}(X)$ positive, or effective, if $n_P \ge 0$ for all $P \in X(\mathbb{C})$, and write $D \ge 0$. For every divisor $D \in \operatorname{Div}(X)$ we have an associated Riemann–Roch space

$$L(D) := \{f \in \mathbb{C}(X) : f = 0 \text{ or } \operatorname{div}(f) + D \ge 0\},$$

which is a finite-dimensional $\mathbb{C}$-vector space, with $L(0) = \mathbb{C}$ and $L(D) = \{0\}$ for $\deg(D) < 0$. We denote its dimension by

$$\ell(D) := \dim_{\mathbb{C}} L(D).$$

Theorem 7.3 (Riemann-Roch)**.** *Let X be a compact Riemann surface of genus g and let $\omega \in \Omega^1(X)$ be a nonzero meromorphic differential. Then for every $D \in \operatorname{Div}(X)$ we have*

$$\ell(D) = \deg(D) - g + 1 + \ell(\operatorname{div}(\omega) - D).$$

Corollary 7.4. *Let X be a compact Riemann surface of genus g. The following hold*

- *For each nonzero $\omega \in \Omega^1(X)$ we have* $\deg(\operatorname{div}(\omega)) = 2g-2$ *and* $\ell(\operatorname{div}(\omega)) = g$.
- *For each nonzero $\omega \in \Omega^n(X)$ we have* $\deg(\operatorname{div}(\omega)) = 2n(g-1)$.
- *For each $D \in \operatorname{Div}(X)$ with* $\deg(D) > 2g-2$ *we have* $\ell(D) = \deg(D) - g + 1$.

Definition 7.5. A meromorphic differential $\omega \in \Omega^n(X)$ is holomorphic if $\omega = 0$ or $\operatorname{div}(\omega) \geq 0$.

Corollary 7.6. *Let X be a compact Riemann surface of genus g. Then* $\dim_{\mathbb{C}} \Omega_0^1(X) = g$.

Proof. Let $\omega_1 \in \Omega^1(X)$ be nonzero. Then $L(\operatorname{div}(\omega_1)) = \{f \in \mathbb{C}(X) : f = 0 \text{ or } \operatorname{div}(f\omega) \geq 0\}$. The isomorphism $\mathbb{C}(X) \xrightarrow{\sim} \Omega^1(X)$ defined by $f \mapsto f\omega_1$ implies

$$L(\operatorname{div}(\omega_1)) \simeq \{\omega \in \Omega^1(X) : \omega = 0 \text{ or } \operatorname{div}(\omega) \geq 0\} = \Omega_0^1(X),$$

and $\dim_{\mathbb{C}} \Omega_0^1(X) = \ell(\operatorname{div}(\omega_1)) = g$. □

7.3 Automorphic forms and differentials

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a lattice and consider the Riemann surface $X_\Gamma := \Gamma\backslash\mathbf{H}_\Gamma^*$. Let $\pi := \pi_\Gamma : \mathbf{H}_\Gamma^* \to X_\Gamma$ be the projection map. Assume $k = 2n$ is even and fix an automorphic form $f \in A_k(\Gamma)$. Then f is meromorphic on $\mathbf{H}_\Gamma^*$ and $f|_k\gamma = f$ for all $\gamma \in \Gamma$, If $k = 0$ then f defines a meromorphic function $\alpha \in \Omega^0(X)$. Our goal is to define a meromorphic differential $\omega_f \in \Omega^n(X)$ for any $k = 2n$, which will coincide with f when $k = 0$. Recall that for any $\gamma \in \Gamma$ and $z \in \mathbf{H}$ we have

$$(f|_k\gamma)(z) = (\det\gamma)^{k/2} j(\gamma,z)^{-k} f(\gamma z) = j(\gamma,z)^{-k} f(\gamma z) = (cz+d)^{-k} f\left(\tfrac{az+b}{cz+d}\right),$$

thus $f(\gamma z) = j(\gamma,z)^k f(z)$.

For each $P \in X_\Gamma(\mathbb{C})$, pick $z_P \in \pi^{-1}(P)$, and an open neighborhood $z_P \in U_P \subseteq \mathbf{H}_\Gamma^*$ for which $\gamma U_P \cap U_P \neq \emptyset$ if and only if $\gamma \in \Gamma_P := \Gamma_{z_P}$ and $\gamma U_P = U_P$ for all $\gamma \in \Gamma_P$, and let $\psi_P : V_P \xrightarrow{\sim} W_P \subseteq \mathbb{C}$ be a chart for $V_P = \pi(U_P)$, with transition maps $\psi_{PQ} := \psi_Q \circ \psi_P^{-1} : W_{PQ} \to W_Q$ defined on $W_{PQ} := \psi_P(V_P \cap V_Q)$ whenever $V_P \cap V_Q \neq \emptyset$.

Let $\alpha = \{\alpha_P : W_P \to \mathbb{C}\}$, and define $f_P := f_{|_{U_P}}$ and $\pi_P := \pi_{|_{U_P}}$ so that $f_P = \alpha_P \circ \psi_P \circ \pi_P$. If $z_P \in \mathbf{H}$ is not a cusp, we define $g_P : U_P \to \mathbb{C}$ via

$$g_P(z) = f_P(z)(d(\psi_P \circ \pi_P(z))/dz)^{-n},$$

so that $g_P = f_P$ when $n = 0$, and we note that

$$(\gamma z)' = \left(\frac{az+b}{cz+d}\right)' = \frac{a(cz+d)-(az+b)c}{(cz+d)^2} = \frac{1}{(cz+d)^2} = j(\gamma,z)^{-2}.$$

For $\gamma \in \Gamma_P$ and $z \in U_P$ we have $\psi_P \circ \pi_P(\gamma z) = \psi_P \circ \pi_P(z)$ thus

$$\begin{aligned} g_p(\gamma z) &= f_P(\gamma z)(d(\psi_p \circ \pi_P(\gamma z))/d(\gamma z))^{-n} \\ &= j(\gamma,z)^k f_P(z)(d(\psi_P \circ \pi_P(z))/(j(\gamma,z)^{-2}dz))^{-n} \\ &= f_P(z)(d(\psi_P \circ \pi_P(z))/dz)^{-n} = g_P(z) \end{aligned}$$

for all $\gamma \in \Gamma_P$. There is thus a meromorphic function $\omega_P : W_P \to \mathbb{C}$ for which $g_P = \omega_P \circ \psi_P \circ \pi_P$, and $\omega_P \circ \psi_P \circ \pi_P(z) = f_P(z)(d(\psi_P \circ \pi_P(z))/dz)^{-n}$.

If $Q \in X_\Gamma(\mathbb{C})$ is not a cusp then $\omega_Q \circ \psi_Q \circ \pi_Q(z) = f_Q(z) d(\psi_Q \circ \pi_Q(z))/dz)^{-n}$ and if $V_P \cap V_Q \neq \emptyset$ then the compatibility of $\alpha_P = \alpha_Q \circ \psi_{PQ}$ implies

$$\omega_P = (\omega_Q \circ \psi_{PQ})(d\psi_P/d\psi_Q)^n. \tag{1}$$

If $z_P \in \mathbf{H}_\Gamma^* - \mathbf{H}$ is a cusp, we let $U_P^0 := U_P - \{z_P\}$ and proceed as above to obtain meromorphic functions g_P on $U_P - \{z_P\}$ and ω_P on $W_P - \{\psi_P(P)\}$ with $g_P = \omega_P \circ \psi_P \circ \pi_P$. Now choose $\sigma \in \mathrm{SL}_2(\mathbb{R})$ so $\sigma z_P = \infty$ so that $\pm\sigma\Gamma_P\sigma^{-1} = \pm\left\langle\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)\right\rangle$ for some $h > 0$, and define $\varphi_P \colon V_P \to W_P$ via

$$\varphi_P \circ \pi(z) = e^{2\pi i \sigma z/h}$$

If we put $c := (2\pi i/h)^{-n}$ then

$$\begin{aligned} g_P(z) &= cf(z)(d(\sigma z)/dz)^{-n}(\varphi_P \circ \pi(z))^{-n}\\ &= cf(z)j(\sigma,z)^{2n}(\varphi_P \circ \pi(z))^{-n}\\ &= c(f|_k\sigma^{-1})(\sigma z)(\varphi_P \circ \pi(z))^{-n} \end{aligned}$$

is meromorphic at z_P, and ω_P is meromorphic at $\psi_P(P)$. Moreover if $Q \in X_\Gamma(\mathbb{C})$ with $V_P \cap V_Q \neq \emptyset$ then (1) holds as above.

It follows that $\{\omega_P\}$ is a differential $\omega_f \in \Omega^n(X_\Gamma)$, and we have

- $\omega_{fg} = \omega_f \omega_g$ for all $f \in A_{2m}(\Gamma)$ and $g \in A_{2n}(\Gamma)$;
- $\omega_{f+g} = \omega_f + \omega_g$ for all $f, g \in A_{2m}(\Gamma)$.

Conversely, for each $\omega = \{\omega_i\} \in \Omega^n(X_\Gamma)$ we can define an automorphic form

$$f(z) = \psi_i(\pi(z))(d(\psi_i \circ \pi(z))/dz)^n \in A_{2n}(\Gamma)$$

for which $\omega = \omega_f$. This yields the following theorem.

Theorem 7.7. *Let $A(\Gamma)_{\mathrm{even}} = \bigoplus_{n\in\mathbb{Z}} A_{2n}(\Gamma)$. Then $A(\Gamma)_{\mathrm{even}}$ is isomorphic to $\Omega(X_\Gamma)$ as a graded $\mathbb{C}$-algebra via the isomorphism $f \mapsto \omega_f$. In particular, $A_{2n}(\Gamma) \neq \{0\}$.*

Let $f \in A_{2n}(\Gamma)$ be nonzero and choose $P \in X_\Gamma(\mathbb{Z})$, $z_P \in \pi^{-1}(P)$, and $z_P \in U_P \subseteq \mathbf{H}_\Gamma^*$ as above. For $z_P \in \mathbf{H}$ choose $\rho \in \mathrm{SL}_2(\mathbb{C})$ so that $\rho\mathbf{H} = \mathbf{D}$ and $\rho z_P = 0$. Choose $\psi_P \colon V_P \xrightarrow{\sim} W_P \subseteq \mathbb{D}$ so that $\psi_P \circ \pi(z) = (\rho z)^e$ for all $z \in U_P$, where e is the ramification index at P. Then

$$g_P = f(z)(d(\rho z)/dz)^{-n}(e(\rho z)^{e-1})^{-n}$$

and if we put $w = \rho z$ we have

$$g_P \circ \rho^{-1}(w) = e^{-m} f(\rho^{-1}w)(d(\rho^{-1}w)/dw)^n w^{-n(e-1)}$$

which implies that

$$\mathrm{ord}_w(g_P \circ \rho^{-1}) = \mathrm{ord}_w(f(\rho^{-1}w)(d(\rho^{-1}w)/dw)^n) - n(e-1).$$

Now $g_P \circ \rho^{-1}(w) = w^e$, so

$$\mathrm{ord}_w(g_P \circ \rho^{-1}) = e\,\mathrm{ord}_P(g_P) = e\,\mathrm{ord}_P(\omega_f).$$

We also have

$$\mathrm{ord}_w(f(\rho^{-1}w)(d(\rho^{-1}w)/dw)^n) = \mathrm{ord}_{z_0}(f),$$

since $d(\rho z)/dz$ has neither a zero or pole at z_0. We thus define

$$\mathrm{ord}_P(f) := \frac{1}{e}\mathrm{ord}_{z_0}(f),$$

and we then have

$$\mathrm{ord}_P(\omega_f) = \mathrm{ord}_P(f) - n(1 - 1/e).$$

which shows that the definition of $\mathrm{ord}_P(f)$ does not depend on the choice of z_0 or ρ. When $n = 1$ and $z_0 \in \mathbf{H}$ we have

$$\omega_f \text{ is holomorphic at } P \Leftrightarrow \mathrm{ord}_P(\omega_F) \geq 0 \Leftrightarrow \mathrm{ord}_P(f) \geq 0 \Leftrightarrow f \text{ is holomorphic at } z_0.$$

For z_0 a cusp of Γ we choose $\sigma \in \mathrm{SL}_2(\mathbb{R})$ so $\sigma z_0 = \infty$ and define $h > 0$ as above so that

$$(f|_{2n}\sigma^{-1})(\sigma z) = \sum_{m \geq m_0} a_m (g_P(z))^m,$$

with m_0 maximal. We have $\mathrm{ord}_P(g_P) = m_0 - n$ and define $\mathrm{ord}_P(f) := m_0$ so that we have $\mathrm{ord}_P(\omega_f) = \mathrm{ord}_P(f) - n$. When $P = \pi(z_0)$ is a cusp and $n = 1$ we have

$$\omega_f \text{ is holomorphic at } P \Leftrightarrow \mathrm{ord}_P(\omega_F) \geq 0 \Leftrightarrow \mathrm{ord}_P(f) \geq 1 \Leftrightarrow f \text{ vanishes at } z_0.$$

Theorem 7.8. *The correspondence $f \mapsto \omega_f$ induces an isomorphism $S_2(\Gamma) \simeq \Omega_0^1(X_\Gamma)$.*

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.

[2] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Springer, 2005.

[3] John Voight, *Quaternion algebras*, Springer, 2021.