These notes summarize the material in §2.1 of [1] presented in lecture.

6.1 Automorphic forms

Definition 6.1. The automorphy factor $j\colon \mathrm{GL}_2^+(\mathbb{R})\times \mathbf{H}\to\mathbb{C}$ is defined by

$$j(\gamma,z) := cz+d, \qquad \text{where } \gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right),$$

in other words, the denominator of $\gamma z = \frac{az+b}{cz+d}$.

Lemma 6.2 (cocycle relation)**.** *For all $\gamma_1,\gamma_2\in \mathrm{GL}_2^+(\mathbb{R})$ and $z\in\mathbf{H}$ we have*

$$j(\gamma_1\gamma_2,z) = j(\gamma_1,\gamma_2 z)j(\gamma_2,z)$$

Proof. Check. □

Definition 6.3. Let $k\in\mathbb{Z}_{\ge 0}$ and $\gamma\in \mathrm{GL}_2^+(\mathbb{R})$. The weight-$k$ slash operator for γ acts on functions $f:\mathbf{H}\to\mathbb{C}$ via

$$(f|_k\gamma)(z) := (\det\gamma)^{k/2} j(\gamma,z)^{-k} f(\gamma z).$$

We have $(f+g)|_k\gamma = f|_k\gamma + g|_k\gamma$ and $(cf)|_k\gamma = c(f|_k\gamma)$ for all $\gamma\in \mathrm{GL}_2^+(\mathbb{R})$, $f,g\colon\mathbf{H}\to\mathbb{C}$, and $a\in\mathbb{C}$, thus the map $f\mapsto f|_k\gamma$ is a linear operator on the vector space of functions $\mathbf{H}\to\mathbb{C}$.

Moreover, for every $\gamma_1,\gamma_2\in \mathrm{GL}_2^+(\mathbb{R})$ we have

$$f|_k(\gamma_1\gamma_2) = (f|_k\gamma_1)|_k\gamma_2.$$

In other words, for each $k\in\mathbb{Z}$ the weight-k slash operator defines a linear right group action of $\mathrm{GL}_2^+(\mathbb{R})$ on the vector space of functions $f:\mathbf{H}\to\mathbb{C}$.

We can omit the factor $(\det\gamma)^{k/2}$ if we only care about $\gamma\in \mathrm{SL}_2(\mathbb{R})$. For scalars $a\in\mathbb{R}^\times$ we have $f|_k a = \operatorname{sgn}(a)^k f$ (so scalars, including -1, act trivially only when k is even).

Definition 6.4. Let Γ be a lattice in $\mathrm{SL}_2(\mathbb{R})$ (equivalently, a finitely generated Fuchsian group of the first kind), and let $k\in\mathbb{Z}$. We say that a meromorphic function $f:\mathbf{H}\to\mathbb{C}$ is an automorphic function of weight k with respect to Γ if for every $\gamma\in\Gamma$ we have

$$f|_k\gamma = f.$$

The $\mathbb{C}$-vector space of automorphic functions of weight k for Γ is denoted $\Omega_k(\Gamma)$.

We note the following (easily checked) facts:

- If k is odd and $-1\in\Gamma$ then $\Omega_k(\Gamma)=\{0\}$, since $f_k|-1=-f$ when k is odd.
- For lattices $\Gamma'\subseteq\Gamma$ we have $\Omega_k(\Gamma)\subseteq\Omega_k(\Gamma')$ (inclusions are reversed).
- If $f\in\Omega_k(\Gamma)$ and $g\in\Omega_l(\Gamma)$ then $fg\in\Omega_{k+\ell}(\Gamma)$.
- For $f\in\Omega_k(\Gamma)$ and $\alpha\in \mathrm{GL}_2^+(\mathbb{R})$ we have $f|_k\alpha\in\Omega_k(\alpha^{-1}\Gamma\alpha)$.

Recall that a graded ring is a ring R whose additive group is an internal direct sum

$$R = \bigoplus_{i=I} R_i$$

indexed by a monoid I (typically $\mathbb{Z}$ or $\mathbb{Z}_{\ge0}$) such that $R_iR_j\subseteq R_{i+j}$ for all $i,j\in I$. Note that this means each R_i is an R_0-module.

Thus $\Omega(\Gamma) := \bigoplus_{k\in\mathbb{Z}} \Omega_k(\Gamma)$ is a graded ring and each $\Omega_k(\Gamma)$ is an $\Omega_0(\Gamma)$-module (this generalizes the $\mathbb{C}$-vector space structure of $\Omega_k(\Gamma)$, which can be viewed as multiplication by constant functions $\mathbb{C} \subseteq \Omega_0(\Gamma)$).

Suppose Γ has a cusp x with stabilizer Γ_x, and pick $\sigma \in \mathrm{SL}_2(\mathbb{R})$ so that $\sigma x = \infty$. Then

$$\pm\sigma\Gamma_x\sigma^{-1} = \pm\left\langle\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)\right\rangle$$

for some $h > 0$. Recall that a cusp x with $\sigma x = \infty$ is regular if $\left(\begin{smallmatrix}1 & \pm h\\ 0 & 1\end{smallmatrix}\right) \in \sigma\Gamma_x\sigma^{-1}$ and irregular otherwise (in which case $\left(\begin{smallmatrix}-1 & \pm h\\ 0 & -1\end{smallmatrix}\right) \in \sigma\Gamma_x\sigma^{-1}$). For $f \in \Omega_k(\Gamma)$ we have $f|_k\sigma^{-1} \in \Omega_k(\sigma\Gamma\sigma^{-1})$, and if x is regular this implies that

$$(f|_k\sigma^{-1})(z+h) = \left((f|_k\sigma^{-1})|_k\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)\right)(z) = (f|_k\sigma^{-1})(z)$$

for all $z \in \mathbf{H}$; this also holds when x is irregular and k is even, since $f|_k\sigma^{-1}|_k - 1 = f|_k\sigma^{-1}$ for even k. There is then a function $g\colon \mathbb{D}_0 \to \mathbb{C}$ (where $\mathbb{D}_0 := \{z \in \mathbb{C}^\times : |z| < 1\}$) such that

$$(f|_k\sigma^{-1})(z) = g(e^{2\pi iz/h})$$

for all $z \in \mathbf{H}$. We say $f \in \Omega_k(\Gamma)$ is meromorphic/holomorphic/vanishing at x if g is meromorphic/holomorphic/vanishing at 0. For x irregular and k odd we adopt the same terminology, using the g for f^2 (which has even weight $2k$). This definition does not depend on σ.

If $g(w) = \sum_{n\ge n_0} a_n w^n$ is the Laurent series expansion of g in a neighborhood about 0, then

$$(f|_k\sigma^{-1})(x) = \sum_{n\ge n_0} a_n e^{2\pi iz/h}$$

on some neighborhood $U_l := \{z \in \mathbf{H} : \mathrm{Im}(z) > l\}$ of ∞. We thus have

$$(f|_k\sigma^{-1})(z+h) = \begin{cases}(f|_k\sigma^{-1})(z) & \text{if } k \text{ is even or } x \text{ is regular,}\\ -(f|_k\sigma^{-1})(z) & \text{if } k \text{ is odd and } x \text{ is irregular,}\end{cases}$$

which yields the Fourier expansion of f at the cusp x:

$$(f|_k\sigma^{-1})(z) = \begin{cases}\sum_{\text{even } n\ge 0} a_n e^{\pi inz/h} & \text{if } k \text{ is even or } x \text{ is regular,}\\ \sum_{\text{odd } n\ge 0} a_n e^{\pi inz/h} & \text{if } k \text{ is odd and } x \text{ is irregular.}\end{cases}$$

Writing this in terms of $q := e^{2\pi inz}$ yields the q-expansion of f at the cusp x.

Definition 6.5. We call an automorphic function $f \in \Omega_k(\Gamma)$

- an automorphic form (of weight k for Γ) if f is meromorphic at every cusp of Γ;
- a modular form (of weight k for Γ) if f is holomorphic on $\mathbf{H}$ and at every cusp of Γ;
- a cusp form (of weight k for Γ) if f is holomorphic on $\mathbf{H}$ and vanishes at every cusp of Γ;

The $\mathbb{C}$-vector spaces of automorphic/modular/cusp forms of weight k for Γ are denoted $A_k(\Gamma)$, $M_k(\Gamma)$, $S_k(\Gamma)$, respectively. For lattices $\Gamma' \subseteq \Gamma$ of finite index we get corresponding inclusions in the reverse directions (when Γ has cusps this is not immediate, see [1, Lemma 2.1.3] for a proof), and we always have the inclusions

$$S_k(\Gamma) \subseteq M_k(\Gamma) \subseteq A_k(\Gamma) \subseteq \Omega_k(\Gamma).$$

If Γ has no cusps then $\Omega_k(\Gamma) = A_k(\Gamma)$ and $M_k(\Gamma) = S_k(\Gamma)$, and for any $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ the map $f \mapsto f|_k\alpha$ induces isomorphisms

$$A_k(\Gamma) \simeq A_k(\Gamma^\alpha), \qquad M_k(\Gamma) \simeq M_k(\Gamma^\alpha), \qquad S_k(\Gamma) \simeq S_k(\Gamma^\alpha),$$

where $\Gamma^\alpha := \alpha^{-1}\Gamma\alpha$. The product of two automorphic/modular/cusp forms of weights k and l is an automorphic/modular/cusp of weight $k+l$ and we thus have graded rings

$$A(\Gamma) = \bigoplus_{k\in\mathbb{Z}} A_k(\Gamma), \qquad M(\Gamma) = \bigoplus_{k\in\mathbb{Z}} M_k(\Gamma), \qquad S(\Gamma) = \bigoplus_{k\in\mathbb{Z}} S_k(\Gamma).$$

The ratio of automorphic forms of weight k and l is an automorphic form of weight $k-l$, thus $A(\Gamma)$ is a field, as is its subspace $A_0(\Gamma)$, and all the $A_k(\Gamma)$ are $A_0(\Gamma)$-vector spaces. The field $A_0(\Gamma)$ is isomorphic to the field of meromorphic functions on the Riemann surface $X_\Gamma := \Gamma\backslash\mathbf{H}_\Gamma^*$.

Theorem 6.6. *Let $f \in \Omega_k(\Gamma)$ be holomorphic on $\mathbf{H}$. If $f(z) = O(\mathrm{Im}(z)^{-\nu})$ as $\mathrm{Im}(z) \to 0$, uniformly with respect to $\mathrm{Re}(z)$, for some $\nu > 0$ then $f \in M_k(\Gamma)$, and if this holds with $\nu < k$ then $f \in S_k(\Gamma)$.*

Proof. This is [1, Thm. 2.1.4]. □

Theorem 6.7. *Let $f \in \Omega_k(\Gamma)$. Then $f \in S_k(\Gamma)$ if and only if $f(z)\,\mathrm{Im}(z)^{k/2}$ is bounded on $\mathbf{H}$.*

Proof. This is [1, Thm. 2.1.5]. □

Corollary 6.8. *Let $\sum_{n\geq 1} a_n e^{\pi i n z/h}$ be the Fourier expansion of a cusp form $f \in S_k(\Gamma)$ at a cusp of Γ. Then $a_n = O(n^{k/2})$.*

Remark 6.9. The bound in Corollary 6.8 can be improved to $a_n = O(n^{(k-1)/2})$ when Γ is a congruence subgroup [1, Thm. 4.5.17].

6.2 Automorphic forms with character

Definition 6.10. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a lattice. A character of Γ is a group homomorphism $\chi\colon \Gamma \to \mathbb{C}^\times$ with finite image. This implies $\chi(\Gamma) = \langle \zeta_n \rangle$, where $\zeta_n = e^{2\pi i/n}$; n is the order of χ.

Definition 6.11. a Dirichlet character ε is a periodic totally multiplicative function $\mathbb{Z} \to \mathbb{C}$, equivalently, the extension by zero of a character of $(\mathbb{Z}/N\mathbb{Z})^\times$ for some modulus N.

Dirichlet characters ε are commonly used to define characters of $\Gamma_0(N)$ via

$$\chi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) := \varepsilon(d).$$

Definition 6.12. Let χ be a character of a lattice $\Gamma \in \mathrm{SL}_2(\mathbb{R})$. An automorphic function $f \in \Omega_k(\Gamma_\chi)$ which satisfies

$$f|_k\gamma = \chi(\gamma)f$$

for all $\gamma \in \Gamma$ is an automorphic function for Γ with character χ, and we use $\Omega_k(\Gamma, chi)$ to denote the $\mathbb{C}$-vector space of all such functions. We similarly define the $\mathbb{C}$-vector spaces

$$A(\Gamma,\chi) := A(\Gamma_\chi) \cap \Omega_k(\Gamma,\chi), \qquad M(\Gamma,\chi) := M(\Gamma_\chi) \cap \Omega_k(\Gamma,\chi), \qquad S(\Gamma,\chi) := S(\Gamma_\chi) \cap \Omega_k(\Gamma,\chi),$$

of automorphic/modular/cusp forms for Γ with character χ.

Note that Γ_χ has finite index in Γ, hence the same cusps as Γ (see [1, Cor. 1.5.5]), so automorphic forms in $A(\Gamma,\chi)$ are meromorphic at the cusps of Γ, modular forms in $M(\Gamma,\chi)$ are holomorphic at the cusps of Γ, and cusp forms in $S(\Gamma,\chi)$ vanish at the cusps of Γ.

For any finite index $\Gamma' \le \Gamma$ we have corresponding inclusions

$$A(\Gamma,\chi) \subseteq A(\Gamma',\chi'), \qquad M(\Gamma,\chi) \subseteq M(\Gamma',\chi'), \qquad S(\Gamma,\chi) \subseteq S(\Gamma',\chi')$$

where χ' is the restriction of χ to Γ'. If $f \in A_k(\Gamma,\chi)$ and $g \in A_l(\Gamma,\psi)$ then $fg \in A_{k+l}(\Gamma,\chi\psi)$, and similarly for M_k and S_k. The product of a modular form and a cusp form is a cusp form, so we also have

$$M_k(\Gamma,\chi)S_l(\Gamma,\psi) \subseteq S_{k+l}(\Gamma,\chi\psi).$$

for any lattice $\Gamma \in \mathrm{SL}_2(\mathbb{R})$ with characters χ,ψ.

Example 6.13. Let $N \in \mathbb{Z}_{>0}$. As we will prove in a later lecture, for the congruence subgroups $\Gamma_0(N), \Gamma_1(N) \le \mathrm{SL}_2(\mathbb{Z})$, we have the decompositions

$$M_k(\Gamma_1(N)) = \bigoplus_\chi M_k(\Gamma_0(N),\chi) \qquad \text{and} \qquad S_k(\Gamma_1(N)) = \bigoplus_\chi S_k(\Gamma_0(N),\chi),$$

where the direct sums range over Dirichlet characters of modulus N.

6.3 The Petersson inner product

Let χ be a character for a lattice $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ and let $f,g \in M_k(\Gamma,\chi)$ with at least one in $S_k(\Gamma,\chi)$. Then $fg \in S_{2k}(\Gamma,\chi^2)$ and for any $\gamma \in \Gamma$ we have

$$f(\gamma z)\overline{g}(\gamma z)\,\mathrm{Im}(\gamma z)^k = f(z)\overline{g}(z)\,\mathrm{Im}(z)^k,$$

where $\overline{g}(z) := \overline{g(z)}$ denotes complex conjugation. By Theorem 6.7, $|f(z)g(z)|\,\mathrm{Im}(z)^k$ is bounded on $\mathbf{H}$, thus the integral

$$\int_{\Gamma\backslash\mathbf{H}} f(z)\overline{g}(z)\,\mathrm{Im}(z)^k dv(z)$$

is converges. Here $dv(z) := \frac{dx\,dy}{y^2}$ is the hyperbolic measure on $\mathbf{H}$ (with $z = x + iy$). We define

$$v(\Gamma\backslash\mathbf{H}^*_\Gamma) := v(\Gamma\backslash\mathbf{H}) := v(F)$$

where F is any measurable fundamental domain for Γ. The fact that Γ is a lattice (a finitely generated Fuchsian group of the first kind) implies that such an F exists (indeed, we can use any Dirichlet domain for Γ), and that $v(F) \in \mathbb{R}_{>0}$ is finite and independent of the choice of F.

We now define the Petersson inner product of f and g via

$$\langle f,g\rangle := \frac{1}{v(\Gamma\backslash\mathbf{H})}\int_{\Gamma\backslash\mathbf{H}} f(z)\overline{g}(z)^k dv(z) \in \mathbb{C},$$

which induces a positive definite Hermitian inner product on the $\mathbb{C}$-vector space $S_k(\Gamma,\chi)$, making $S_k(\Gamma,\chi)$ into a Hermitian space.

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.

[2] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Springer, 2005.

[3] John Voight, *Quaternion algebras*, Springer, 2021.