3 Fuchsian groups and Dirichlet domains

This is summary of the material from §34,§36-37 of [2] presented in lecture.

3.1 Manifolds and orbifolds

Recall that a (topological) manifold is a second countable Hausdorff space X that is locally homeomorphic to $\mathbb{R}^n$, for some fixed $n \geq 1$ (the dimension of the manifold). This that means every $x \in X$ has an open neighborhood U equipped with a homeomorphism $\phi\colon U \xrightarrow{\sim} \phi(U) \subseteq \mathbb{R}^n$. The map ϕ is a chart, and and an open cover of charts is an atlas. For every pair of overlapping charts $\phi_1\colon U_1 \to \mathbb{R}^n$ and $\phi_2\colon U_2 \to \mathbb{R}^n$, meaning $U_1 \cap U_2 \neq \emptyset$, we have a homeomorphic transition map

$$\phi_{12} = \phi_2\phi_1^{-1}\colon \phi_1(U_1 \cap U_2) \xrightarrow{\sim} \phi_2(U_1 \cap U_2)$$

between two open subsets of $\mathbb{R}^n$. If X has an atlas for which all the transition maps are smooth (continuous partial derivatives of all orders), then X is a smooth manifold. Replacing $\mathbb{R}^n$ with $\mathbb{C}^n$ and "smooth" with "holomorphic" yields a complex manifold. A Riemann surface is a complex manifold of dimension one. Every compact Riemann surface can be realized as an algebraic curve.

A real Lie group is a topological group G that is also a smooth manifold for which the maps $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth; a complex Lie group is defined similarly.

Theorem 3.1 (Riemann uniformization theorem). *Every simply connected Riemann surface is isomorphic to either the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, the complex plane $\mathbb{C}$, or the upper half-plane $\mathbf{H}$.*

The universal cover $\tilde{X}$ of a compact Riemann surface X is simply connected, which means that $X \simeq G\backslash\tilde{X}$, where $G = \pi_1(X, x_0)$ is the fundamental group of X acting on $\tilde{X}$ as a covering space action. This leads to the following possibilities:

- $\tilde{X} = \mathbb{P}^1(\mathbb{C})$ and $X \simeq \mathbb{P}^1(\mathbb{C})$ (elliptic);
- $\tilde{X} = \mathbb{C}$ and $X \simeq \mathbb{C}$ or $X \simeq \mathbb{C}/\mathbb{Z}$ or $X \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with $\tau \in \mathbf{H}$ (parabolic).
- $\tilde{X} = \mathbf{H}$ and $X \simeq \Gamma\backslash\mathbf{H}$ with Γ a torsion-free Fuchsian group (hyperbolic).

Recall that a Fuchsian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, which need not be torsion free!

Remark 3.2. It will often be convenient to work with subgroups Γ of $\mathrm{SL}_2(\mathbb{R})$ rather than $\mathrm{PSL}_2(\mathbb{R})$, but we can ignore the trivial action of -1 (so when we speak of torsion, we always refer to the image of Γ in $\mathrm{PSL}_2(\mathbb{R})$).

Definition 3.3. An orbifold is a second countable Hausdorff space X locally homeomorphic to $G\backslash\mathbb{R}^n$ for some finite group G acting continuously on $\mathbb{R}^n$. An atlas for an orbifold is an open cover of charts U_i, closed under finite intersection, and a corresponding collection of open $V_i \subseteq \mathbb{R}^n$ equipped with the continuous action of a finite group G_i and a homeomorphism $\phi_i\colon U_i \xrightarrow{\sim} G_i\backslash V_i$ such that whenever $U_i \subseteq U_j$ there exists an injective homomorphism $f_{ij}\colon G_i \hookrightarrow G_j$ and a G_i-equivariant gluing map $\psi_{ij}\colon V_i \hookrightarrow V_j$ for which $\psi_{ij} \circ \phi_i = \phi_j$. If the gluing maps and G-action are C^∞-smooth we have a smooth orbifold, and replacing "C^∞-smooth" with "holomorphic" yields a complex orbifold.

If X is a manifold with a proper action by a discrete group G, we can make $G\backslash X$ into an orbifold using an atlas defined by neighborhoods $U \ni x$ chosen so that $\{g \in G : U \cap gU \neq \emptyset\} = G_x$ is finite; orbifolds that arise in this way are good. If $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group, then $\Gamma\backslash\mathbf{H}$ is a good complex orbifold of dimension 1 (and also a good 2-orbifold, a real orbifold of dimension 2).

3.2 Dirichlet domains

For a group G acting on a topological space X, we call an element of G trivial if it acts trivially on every element of X. For example, ± 1 are the trivial elements of $\mathrm{SL}_2(\mathbb{R})$ (as noted above, we do not require that G act faithfully).

Definition 3.4. A fundamental set for a group G acting on a topological space X is a subset $S \subseteq X$ for which (i) $\overline{S^0} = S$, (ii) $GS = X$, (iii) $S^0 \cap (gS)^0$ for all nontrivial g.

Let X be a completely locally compact geodesic space with metric d, and let Γ be a discrete group of isometries acting properly on X.

Definition 3.5. The Dirichlet domain for Γ centered at $x_0 \in X$ is the set

$$D(x_0) := D(\Gamma; x_0) := \{x \in X : d(x_0, x) \le d(x_0, \gamma x) \text{ for all } \gamma \in \Gamma\}.$$

For $x_1, x_2 \in X$ the Leibniz half-space is the set

$$H(x_1, x_2) := \{x \in X : d(x_1, x) \le d(x_2, x)\},$$

with $H(x_1, x_2) = X$ if $x_1 = x_2$, and $D(x_0) = \cap_\gamma H(x_0, \gamma x_0)$ is an intersection of Leibniz half-spaces over $\gamma \in \Gamma - \Gamma_{x_0}$. For $x_1 \ne x_2$ the boundary

$$L(x_1, x_2) := H(x_1, x_2) - H(x_1, x_2)^0 = \{x \in X : d(x_1, x) = d(x_2, x)\}$$

is the hyperplane bisector separating x_1 and x_2, and $H(x_1, x_2)^0 = \{x \in X : d(x_1, x) < d(x_2, x)\}$.

Remark 3.6. Hyperplane bisectors need not be geodesics, but hyperplane bisectors in $\mathbf{H}$ are.

Definition 3.7. A set $S \subseteq X$ is convex if it contains every geodesic segment connecting points in S, and star-shaped with respect to $x_0 \in S$ if it contains every geodesic segment connecting a point in S to x_0.

Lemma 3.8. *Every Leibniz half-plane $H(x_1, x_2)$ in X is star-shaped with respect to x_1.*

Proof. Consider any $x \in H(x_1, x_2)$ and any y on a geodesic segment from x to x_1. Then $d(x_1, y) + d(y, x) = d(x_1, x)$. If $y \notin H(x_1, x_2)$ then $d(x_2, y) < d(x_1, y)$ and

$$d(x_2, x) \le d(x_2, y) + d(y, x) < d(x_1, y) + d(y, x) = d(x_1, x),$$

but this contradicts $x \in H(x_1, x_2)$. □

Corollary 3.9. *The Dirichlet domain $D(x_0)$ is closed and star-shaped with respect to x_0.*

Proof. $D(x_0)$ is an intersection of Leibniz half-spaces that are closed and star-shaped. □

Lemma 3.10. *If $x_0 \in S \subseteq X$ is bounded then the set $\{\gamma \in \Gamma : S \not\subseteq H(x_0, \gamma x_0)\}$ is finite.*

Proof. If S is bounded then $\sup\{d(x_0, x) : x \in S\} = r < \infty$, which implies $\Gamma x_0 \subseteq X$ is discrete and $\Gamma_{x_0} \subseteq \Gamma$ is finite, since Γ is discrete and acts properly on X. Thus only finitely many $\gamma \in \Gamma$ satisfy $\rho(x_0, \gamma x_0) \le 2r$. For every other $\gamma \in \Gamma$ and all $x \in S$ we have

$$d(x, \gamma x_0) \ge d(x_0, \gamma x_0) - d(x, x_0) > 2r - r = r \ge d(x, x_0)$$

which implies $x \in H(x_0, \gamma x_0)$. □

Corollary 3.11. *We have* $D(x_0)^0 = \{x \in D(x_0) \colon d(x_0, x) < d(x_0, \gamma x)$ *for all* $\gamma \in \Gamma - \Gamma_{x_0}\}$ *and* $\overline{D(x_0)^0} = D(x_0)$.

Proof. For any $x \in D(x_0)$ we can pick a bounded open $U \ni x$ for which

$$U \cap D(x_0) = U \cap \bigcap_{\gamma_i \in F} H(x_0, \gamma x_0)$$

where F is the finite set of Lemma 3.10 for $S = U$ with elements of Γ_{x_0} removed. Therefore

$$U \cap D(x_0)^0 = U \cap \bigcap_{\gamma \in F} H(x_0, \gamma x_0)^0$$

For $\gamma \in \Gamma - (F \cup \Gamma_{x_0})$ we have $U \subseteq H(x_0, \gamma x_0)^0$, and if V is the set on the RHS of the corollary,

$$U \cap D(x_0)^0 = U \cap \bigcap_{\gamma \in \Gamma - \Gamma_{x_0}} H(x_0, \gamma_i x_0)^0 = U \cap V,$$

since $H(x_0, \gamma x_0)^0 = \{x \in X : d(x, x_0) < d(x, \gamma x_0)\}$ for $\gamma \in \Gamma - \Gamma_{x_0}$.

This implies the first statement in the corollary, and the second then follows, since we have $\overline{D(x_0)^0} = \bigcap_{\gamma \in \Gamma - \Gamma_{x_0}} H(x_0, \gamma x_0) = D(x_0)$. $\square$

Definition 3.12. $S \subseteq X$ is locally finite if for every compact $C \subseteq X$ the set $\{\gamma \in \Gamma \colon S \cap \gamma C \neq \emptyset\}$ is finite.

Lemma 3.13. *If S is a locally finite fundamental set then Γ is generated by* $\{\gamma \in \Gamma : S \cap \gamma S \neq \emptyset\}$.

Proof. Let $\Gamma^* \leq \Gamma$ be the subgroup generated by $\{\gamma \in \Gamma : S \cap \gamma S \neq \emptyset\}$. Define $f : X \to \Gamma^* \backslash \Gamma$ via $x \mapsto \Gamma^* \gamma$, with $\gamma \in \Gamma$ chosen so $\gamma x \in S$; such a γ exists because S is fundamental, and it doesn't matter which one we pick because if $\gamma' x \in S$ then $\gamma' x \in S \cap \gamma' \gamma^{-1} S \neq \emptyset$, which implies $\gamma' \gamma^{-1} \in \Gamma^*$ and $\gamma' \Gamma^* = \gamma \Gamma^*$.

We claim f is locally constant. Consider any $x \in X$ with compact neighborhood C. Since S is locally finite, we can pick a finite set of $\{\gamma_i\} \subseteq \Gamma$ so that $C \subseteq \bigcup_i \gamma_i S$. By replacing C by its intersection with the union of the $\gamma_i S$ that contain x, we can assume $x \in \gamma_i S$ for all i, which implies that $f(x) = \Gamma^* \gamma_i^{-1}$ for all i. Any $y \in C$ lies in $\gamma_i S$ for some i, which means $f(y) = f(x)$.

But X is connected (it is a geodesic space), so f is actually constant, and for any $\gamma \in \Gamma$ and any $x \in S$ we have $\Gamma^* = f(x) = f(\gamma^{-1} x) = \Gamma^* \gamma$, which implies $\Gamma^* = \Gamma$. $\square$

In the theorem below, by trivial, we mean an element or subgroup of Γ that acts trivially on every $x \in X$ (so for $\Gamma \in \mathrm{SL}_2(\mathbb{R})$ this would include ± 1).

Theorem 3.14. *Let X be a complete, locally compact geodesic space and let Γ be a discrete group of isometries acting properly on X. If $x_0 \in X$ has trivial stabilizer then $D(x_0)$ is a locally finite fundamental set for Γ that is star-shaped with respect to x_0 whose boundary consists of segments of hyperplane bisectors.*

Proof. Let $x_0 \in X$ have trivial stabilizer. It follows from Lemma 3.8 that $D(x_0)$ is star-shaped with respect to x_0, and Corollary 3.11 implies that its boundary is a union of segments of hyperplane bisectors. We now show that $D(x_0)$ is a fundamental set. Corollary 3.11 implies that $\overline{D(x_0)^0}$, so (i) holds. Now consider any $x \in X$. Then Γx is discrete and

$$d(\Gamma x, x_0) = \inf\{d(\gamma x, x_0) : \gamma \in \Gamma\}$$

is realized by some $\gamma x \in D(x_0)$. It follows that $D(x_0)$ intersects every Γ-orbit, so $X = \Gamma D(x_0)$ and (ii) holds. Finally, if $x \in D(x_0)^0$ and $x \neq \gamma x \in D(x_0)^0$, then

$$d(x, x_0) < d(\gamma x, x_0) \leq d(\gamma^{-1}\gamma x, x_0) = d(x, x_0),$$

a contradiction, so (iii) holds.

To show that $D(x_0)$ is locally finite, it suffices to check a bounded closed disc $C \subseteq X$ with center x_0 and radius $r \geq 0$. If $\gamma C \cap D(x_0) \neq \emptyset$ then for some $x \in D(x_0)$ we have $d(x_0, \gamma x) \leq r$ and

$$d(x_0, \gamma x_0) \leq d(x_0, \gamma x) + d(\gamma x, \gamma x_0) \leq r + d(x, x_0),$$

but $x \in S$ implies $d(x, x_0) = d(x_0, x) \leq d(x_0, \gamma x) \leq r$, so $d(x_0, \gamma x_0) \leq r + r = 2r$. This can hold for only finitely many γ, since Γx_0 is discrete, so $D(x_0)$ is locally finite. □

3.3 Hyperbolic space

Hyperbolic space $\mathbf{H}^3$ is the set

$$\mathbb{C} \times \mathbb{R}_{>0} = \{(x, y) = (x_1 + x_2 i, y) \in \mathbb{C} \times \mathbb{R} : y > 0\}.$$

equipped with the metric induced by the hyperbolic length element and volume

$$ds = \frac{\sqrt{dx_1^2 + dx_2^2 + dy^3}}{y} \qquad dV = \frac{dx_1 dx_2 dy}{y^3}.$$

Its boundary is the sphere at infinity $\mathbb{P}^1(\mathbb{C})$, analogous to the circle at infinity $\mathbb{P}^1(\mathbb{R})$ that forms the boundary of the hyperbolic plane $\mathbf{H}^2 := \mathbf{H}$.

The group $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ via linear fractional transformations, and we extend this to hyperbolic space by embedding $\mathbb{C} \times \mathbb{R}_{>0}$ in the Hamiltonians $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ (where $jz = \bar{x}j$ for $z \in \mathbb{C} \subseteq \mathbf{H}$) via $(x, y) \mapsto x + yj$ and defining

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}$$

for $z = x + yj \in \mathbb{H}$. One can then show that $\mathrm{PSL}_2(\mathbb{C})$ acts faithfully and transitively on $\mathbf{H}^3$ via isometries, and that every orientation preserving isometry of $\mathbf{H}^3$ corresponds to the action of an element of $\mathrm{PSL}_2(\mathbb{C})$ (and in general $\mathrm{Isom}(\mathbf{H}^3) \simeq \mathrm{PSL}_2(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z})$), just as every orientation preserving isometry of $\mathbf{H}^2$ corresponds to the action of an element of $\mathrm{PSL}_2(\mathbb{R}) \simeq \mathrm{PGL}_2^+(\mathbb{R})$ (and in general, $\mathrm{Isom}(\mathbf{H}^2) \simeq \mathrm{PGL}_2(\mathbb{R})$)

Definition 3.15. A Kleinian group is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{PSL}_2(\mathbb{C})$.

As with Fuchsan groups (discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{R})$), Kleinian groups act properly on $\mathbf{H}^3$ via isometries.

3.4 Hyperbolic Dirichlet domains

We now specialize to the case X is either hyperbolic space $\mathbf{H}^2$ or the hyperbolic plan $\mathbf{H}^3$, which we note are both complete, locally compact, uniquely geodesic spaces, and Γ is a Fuchsian or Kleinian group.

Definition 3.16. A fundamental domain for Γ is a connected fundamental set S whose boundary has measure zero; if the boundary of S is a countable union of geodesic segments then we say that S has geodesic boundary.

Theorem 3.17. *Let Γ be a Fuchsian group acting on $X = \mathbf{H}^2$ or a Kleinian group acting on $X = \mathbf{H}^3$. Let $z_0 \in X$ have stabilizer Γ_0 and let $u_0 \in X$ have trivial stabilizer in Γ_0. Then*

$$D(\Gamma; z_0) \cap D(\Gamma_0; u_0)$$

is a convex locally finite fundamental domain for Γ with geodesic boundary.

Proof. We first note that any Dirichlet domain $D(x_0)$ for a Fuchsian or Kleinian group G (including Γ and Γ_0) is convex with geodesic boundary because it is a countable intersection of Leibniz half-spaces $H(x_0, \gamma x_0)$ over $\gamma \in G$; Leibniz half-spaces in $\mathbf{H}^2$ or $\mathbf{H}^3$ are convex with geodesic boundary and G is countable (because G is discrete and X is locally compact second countable). It follows that $D(\Gamma; z_0) \cap D(\Gamma_0; u_0)$ is a convex set with geodesic boundary.

Theorem 3.14 implies that $D(\Gamma_0; u_0)$ is a locally finite fundamental set for Γ_0, and this implies that $D(\gamma; z_0) \cap D(\Gamma_0; u_0)$ is locally finite, since a subset of a locally finite set is locally finite. It remains only to show that $S := D(\Gamma; z_0) \cap D(\Gamma_0; u_0)$ is a fundamental set; we have already shown it is convex with geodesic boundary, so this will imply it is a fundamental domain.

The proof of Corollary 3.11 implies $\overline{S^0} = S$, since S is an intersection of Leibniz half-spaces, so (i) holds. Now consider any $z \in X$, and choose $\gamma \in \Gamma$ to minimize $d(z_0, \gamma z)$ and choose $\gamma_0 \in \Gamma_0$ so $\gamma_0 \gamma z \in D(\Gamma_0; u_0)$ (note $D(\Gamma_0; u_0)$ is a fundamental set). Then $d(z_0, \gamma_0 \gamma z) = d(z_0 \gamma z)$, by the minimality of γ, which implies that $\gamma_0 \gamma z \in D(\Gamma; z_0)$ and therefore in S, thus $\Gamma S = X$ and (ii) holds. Finally, if $\gamma \in \Gamma$ is nontrivial, either $\gamma \in \Gamma_0$ and $S^0 \cap (\gamma S)^0 = \emptyset$ because $D(\Gamma_0; u_0)$ is a fundamental set for Γ_0, or $\gamma \notin \Gamma_0$ and $S^0 \cap (\gamma S)^0 \neq \emptyset$ implies $d(z, z_0) < d(\gamma z, z_0) \leq d(z, z_0)$, a contradiction, as in the proof of Theorem 3.14; so (iii) holds. □

Corollary 3.18. *Let Γ by a Fuchsian group. If $z_0 \in \mathbf{H}$ has trivial stabilizer then the Dirichlet domain $D(\Gamma; z_0)$ is a connected, convex, locally finite fundamental domain for Γ with geodesic boundary.*

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.

[2] John Voight, *Quaternion algebras*, Springer, 2021.