5.1 Lattices in locally compact groups

Let G be a locally compact (Hausdorff) group equipped with a left Haar measure μ. Then G contains an open set U of finite measure: pick $g \in U \subseteq C$ with C compact, finitely many G-translates of U cover C, and if $\mu(U) = 0$ then $\mu(C) = 0$ (by additivity of μ); if $\mu(C) = 0$ for every C then $\mu = 0$, a contradiction.

Definition 5.1. Let U be an open set with $0 < \mu(U) < \infty$. The Haar modulus for G (also called modular function for G) is the continuous group homomorphism

$$\begin{aligned}\Delta_G\colon G &\to \mathbb{R}_{>0}^{\times}\\ g &\mapsto \frac{\mu(Ug)}{\mu(U)}\end{aligned}$$

which does not depend on the choice of U (since μ is unique up to scaling).

Note that G is unimodular (meaning μ is also a right Haar measure) precisely when $\Delta_G(G) = \{1\}$. The center of G necessarily lies in the kernel of Δ_G, as does its commutator subgroup and all torsion elements, since $\mathbb{R}_{>0}^{\times}$ is abelian and torsion free. This implies that all perfect groups are unimodular (including $\mathrm{SL}_n(\mathbb{R})$ for all $n \geq 1$), as are all compact groups, since Δ_G is continuous and the only compact subgroup of $\mathbb{R}_{>0}^{\times}$ is the trivial group. For $n > 1$ the subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathbb{R})$ is an example of a locally compact group that is not unimodular.

Now let H be a closed subgroup of G. Then the (left) coset space G/H is a locally compact Hausdorff space with a continuous (left) G-action.

Definition 5.2. A (left) Haar measure μ on G/H is a Radon measure with $\mu(gS) = \mu(S)$ for every $S \in \Sigma(G/H)$. If $\mu(G/H) < \infty$ then μ is finite. When G/H admits a finite Haar measure we say that H is cofinite.

Definition 5.3. A lattice in a locally compact group G is a discrete cofinite subgroup Γ.

As shown in Problem 3 of Problem Set 2, the lattices in $\mathrm{SL}_2(\mathbb{R})$ are the finitely generated Fuchsian groups of the first kind (they satisfy both $\Lambda(\Gamma) = \mathbb{P}^1(\mathbb{R})$ and $X_\Gamma := \mathbf{H}_\Gamma^*/\Gamma$ compact).

Lemma 5.4. *Let Γ be a discrete subgroup of a second countable locally compact group G. Then Γ is countable and G contains a measurable set of unique Γ-coset representatives with nonzero measure.*

Proof. Let $\pi\colon G \to G/\Gamma$ be the projection map. Let U be an open neighborhood of the identity $1 \in G$ such that $U \cap \Gamma = \{1\}$, let $A \times B$ be the inverse image of U under the multiplication map $G \times G \to G$, with $A, B \subseteq G$ open, and let $V := (A \cap B)^{-1} \cap (A \cap B)$. Then V is an open neighborhood of the identity with $V^{-1}V \cap \Gamma = \{1\}$. For any $g \in G$ the restriction of π to gV is injective, since for $v_1, v_2 \in V$, if $\pi(gv_1) = \pi(gv_2)$ then $gv_1\gamma_1 = gv_2\gamma_2$ for some $\gamma_1, \gamma_2 \in \Gamma$ and then $\gamma_2\gamma_1^{-1} = v_2^{-1}v_1 \in V^{-1}V \cap \Gamma = \{1\}$.

Let $\{U_n\}_{n\geq 1}$ be a countable subcover of $G = \{gV : g \in G\}$ (second countable implies Lindelöf) with $\mu(U_1) > 0$; then π restricts to an injection on each U_n and Γ is countable. Now let $F_1 := U_1$ and define

$$F_n := U_n - U_n \cap \pi^{-1}(\pi(U_1 \cup \cdots \cup U_{n-1}))$$

for $n \geq 1$. Then π restricts to an injection on each $F_n \subseteq U_n$ and the images $\pi(F_m)$ and $\pi(F_n)$ are disjoint for $m \neq n$, since $\pi(F_n) = \pi(U_n) - \pi(U_n) \cap (\pi(F_1) \cup \cdots \cup \pi(F_{n-1}))$. Now let $F := \bigcup_{n\geq 1} F_n$. The restriction of π to F is injective, and also surjective, since $\pi(F) = \pi(\cup_{n\geq 1} U_n) = \pi(G)$. Thus F is a set of unique Γ-coset representatives which is measurable, since each F_n is (note that $\pi^{-1}(\pi(U)) = U\Gamma$ is open for any open $U \subseteq G$). The measure of F is nonzero, since it contains the open set U_1 with positive measure. □

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A set of unique Γ-coset representatives in G is a strict fundamental domain for Γ.

Lemma 5.5. *Let Γ be a discrete subgroup of a second countable locally compact group G. Then G/Γ admits a Haar measure if and only if $\Delta_G(\Gamma) = \{1\}$.*

Proof. Let $\pi\colon G \to G/\Gamma$ be the quotient map, let F be a measurable strict fundamental domain for G/Γ as in Lemma 5.4, and let ν be a left Haar measure on G. For $S \in \Sigma(G/\Gamma)$ we define

$$\mu(S) := \nu(F \cap \pi^{-1}(S)).$$

For any $g \in G$, the left G-invariance of ν implies

$$\mu(gS) = \nu(F \cap \pi^{-1}(gS)) = \nu(F \cap g\pi^{-1}(S)) = \nu(g^{-1}F \cap \pi^{-1}(S)) = \nu(F' \cap T),$$

where $F' := g^{-1}F$ and $T := \pi^{-1}(S)$. Now F' is also a measurable strict fundamental domain for G/Γ, thus $G = F'\Gamma = F\Gamma$, and $T\Gamma = T$. If $\Delta_G(\Gamma) = \{1\}$ then ν is Γ-right invariant and

$$\mu(gS) = \nu(G \cap F' \cap T) = \sum_{\gamma \in \Gamma} \nu(F\gamma \cap F' \cap T) = \sum_{\gamma \in \Gamma} \nu(F \cap F'\gamma^{-1} \cap T) = \nu(F \cap G \cap T) = \mu(S),$$

implying that μ is a left G-invariant. The function μ is a nonzero Radon measure on G/Γ, since ν restricts to a nonzero Radon measure on F (because F is measurable and G is second countable, hence σ-compact). It follows that μ is a left Haar measure on G/Γ.

Conversely, given a left Haar measure μ on G/Γ, for any $S \in \Sigma(G)$ we may define

$$\nu(S) := \sum_{\gamma \in \Gamma} \mu(\pi(F\gamma \cap S)).$$

The Haar measure μ on G/Γ lifts to a nonzero Radon measure on each of the measurable strict fundamental domains $F\gamma$, which form a countable partition of G. It follows that ν is a nonzero Radon measure on G. The G-invariance of μ implies that ν is a Haar measure, and we note that ν is right Γ-invariant, since π is; it follows that $\Delta_G(\Gamma) = \{1\}$. □

Corollary 5.6. *Let Γ be a lattice in a second countable locally compact group G. Then G is unimodular.*

Proof. $\Gamma \in \ker \Delta_G$, so Δ_G factors through the map $G/\Gamma \to \mathbb{R}^\times_{>0}$ induced by μ, and $\Delta_G(G)$ must be bounded, since μ is finite. But the only bounded subgroup of $\mathbb{R}^\times_{>0}$ is the trivial group. □

Lemma 5.7. *Let G be a second countable locally compact group with a discrete cocompact subgroup Γ. Then Γ is a lattice.*

In other words, for discrete subgroups of second countable locally compact groups, cocompact implies cofinite. The converse does not hold: $\mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{SL}_2(\mathbb{R})$ is discrete and $\mathrm{SL}_2(\mathbb{R})$ is unimodular, so $\mathrm{SL}_2(\mathbb{Z})$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$, but it is not cocompact because the diagonal subgroup of $\mathrm{SL}_2(\mathbb{R})$ has unbounded image in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. Cocompact lattices are uniform

5.2 Arithmetic groups

Recall that a Lie group is a smooth (second countable) manifold with a compatible group structure (so group operations are smooth maps). An algebraic group is an algebraic variety with a compatible group structure (so group operations are rational maps). Projective algebraic groups are abelian varieties (being projective forces the group to be abelian). Affine algebraic groups are linear algebraic groups, and include GL_n, which can be defined as an affine varieties in $\mathbb{A}^{n^2+1}$ over any field k, using variables t and A_{ij} for $1 \le i, j \le n$ and the equation $t \det A = 1$ (where $\det A$ is the polynomial in the a_{ij} corresponding to the determinant), along with polynomial maps corresponding to matrix multiplication and inversion (using t as the inverse of $\det A$). The classical groups SL_n, Sp_{2n}, U_n, SU_n, O_n, and SO_n are all examples of linear algebraic groups (assume $\mathrm{char}(k) \ne 2$ for O_n and SO_n).

If G is an algebraic group over $\mathbb{Q}$ (a $\mathbb{Q}$-algebraic group), then $G(\mathbb{R})$ and $G(\mathbb{C})$ are Lie groups.

Definition 5.8. Subgroups H_1 and H_2 of a group G are commensurable if $H_1 \cap H_2$ has finite index in both H_1 and H_2.

Definition 5.9. Let G be a Lie group. A subgroup $\Gamma \le G$ is arithmetic if there is a $\mathbb{Q}$-algebraic linear group H and a morphism of Lie groups $\phi\colon H(\mathbb{R}) \to G$ with compact kernel and finite cokernel for which $\phi(H(\mathbb{Z})$ is commensurable to Γ.

Arithmetic groups are discrete (since $\mathbb{Z}$ is discrete in $\mathbb{R}$) and cofinite, which implies they are lattices (which may be uniform or non-uniform). Not all lattices are arithmetic (not even all uniform lattices); triangle groups in $\mathrm{SL}_2(\mathbb{R})$ provide many examples (see the next section). But in a semisimple Lie group G (no nontrivial connected normal abelian subgroup) of real rank at least 2, every irreducible lattice Γ (infinite lattices for which ΓN is dense in G for every noncompact closed $N \trianglelefteq G^0$) is arithmetic, by a theorem of Margulis [1].

The arithmetic lattices in $\mathrm{SL}_2(\mathbb{R})$ arise from orders O in quaternion algebras B over a totally real field F in which exactly one of the real places splits (this includes both uniform and non-uniform lattices). As in the previous lecture, we embed F in $\mathbb{R}$ via the unique real place that splits and use this to embed B in $\mathrm{M}_2(\mathbb{R})$, then take the image of the norm-1 elements of O under this embedding to get a discrete cofinite subgroup $\Gamma(O^1) \le \mathrm{SL}_2(\mathbb{R})$.

When B is a division algebra, the arithmetic lattice $\Gamma(O^1)$ is uniform (cocompact) and has no cusps, otherwise B is a matrix algebra and $\Gamma(O^1)$ is non-uniform and has at least one cusp.

5.2.1 Triangle groups

Let Γ be a lattice in $\mathrm{SL}_2(\mathbb{R})$, equivalently, a finitely generated Fuchsian group of the first kind with $Z(\Gamma) := \Gamma \cap \{\pm 1\}$. Then $\Gamma/Z(\Gamma)$ is generated by hyperbolic elements $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$, elliptic elements $\{\gamma_1, \ldots, \gamma_s\}$, and parabolic elements $\{\gamma_{s+1}, \ldots, \gamma_{s+t}\}$, which satisfy

$$\begin{aligned} \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\gamma_1\cdots\gamma_{s+t} &= 1 \\ \gamma_i^{e_i} &= 1 \qquad (1 \le i \le s), \end{aligned}$$

where $1 < e_1 \le \cdots \le e_s < \infty$ are the orders of the elliptic elements.

Definition 5.10. The signature of Γ is the tuple $(g; e_1, \ldots, e_{s+t})$, with $e_{s+1} = \cdots = e_{s+t}$. It satisfies the inequality

$$2g - 2 + \sum_{j=1}^{s+t}\left(1 - \frac{1}{e_i}\right) > 0.$$

When $g = 0$ and $s + t = 3$ we call Γ a triangle group of type (e_1, e_2, e_3). For triangle groups of type (e_1, e_2, e_3) we necessarily have

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} < 1.$$

Proposition 5.11. *Let (e_1, e_2, e_3) and s be as above. Up to conjugacy in $\mathrm{SL}_2(\mathbb{R})$, the following hold:*

- *If $s \geq 1$ and e_i is even for some $i \leq s$ then there is a unique triangle group of type (e_1, e_2, e_3) and it contains -1.*
- *If $g \geq 2$ and e_i is odd for all $i \leq s$ then there are exactly two triangle groups of type (e_1, e_2, e_3). One contains -1 and the other of does not.*
- *If $s = 0$ or $s = 1$ and e_1 is odd then there are exactly three triangle groups of type (e_1, e_2, e_3). One contains -1 and the other two do not and have index 2 in the one that does.*

Proof. This is [4, Prop. 1]. □

This implies that there are infinitely many triangle groups.

Theorem 5.12. *Let Γ be a triangle group of type (e_1, e_2, e_3) and let*

$$k := \mathbb{Q}\big(\cos(\pi/e_1)^2, \cos(\pi/e_2)^2, \cos(\pi/e_3)^2, \cos(\pi/e_1)\cos(\pi/e_2)\cos(\pi/e_3)\big).$$

If $t > 0$ then Γ is arithmetic if and only if $k = \mathbb{Q}$. If $t = 0$ then Γ is arithemtic if and only if $k = \mathbb{Q}$ or $k \supsetneq \mathbb{Q}$ and under every embedding of k into $\mathbb{R}$ we have

$$\cos(\pi/e_1)^2 + \cos(\pi/e_2)^2 + \cos(\pi/e_3)^2 + 2\cos(\pi/e_1)\cos(\pi/e_2)\cos(\pi/e_3) < 1$$

Proof. This is [4, Thm. 1]. □

Theorem 5.13. *Let Γ be an arithmetic triangle group. If $t > 0$ there are 9 possible types for Γ:*

$(2,3,\infty), (2,4,\infty), (2,6,\infty), (2,\infty,\infty), (3,3,\infty), (3,\infty,\infty), (4,4,\infty), (6,6,\infty), (\infty,\infty,\infty)$

and otherwise there are 76 possible types:

- $(2,3,n)$ *with* $n \in \{7,\ldots,14,16,18,24,30\}$ *or* $(2,4,n)$ *with* $n \in \{5,\ldots,8,10,12,18\}$*;*
- $(2,5,n)$ *with* $n \in \{5,6,8,10,20,30\}$ *or* $(2,6,n)$ *with* $n \in \{6,8,12\}$*;*
- *one of* $(2,7,7)$, $(2,7,14)$, $(2,8,8)$, $(2,8,16)$, $(2,9,18)$, $(2,10,10)$, $(2,12,12)$, $(2,12,24)$, $(2,15,30)$, $(2,18,18)$*;*
- $(3,3,n)$ *with* $n \in \{4,\ldots,9,12,15\}$ *or* $(3,4,n)$ *with* $n \in \{4,6,12\}$*;*
- *one of* $(3,5,5)$, $(3,6,6)$, $(3,6,18)$, $(3,8,8)$, $(3,8,24)$, $(3,10,30)$, $(3,12,12)$*;*
- $(4,4,n)$ *with* $n \in \{4,5,6,9\}$ *or one of* $(4,5,5)$, $(4,6,6)$, $(4,8,8)$, $(4,16,16)$*;*
- *one of* $(5,5,5)$, $(5,5,10)$, $(5,5,15)$, $(5,10,10)$, $(6,6,6)$, $(6,12,12)$, $(6,24,24)$, $(7,7,7)$, $(8,8,8)$, $(9,9,9)$, $(9,18,18)$, $(12,12,12)$, $(15,15,15)$.

Proof. This is [4, Thm. 3]. □

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