These notes summarize the material in §2.1 of [1] presented in lecture.

6.1 Automorphic forms

Definition 6.1. The automorphy factor $j: \text{GL}_2^+(\mathbb{R}) \times \mathbb{H} \to \mathbb{C}$ is defined by

$$j(\gamma, z) := cz + d, \quad \text{where} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

in other words, the denominator of $\gamma z = \frac{az + b}{cz + d}$.

Lemma 6.2 (cocycle relation). For all $\gamma_1, \gamma_2 \in \text{GL}_2^+(\mathbb{R})$ and $z \in \mathbb{H}$ we have

$$j(\gamma_1 \gamma_2, z) = j(\gamma_1, \gamma_2 z) j(\gamma_2, z)$$

Proof. Check. \hfill \Box

Definition 6.3. Let $k \in \mathbb{Z}_{\geq 0}$ and $\gamma \in \text{GL}_2^+(\mathbb{R})$. The weight-$k$ slash operator for $\gamma$ acts on functions $f: \mathbb{H} \to \mathbb{C}$ via

$$(f \mid_k \gamma)(z) := (\det \gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma z).$$

We have $(f + g) \mid_k \gamma = f \mid_k \gamma + g \mid_k \gamma$ and $(cf) \mid_k \gamma = c(f \mid_k \gamma)$ for all $\gamma \in \text{GL}_2^+(\mathbb{R})$, $f, g: \mathbb{H} \to \mathbb{C}$, and $a \in \mathbb{C}$, thus the map $f \to f \mid_k \gamma$ is a linear operator on the vector space of functions $\mathbb{H} \to \mathbb{C}$.

Moreover, for every $\gamma_1, \gamma_2 \in \text{GL}_2^+(\mathbb{R})$ we have

$$f \mid_k (\gamma_1 \gamma_2) = (f \mid_k \gamma_1) \mid_k \gamma_2.$$

In other words, for each $k \in \mathbb{Z}$ the weight-$k$ slash operator defines a linear right group action of $\text{GL}_2^+(\mathbb{R})$ on the vector space of functions $f: \mathbb{H} \to \mathbb{C}$.

We can omit the factor $(\det \gamma)^{k/2}$ if we only care about $\gamma \in \text{SL}_2(\mathbb{R})$. For scalars $a \in \mathbb{R}^\times$ we have $f \mid_k a = \text{sgn}(a)^k f$ (so scalars, including $-1$, act trivially only when $k$ is even).

Definition 6.4. Let $\Gamma$ be a lattice in $\text{SL}_2(\mathbb{R})$ (equivalently, a finitely generated Fuchsian group of the first kind), and let $k \in \mathbb{Z}$. We say that a meromorphic function $f: \mathbb{H} \to \mathbb{C}$ is an automorphic function of weight $k$ with respect to $\Gamma$ if for every $\gamma \in \Gamma$ we have

$$f \mid_k \gamma = f.$$

The $\mathbb{C}$-vector space of automorphic functions of weight $k$ for $\Gamma$ is denoted $\Omega_k(\Gamma)$.

We note the following (easily checked) facts:

- If $k$ is odd and $-1 \in \Gamma$ then $\Omega_k(\Gamma) = \{0\}$, since $f_k \mid -1 = -f$ when $k$ is odd.
- For lattices $\Gamma' \subseteq \Gamma$ we have $\Omega_k(\Gamma) \subseteq \Omega_k(\Gamma')$ (inclusions are reversed).
- If $f \in \Omega_k(\Gamma)$ and $g \in \Omega_l(\Gamma)$ then $f g \in \Omega_{k+l}(\Gamma)$.
- For $f \in \Omega_k(\Gamma)$ and $\alpha \in \text{GL}_2^+(\mathbb{R})$ we have $f \mid_k \alpha \in \Omega_k(\alpha^{-1} \Gamma \alpha)$.

Recall that a graded ring is a ring $R$ whose additive group is an internal direct sum

$$R = \bigoplus_{i=1}^\infty R_i$$

indexed by a monoid $I$ (typically $\mathbb{Z}$ or $\mathbb{Z}_{\geq 0}$) such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in I$. Note that this means each $R_i$ is an $R_0$-module.
Thus $\Omega(\Gamma) := \bigoplus_{k \in \mathbb{Z}} \Omega_k(\Gamma)$ is a graded ring and each $\Omega_k(\Gamma)$ is an $\Omega_0(\Gamma)$-module (this generalizes the $\mathbb{C}$-vector space structure of $\Omega_k(\Gamma)$, which can be viewed as multiplication by constant functions $\mathbb{C} \subseteq \Omega_0(\Gamma)$).

Suppose $\Gamma$ has a cusp $x$ with stabilizer $\Gamma_x$, and pick $\sigma \in \text{SL}_2(\mathbb{R})$ so that $\sigma x = \infty$. Then

$$\pm \sigma \Gamma_x \sigma^{-1} = \pm \left( \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right)$$

for some $h > 0$. Recall that a cusp $x$ with $\sigma x = \infty$ is regular if $\left( \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right) \in \sigma \Gamma_x \sigma^{-1}$ and irregular otherwise (in which case $\left( \begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix} \right) \in \sigma \Gamma_x \sigma^{-1}$). For $f \in \Omega_k(\Gamma)$ we have $f|_k \sigma^{-1} \in \Omega_k(\sigma \Gamma \sigma^{-1})$, and if $x$ is regular this implies that

$$(f|_k \sigma^{-1})(z + h) = \left( f|_k \sigma^{-1} \right) \left( \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \right) = (f|_k \sigma^{-1})(z)$$

for all $z \in \mathbb{H}$; this also holds when $x$ is irregular and $k$ is even, since $f|_k \sigma^{-1}|_k = f|_k \sigma^{-1}$ for even $k$. There is then a function $g : \mathbb{D}_0 \to \mathbb{C}$ (where $\mathbb{D}_0 := \{ z \in \mathbb{C}^\times : |z| < 1 \}$) such that

$$(f|_k \sigma^{-1})(z) = g(e^{2\pi iz/h})$$

for all $z \in \mathbb{H}$. We say $f \in \Omega_k(\Gamma)$ is meromorphic/holomorphic/vanishing at $x$ if $g$ is meromorphic/holomorphic/vanishing at $0$. For $x$ irregular and $k$ odd we adopt the same terminology, using the $g$ for $f^2$ (which has even weight $2k$). This definition does not depend on $\sigma$.

If $g(w) = \sum_{n \geq n_0} a_n w^n$ is the Laurent series expansion of $g$ in a neighborhood about 0, then

$$(f|_k \sigma^{-1})(x) = \sum_{n \geq n_0} a_n e^{2\pi iz/h}$$

on some neighborhood $U_l := \{ z \in \mathbb{H} : \text{Im}(z) > l \}$ of $\infty$. We thus have

$$\begin{cases} (f|_k \sigma^{-1})(z) & \text{if } k \text{ is even or } x \text{ is regular}, \\ -(f|_k \sigma^{-1})(z) & \text{if } k \text{ is odd and } x \text{ is irregular}, \end{cases}$$

which yields the Fourier expansion of $f$ at the cusp $x$:

$$(f|_k \sigma^{-1})(z) = \begin{cases} \sum_{\text{even } n \geq n_0} a_n e^{\pi n z/h} & \text{if } k \text{ is even or } x \text{ is regular}, \\ \sum_{\text{odd } n \geq n_0} a_n e^{\pi n z/h} & \text{if } k \text{ is odd and } x \text{ is irregular}. \end{cases}$$

Writing this in terms of $q := e^{2\pi iz}$ yields the $q$-expansion of $f$ at the cusp $x$.

**Definition 6.5.** We call an automorphic function $f \in \Omega_k(\Gamma)$

- an **automorphic form** (of weight $k$ for $\Gamma$) if $f$ is meromorphic at every cusp of $\Gamma$;
- a **modular form** (of weight $k$ for $\Gamma$) if $f$ is holomorphic on $\mathbb{H}$ and at every cusp of $\Gamma$;
- a **cusp form** (of weight $k$ for $\Gamma$) if $f$ is holomorphic on $\mathbb{H}$ and vanishes at every cusp of $\Gamma$;

The $\mathbb{C}$-vector spaces of automorphic/modular/cusp forms of weight $k$ for $\Gamma$ are denoted $A_k(\Gamma)$, $M_k(\Gamma)$, $S_k(\Gamma)$, respectively. For lattices $\Gamma' \subseteq \Gamma$ of finite index we get corresponding inclusions in the reverse directions (when $\Gamma$ has cusps this is not immediate, see [1, Lemma 2.1.3] for a proof), and we always have the inclusions

$$S_k(\Gamma) \subseteq M_k(\Gamma) \subseteq A_k(\Gamma) \subseteq \Omega_k(\Gamma).$$
If $\Gamma$ has no cusps then $\Omega_k(\Gamma) = A_k(\Gamma)$ and $M_k(\Gamma) = S_k(\Gamma)$, and for any $\alpha \in \text{GL}_2^+(\mathbb{R})$ the map $f \mapsto f|_{\kappa} \alpha$ induces isomorphisms

$$A_k(\Gamma) \cong A_k(\Gamma^\alpha), \quad M_k(\Gamma) \cong M_k(\Gamma^\alpha), \quad S_k(\Gamma) \cong S_k(\Gamma^\alpha),$$

where $\Gamma^\alpha := \alpha^{-1} \Gamma \alpha$. The product of two automorphic/modular/cusp forms of weights $k$ and $l$ is an automorphic/modular/cusp of weight $k + l$ and we thus have graded rings

$$A(\Gamma) = \bigoplus_{k \in \mathbb{Z}} A_k(\Gamma), \quad M(\Gamma) = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma), \quad S(\Gamma) = \bigoplus_{k \in \mathbb{Z}} S_k(\Gamma).$$

The ratio of automorphic forms of weight $k$ and $l$ is an automorphic form of weight $k - l$, thus $A(\Gamma)$ is a field, as is its subspace $A_0(\Gamma)$, and all the $A_k(\Gamma)$ are $A_0(\Gamma)$-vector spaces. The field $A_0(\Gamma)$ is isomorphic to the field of meromorphic functions on the Riemann surface $X$.

**Theorem 6.6.** Let $f \in \Omega_k(\Gamma)$ be holomorphic on $H$. If $f(z) = O(\text{Im}(z)^{-\nu})$ as $\text{Im}(z) \to 0$, uniformly with respect to $\text{Re}(z)$, for some $\nu > 0$ then $f \in M_k(\Gamma)$, and if this holds with $\nu < k$ then $f \in S_k(\Gamma)$.

**Proof.** This is [1, Thm. 2.1.4].

**Theorem 6.7.** Let $f \in \Omega_k(\Gamma)$. Then $f \in S_k(\Gamma)$ if and only if $f(z) \text{Im}(z)^{k/2}$ is bounded on $H$.

**Proof.** This is [1, Thm. 2.1.5].

**Corollary 6.8.** Let $\sum_{n \geq 1} a_n e^{\pi i n z / h}$ be the Fourier expansion of a cusp form $f \in S_k(\Gamma)$ at a cusp of $\Gamma$. Then $a_n = O(n^{k/2})$.

**Remark 6.9.** The bound in Corollary 6.8 can be improved to $a_n = O(n^{(k-1)/2})$ when $\Gamma$ is a congruence subgroup [1, Thm. 4.5.17].

### 6.2 Automorphic forms with character

**Definition 6.10.** Let $\Gamma \leq \text{SL}_2(\mathbb{R})$ be a lattice. A character of $\Gamma$ is a group homomorphism $\chi : \Gamma \to \mathbb{C}^\times$ with finite image. This implies $\chi(\Gamma) = \{\zeta_n\}$, where $\zeta_n = e^{2\pi i / n}$, $n$ is the order of $\chi$.

**Definition 6.11.** A Dirichlet character $\varepsilon$ is a periodic totally multiplicative function $\mathbb{Z} \to \mathbb{C}$, equivalently, the extension by zero of a character of $(\mathbb{Z}/N\mathbb{Z})^\times$ for some modulus $N$.

Dirichlet characters $\varepsilon$ are commonly used to define characters of $\Omega_0(N)$ via

$$\chi \left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) := \varepsilon(d).$$

**Definition 6.12.** Let $\chi$ be a character of a lattice $\Gamma \leq \text{SL}_2(\mathbb{R})$. An automorphic function $f \in \Omega_k(\Gamma, \chi)$ which satisfies

$$f|_{\kappa} \gamma = \chi(\gamma) f$$

for all $\gamma \in \Gamma$ is an automorphic function for $\Gamma$ with character $\chi$, and we use $\Omega_k(\Gamma, \chi)$ to denote the $\mathbb{C}$-vector space of all such functions. We similarly define the $\mathbb{C}$-vector spaces

$$A(\Gamma, \chi) := A(\Gamma, \chi) \cap \Omega_k(\Gamma, \chi), \quad M(\Gamma, \chi) := M(\Gamma, \chi) \cap \Omega_k(\Gamma, \chi), \quad S(\Gamma, \chi) := S(\Gamma, \chi) \cap \Omega_k(\Gamma, \chi),$$

of automorphic/modular/cusp forms for $\Gamma$ with character $\chi$. 

18.786 Spring 2024, Lecture #6, Page 3
Note that $\Gamma_\chi$ has finite index in $\Gamma$, hence the same cusps as $\Gamma$ (see [1, Cor. 1.5.5]), so automorphic forms in $A(\Gamma, \chi)$ are meromorphic at the cusps of $\Gamma$, modular forms in $M(\Gamma, \chi)$ are holomorphic at the cusps of $\Gamma$, and cusp forms in $S(\Gamma, \chi)$ vanish at the cusps of $\Gamma$.

For any finite index $\Gamma' \leq \Gamma$ we have corresponding inclusions

$$A(\Gamma, \chi) \subseteq A(\Gamma', \chi'), \quad M(\Gamma, \chi) \subseteq M(\Gamma', \chi'), \quad S(\Gamma, \chi) \subseteq S(\Gamma', \chi')$$

where $\chi'$ is the restriction of $\chi$ to $\Gamma'$. If $f \in A_k(\Gamma, \chi)$ and $g \in A_l(\Gamma, \psi)$ then $fg \in A_{k+l}(\Gamma, \chi \psi)$, and similarly for $M_k$ and $S_k$. The product of a modular form and a cusp form is a cusp form, so we also have

$$M_k(\Gamma, \chi)S_l(\Gamma, \psi) \subseteq S_{k+l}(\Gamma, \chi \psi).$$

for any lattice $\Gamma \in \text{SL}_2(\mathbb{R})$ with characters $\chi, \psi$.

**Example 6.13.** Let $N \in \mathbb{Z}_{>0}$. As we will prove in a later lecture, for the congruence subgroups $\Gamma_0(N), \Gamma_1(N) \leq \text{SL}_2(\mathbb{Z})$, we have the decompositions

$$M_k(\Gamma_1(N)) = \bigoplus_\chi M_k(\Gamma_0(N), \chi) \quad \text{and} \quad S_k(\Gamma_1(N)) = \bigoplus_\chi S_k(\Gamma_0(N), \chi),$$

where the direct sums range over Dirichlet characters of modulus $N$.

### 6.3 The Petersson inner product

Let $\chi$ be a character for a lattice $\Gamma \leq \text{SL}_2(\mathbb{R})$ and let $f, g \in M_k(\Gamma, \chi)$ with at least one in $S_k(\Gamma, \chi)$. Then $fg \in S_{2k}(\Gamma, \chi^2)$ and for any $\gamma \in \Gamma$ we have

$$f(\gamma z)\overline{g(\gamma z)} \text{Im}(\gamma z)^k = f(z)\overline{g(z)} \text{Im}(z)^k,$$

where $\overline{g(z)} := g(\overline{z})$ denotes complex conjugation. By Theorem 6.7, $|f(z)g(z)| \text{Im}(z)^k$ is bounded on $\mathbb{H}$, thus the integral

$$\int_{\Gamma \backslash \mathbb{H}} f(z)\overline{g(z)} \text{Im}(z)^k \ dv(z)$$

is converges. Here $dv(z) := \frac{dx \ dy}{y^2}$ is the hyperbolic measure on $\mathbb{H}$ (with $z = x + iy$). We define $v(\Gamma \backslash \mathbb{H}) := v(\Gamma \backslash \mathbb{H}) := v(F)$

where $F$ is any measurable fundamental domain for $\Gamma$. The fact that $\Gamma$ is a lattice (a finitely generated Fuchsian group of the first kind) implies that such an $F$ exists (indeed, we can use any Dirichlet domain for $\Gamma$), and that $v(F) \in \mathbb{R}_{>0}$ is finite and independent of the choice of $F$.

We now define the **Petersson inner product** of $f$ and $g$ via

$$\langle f, g \rangle := \frac{1}{v(\Gamma \backslash \mathbb{H})} \int_{\Gamma \backslash \mathbb{H}} f(z)\overline{g(z)} \ dv(z) \in \mathbb{C},$$

which induces a positive definite Hermitian inner product on the $\mathbb{C}$-vector space $S_k(\Gamma, \chi)$, making $S_k(\Gamma, \chi)$ into a Hermitian space.

### References

