This is summary of the material from §1.5, §1.7 of [1] and §37-38 of [2] presented in lecture.

4.1 Cusps and stabilizers

Recall that the action of $\mathrm{SL}_2(\mathbb{R})$ on the extended upper half plane $\overline{\mathbf{H}} := \mathbf{H} \cup \mathbb{P}^1(\mathbb{R}) \subseteq \mathbb{P}^1(\mathbb{C})$ via linear transformations stabilizes the boundary $\partial\mathbf{H} = \mathbb{P}^1(\mathbb{R})$ and is transitive on $\mathbf{H}$ and $\partial\mathbf{H}$. We classified nontrivial elements $\alpha \in \mathrm{SL}_2(\mathbb{R})$ according to $\mathrm{disc}(\alpha) = \mathrm{tr}(\alpha)^2 - 4\det(\alpha)$ as

- elliptic if $\mathrm{disc}(\alpha) < 0$, equivalently, $\alpha \in \mathrm{SL}_2(\mathbb{R})_z$ for exactly one $z \in \mathbf{H}$ and no $z \in \partial\mathbf{H}$;
- parabolic if $\mathrm{disc}(\alpha) = 0$, equivalently, $\alpha \in \mathrm{SL}_2(\mathbb{R})_z$ for no $z \in \mathbf{H}$ and exactly one $z \in \partial\mathbf{H}$;
- hyperbolic if $\mathrm{disc}(\alpha) > 0$, equivalently, $\alpha \in \mathrm{SL}_2(\mathbb{R})_z$ for no $z \in \mathbf{H}$ and exactly two $z \in \partial\mathbf{H}$.

If $\alpha \in \mathrm{SL}_2(\mathbb{R})$ fixes more than two points in $\overline{\mathbf{H}}$ then $\alpha = \pm 1$ trivial.

Definition 4.1. For $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ we classify the $z \in \overline{\mathbf{H}} = \mathbf{H} \cup \partial\mathbf{H}$ as follows:

- $z \in \overline{\mathbf{H}}$ is ordinary if $\Gamma_z \subseteq \{\pm 1\}$ (in which case Γ_z is trivial);
- $z \in \mathbf{H}$ is an elliptic point of Γ_z contains an elliptic $\gamma \in \Gamma$;
- $z \in \partial\mathbf{H}$ is a parabolic point or cusp if Γ_z contains a parabolic $\gamma \in \Gamma$;
- $z \in \partial\mathbf{H}$ is a hyperbolic point if Γ_z contains a hyperbolic $\gamma \in \Gamma$ (so $\Gamma_{z,z'} \subseteq \Gamma_z$ for some z').

For $\Gamma \in \mathrm{SL}_2(\mathbb{R})$ we let $Z(\Gamma) = \Gamma \cap \{\pm 1\}$ denote its trivial subgroup, so that $\Gamma/Z(\Gamma)$ is isomorphic to the image of Γ in $\mathrm{PSL}_2(\mathbb{R})$.

Theorem 4.2. *Let $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group and let $z \in \overline{\mathbf{H}} = \mathbf{H} \cup \partial\mathbf{H}$. The following hold:*

(i) *If z is an elliptic point of Γ then Γ_z is a finite cyclic group that strictly contains $Z(\Gamma)$;*

(ii) *If z is a cusp of Γ then $\Gamma_z/Z(\Gamma) \simeq \mathbb{Z}$ and $\pm\Gamma_z$ is $\mathrm{SL}_2(\mathbb{R})$-conjugate to $\left\langle \pm \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$ with $h > 0$;*

(iii) *If z is a hyperbolic point of Γ then $\Gamma_z \supseteq \Gamma_{z,z'} \supsetneq Z(\Gamma)$ (for some $z' \ne z$) then $\Gamma_{z,z'}/Z(\Gamma) \simeq \mathbb{Z}$ and $\pm\Gamma_{z,z'}$ is $\mathrm{SL}_2(\mathbb{R})$-conjugate to $\left\langle \pm \left(\begin{smallmatrix} u & 0 \\ 0 & 1/u \end{smallmatrix}\right) \right\rangle$ with $u > 0$.*

Proof. (i) $\mathrm{SL}_2(\mathbb{R})_z$ is conjugate to $\mathrm{SL}_2(\mathbb{R})_i = \mathrm{SO}_2(\mathbb{R}) \simeq S^1$, a compact abelian group. Since Γ is discrete, $\Gamma_z = \Gamma \cap \mathrm{SL}_2(\mathbb{R})_z$ is finite, hence cyclic, and it contains an elliptic $\gamma \notin Z(\Gamma)$.

(ii) WLOG $z = \infty$ and $\Gamma_z = \Gamma_\infty$ is upper triangular and contains some $\gamma = \left\langle \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$ with $h \ne 0$ (we can conjugate Γ_z to achieve this). Up to ± 1 every element of Γ_z must be of this form: if $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right) \in \Gamma_x$ with $|a| < 1$ then $\alpha^n \gamma \alpha^{-n} = \left(\begin{smallmatrix} 1 & a^{2n}h \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_z$ for all $n \ge 1$, but Γ_z is discrete. Thus $\Gamma_x/Z(\Gamma)$ is isomorphic to a nontrivial discrete subgroup of $\mathbb{R}$, which implies $\Gamma_x/Z(\Gamma) \simeq \mathbb{Z}$.

(iii) WLOG $z = 0$ and $\Gamma_z \Gamma_0 \supseteq \Gamma_{0,\infty} \supsetneq Z(\Gamma)$, so that $\Gamma_{z,z'} = \Gamma_{0,\infty}$ is diagonal, consisting of elements of the form $\left(\begin{smallmatrix} u & 0 \\ 0 & 1/u \end{smallmatrix}\right)$ with $u \in \mathbb{R}^\times$. It follows that $\Gamma_x/Z(\Gamma)$ is a nontrivial discrete subgroup of $\mathbb{R}^\times/\pm 1$, which implies $\Gamma_x/Z(\Gamma) \simeq \mathbb{Z}$. □

Corollary 4.3. *For any Fuchsian group Γ we may partition $\overline{\mathbf{H}} = \mathbf{H} \cup \partial\mathbf{H}$ into elliptic points, cusps, hyperbolic points, and ordinary points. In this partition, $\mathbf{H}$ consists of elliptic and ordinary points, while $\partial\mathbf{H}$ consists of parabolic, hyperbolic and ordinary points.*

Proof. The theorem implies that a cusp cannot also be a hyperbolic point. □

Corollary 4.4. *Let Γ be a Fuchsian group. Every finite index $\Gamma' \le \Gamma$ has the same set of cusps as Γ.*

Proof. Any cusp x of Γ' is a cusp of Γ, since $\Gamma'_x \subseteq \Gamma_x$. If x is a cusp of Γ then $\Gamma'_x = \Gamma_x \cap \Gamma$ has finite index in Γ_x, which is infinite, so Γ'_x cannot be trivial and x is also a cusp of Γ'. □

Note that even when $\Gamma' \leq \Gamma$ has the same set of cusps as Γ it need not have the same cusp orbits (the cusp orbits of Γ' will be a refinement of the partition of the cusps of Γ into cusp orbits).

Definition 4.5. For a Fuchsian group Γ and $z \in \overline{\mathbf{H}} = \mathbf{H} \cup \partial \mathbf{H}$, the order of z with respect to Γ is

$$e_z := \#\Gamma_z / Z(\Gamma) \in \mathbb{Z}_{>0} \cup \{\infty\}.$$

It follows from the theorem above that $e_z = 1$ if z is ordinary, $1 < e_z < \infty$ if z is elliptic, and $e_z = \infty$ if z is a cusp or hyperbolic point.

Corollary 4.6. *If $-1 \notin \Gamma$ then every elliptic point of Γ has odd order.*

Proof. The group $\pm\Gamma$ is a finite cyclic group whose unique element of order 2 is -1. □

Definition 4.7. Let Γ be a Fuchsian group, let $x \in \partial \mathbf{H}$ be a cusp of Γ, and choose $\sigma \in \mathrm{SL}_2(\mathbb{R})$ so $\sigma x = \infty$. Then $\pm \sigma \Gamma_x \sigma^{-1} = \pm \left\langle \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$ with $h > 0$. If $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \in \sigma \Gamma_x \sigma^{-1}$ then x is a regular cusp, and otherwise x is irregular. The latter can arise only when $-1 \notin \Gamma$, and the regularity of x does not depend on the choice of σ.

4.2 Adjoining cusps

For a Fuchsian group Γ we let P_Γ denote its set of cusps, and define $\mathbf{H}^*_\Gamma := \mathbf{H} \cap P_\Gamma$. We extend the topology of $\mathbf{H}$ to $\mathbf{H}^*_\Gamma$ by taking defining open neighborhoods $U_l := \{z \in \mathbf{H} : \mathrm{Im}(z) > x\} \cup \{\infty\}$ of ∞ for all $l > 0$, and we take $\mathrm{SL}_2(\mathbb{R})$-translates σU_l as open neighborhoods of $\sigma\infty$ for each cusp $\sigma\infty$ of Γ. Then $\mathbf{H}^*_\Gamma$ is a Hausdorff space, on which Γ acts.

This action is not proper, since cusps have infinite stabilizers, but the quotient $X_\Gamma := \Gamma \backslash \mathbf{H}^*_\Gamma$ is Hausdorff (see [1, Lemma 1.7.7]), and we can give it the structure of a Riemann surface (see [1, §1.8]). If Γ is a Fuchsian group and X_Γ has finite volume (under the measure induced by the measure dv on $\mathbf{H}$), then X_Γ is a compact Riemann surface, by Siegel's Theorem [1, Thm. 1.9.1], hence an algebraic curve of some genus $g \geq 0$. In this case the number of elliptic points and cusps is necessarily finite [1, Thm. 1.7.8].

4.3 The modular group

The modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is a Fuchsian group containing -1, generated by the matrices

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

A Dirichlet domain for Γ is given by

$$D(2i) = \left\{ z \in \mathbf{H} : |z| \geq 1 \text{ and } \mathrm{Re}(z) \in [-1/2, 1/2] \right\},$$

with elliptic points $i = \zeta_4$ of order 2, with $\Gamma_i = \langle S \rangle$, and $\omega = \zeta_3$ of order 3 with $\Gamma_\omega = \langle ST \rangle$. The Dirichlet domain $D(2i)$ also contains the elliptic point $T\omega$ on its boundary.

The closure of $D(2i)$ in $\mathbf{H}^*_\Gamma$ contains a single cusp ∞ with $\Gamma_\infty = \langle \pm T \rangle$ and $P_\Gamma = \Gamma\infty = \mathbb{P}^1(\mathbb{Q})$. Then $Y(1) := Y_\Gamma := \Gamma \backslash \mathbf{H} \simeq \mathbb{C}$ and $X(1) := X_\Gamma := \Gamma \backslash \mathbf{H}^*_\Gamma \simeq \mathbb{P}^1(\mathbb{C})$.

Every subgroup of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is also a Fuchsian group, but we will only be interested in those that have finite index. An important example of such groups is given by congruence subgroups, which are subgroups of $\mathrm{SL}_2(\mathbb{Z})$ that contain the group

$$\Gamma(N) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv 1 \bmod N \right\}.$$

Two particularly important examples of congruence subgroups are

$$\Gamma_0(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod N\right\} \quad \text{and} \quad \Gamma_1(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \bmod N\right\}.$$

We note that not all finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are congruence subgroups.

4.3.1 Triangle groups

Definition 4.8. Let k be a field whose characteristic is not 2. A quaternion algebra $B := \left(\frac{a,b}{k}\right)$ over a field k is a (unital associative) k-algebra that has a k-basis of the form $\{1, i, j, ij\}$ with

$$i^2 = a, \qquad j^2 = b, \qquad ij = -ji.$$

A quaternion algebra is either a division ring (nonsplit) or isomorphic to $\mathrm{M}_2(k)$ (split). The standard involution on B is the map $\alpha \mapsto \bar\alpha$ that sends $\alpha = t+xi+yj-zij$ to $\bar\alpha = t-xi-yj-zij$. The (reduced) norm of $\alpha \in B$ is $N(\alpha) = \alpha\bar\alpha = t^2 = ax^2-by^2+abz^2 \in k$, which is multiplicative and nonzero precisely when α is a unit in B. The set of norm-1 elements B^1 form a subgroup of the unit group $B^\times$.

Let $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ be a quaternion algebra over $\mathbb{Q}$. We say that a place p of $\mathbb{Q}$ is ramified if B if $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division ring, and unramified if $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is split. We call B definite if it is ramified at the archimedean place of $\mathbb{Q}$, equivalently, $B_\infty := B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$ and indefinite otherwise. The quaternion algebra $\left(\frac{a,b}{\mathbb{Q}}\right)$ is definite if and only if $a, b < 0$.

The set of ramified places of B is finite of even cardinality; conversely, for every finite set Σ of places of $\mathbb{Q}$ of even cardinality, there is a quaternion algebra with ramified exactly at the places in Σ, which is unique up to isomorphism (this is true more generally for quaternion algebras over any global field, see [2, Theorem 14.6.1]). The discriminant of B is the product of the primes that ramify in B, which we negate when ∞ is ramified; the discriminant uniquely determines the isomorphism class of a quaternion algebra over $\mathbb{Q}$ (and in fact $|B|$ does).

If B is an indefinite quaternion algebra over $\mathbb{Q}$, equivalently $\mathrm{disc}(B) > 0$ we can embed B in $\mathrm{M}_2(\mathbb{R})$ via $B \hookrightarrow B_\infty \simeq \mathrm{M}_2(\mathbb{R})$ by writing $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ with $a > 0$ and defining

$$\iota_\infty : B \hookrightarrow \mathrm{M}_2(\mathbb{Q}(\sqrt{a})) \subseteq \mathrm{M}_2(\mathbb{R})$$
$$t + xi + yj + zij \mapsto \begin{pmatrix} t + x\sqrt{a} & y + z\sqrt{a} \\ b(y - z\sqrt{a}) & t - x\sqrt{a} \end{pmatrix}.$$

For any $\alpha = t + xi + yj + zij \in B$ we then have

$$\det \iota_\infty(\alpha) = t^2 - ax^2 - by^2 + abz^2 = N(\alpha),$$

thus ι_∞ restricts to group homomorphisms $B^\times \hookrightarrow \mathrm{GL}_2(\mathbb{R})$ and $B^1 \hookrightarrow \mathrm{SL}_2(\mathbb{R})$.

An order O in a quaternion algebra B over $\mathbb{Q}$ is a subring that is finitely generated as a $\mathbb{Z}$-module with $O \otimes_{\mathbb{Z}} \mathbb{Q} = B$. It's norm-1 unit group $O^1 := O \cap B^1$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, hence a Fucshian group.

Consider the indefinite quaternion algebra $B = \left(\frac{-1,3}{\mathbb{Q}}\right)$ of discriminant 6. It's unit group is

$$O := \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k, \qquad k = \frac{1 + i + j + ij}{2}.$$

The Fuchsian group $\Gamma := \iota_\infty(O^1)$ has no cusps, and $X_\Gamma = Y_\Gamma = \Gamma\backslash\mathbf{H} = X(\Gamma)$ is a good compact complex 1-orbifold.

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.

[2] John Voight, *Quaternion algebras*, Springer, 2021.