2 Topological group actions and Fuchsian groups

This is summary of material from §33–34 of [3] presented in lecture, with proofs omitted.

2.1 Geodesic spaces

Definition 2.1. Let X be a metric space with distance $d\colon X\to\mathbb{R}_{\ge 0}$. An isometry $g\colon X\xrightarrow{\sim}X$ is a distance preserving homeomorphism; they form a group $\operatorname{Isom}(X)$ under composition.

In Lecture 1 we saw that the isometry group of the upper half-plane $\operatorname{Isom}(\mathbf{H})\simeq \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$.

In any metric space X we can define the length of a path $\phi\colon[0,1]\to X$ as the supremum over all finite subdivisions $0=t_0<t_1<\cdots<t_n<t_{n+1}=1$ of $[0,1]$ of the sum $\sum_{i=0}^n d(\phi(t_i),\phi(t_{i+1}))$ (which need not be finite). Note that this length depends only on the image ϕ, which we denote $C_{x_0\to x_1}$, where $x_0=\phi(0)$ and $x_1=\phi(1)$. Conversely, given a function ℓ assigning lengths to paths $C_{x_0\to x_1}$ in X, we can define

$$d(x,y)=\inf_{C_{x\to y}}\ell(C_{x\to y}),$$

and if the infimum exists for all $x,y\in X$, then d is a distance metric (positive on distinct points, symmetric, and satisfies the triangle inequality), called the intrinsic metric. Metric spaces whose distance functions can be defined in this way are called length spaces (or path metric spaces).

Definition 2.2. Shortest paths in in a length space (those that realize the infimum defining the intrinsic metric) are called geodesic segments. A geodesic in a length space is the image of a continuous map $\phi\colon\mathbb{R}\to X$ for which there exists $\epsilon>0$ such that for every $t_0,t_1\in\mathbb{R}$ with $t_0<t_1<t_0+\epsilon$ the image of the restriction of ϕ to $[t_0,t_1]$ is a geodesic segment. A geodesic space is a length space in which every pair of distinct points are connected by a geodesic segment, and if this geodesic segment is always unique, then X is a uniquely geodesic space.

In Lecture 1 we say that $\mathbf{H}$ is a uniquely geodesic space. More generally, we have the following theorem.

Theorem 2.3 (Hopf-Rinow)**.** *Every complete locally compact metric space is a geodesic space.*

Proof. See [1, Prop. I.3.7]. □

2.2 Haar measures

Definition 2.4. Let X be a locally compact Hausdorff space. The σ-algebra Σ of X is the collection of subsets of X generated by the open and closed sets under countable unions and countable intersections. Its elements are Borel sets, or measurable sets. A Borel measure on X is a countably additive function

$$\mu\colon\Sigma\to\mathbb{R}_{\ge 0}\cup\{\infty\}.$$

A Radon measure on X is a Borel measure on X that additionally satisfies

(i) $\mu(S)<\infty$ if S is compact,

(ii) $\mu(S)=\inf\{\mu(U): S\subseteq U,\ U \text{ open}\}$,

(iii) $\mu(S)=\sup\{\mu(C): C\subseteq S,\ C \text{ compact}\}$, for all Borel sets S.[1]

Definition 2.5. A (left) Haar measure μ on a locally compact Hausdorff group G is a nonzero Radon measure that is translation invariant: $\mu(S) = \mu(gS)$ for all $g \in G$ and measurable $S \subseteq G$. Replacing $\mu(gS)$ with $\mu(Sg)$ yields a right Haar measure, and if μ is both a left and a right Haar measure then μ and G are said to be unimodular.

It follows from the left and right invariance of the measure $d\alpha$ on $\mathrm{SL}_2(\mathbb{R})$ defined in Lecture 1 that $\mathrm{SL}_2(\mathbb{R})$ is unimodular. We used the homeomorphism $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \simeq \mathbf{H}$ defined by $g\mathrm{SO}_2 \to gi$ to define this measure. But $\mathrm{SL}_2(\mathbb{R}) \subseteq \mathrm{M}_2(\mathbb{R}) \simeq \mathbb{R}^4$ inherits a natural metric space structure with distance function $d(\alpha, \beta) = \|\alpha - \beta\|$, where

$$\left\|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right\|^2 = a^2 + b^2 + c^2 + d^2,$$

which is related to the distance metric d on $\mathbf{H}$ via

$$\|\alpha\|^2 = 2\cosh d(i, \alpha i).$$

Theorem 2.6 (Weil)**.** *Every locally compact Hausdorff group G has a Haar measure μ, and if μ' is any other Haar measure on G then $\mu' = \lambda\mu$ for some $\lambda \in \mathbb{R}_{>0}$.*

Proof. See [2, §7.2]. □

2.3 Topological group actions

Recall that a covering map $f: X \to Y$ is a continuous surjection for which every $y \in Y$ has an open neighborhood V whose preimage $f^{-1}(V)$ is a disjoint union of open $U_\alpha \subseteq X$ on which π restricts to a homeomorphism $U_\alpha \to V$, in which case X is a covering space of Y. A deck transformation of is a homeomorphism $\phi: X \xrightarrow{\sim} X$ that preserves f, meaning $f \circ \phi = f = f \circ \phi$. The deck transformations form a group $\mathrm{Deck}(f)$, which acts on X, with each $\mathrm{Deck}(f)$-orbit lying in a fiber of f. If we actually have $\mathrm{Deck}(f)\backslash X \simeq Y$, then f is a normal covering map.

Definition 2.7. The action of a topological group G on a topological space X is a covering space action if every $x \in X$ has an open neighborhood U for which

$$\rho(G, U) := \{g \in G : U \cap gU \neq \emptyset\} = \{1\}.$$

If G is a covering space action, then $\pi: X \to G\backslash X$ is a normal covering map and $G \subseteq \mathrm{Deck}(\pi)$.

Covering space actions are free actions, since $G_x \subseteq \rho(G, U)$ for every open $U \ni x$, but the actions we are most interested have non-trivial stabilizers and cannot be covering space actions. This leads to the following definition.

Definition 2.8. The action of a topological group G on a topological space X is wandering if every $x \in X$ has an open neighborhood U for which $\rho(G, U)$ is finite.

Lemma 2.9. *For a Hausdorff group G acting on a Hausdorff space X the following are equivalent:*

(i) *The action of G is wandering;*

(ii) *Every $x \in X$ has open neighborhood U for which $\rho(G, U) = G_x$ is finite.*

[1] Some authors additionally require X to be σ-compact (a countable union of compact sets); all the X we shall consider are σ-compact.

If G acts freely these equivalent conditions hold if and only if the G-action is a covering space action.

Proof. This is [3, Lemma 34.3.6]. □

Lemma 2.10. *For a Hausdorff group G acting on a Hausdorff space X the following are equivalent:*

(i) *$G\backslash X$ is Hausdorff;*

(ii) *If $Gx \neq Gy$ then there are open $U \ni x$ and $V \ni y$ for which $gU \cap V = \emptyset$ for all $g \in G$;*

(iii) *The set $\{(x, gx) : g \in G, x \in X\} \subseteq X \times X$ is closed.*

Proof. This is [3, Lemma 34.4.1]. □

Recall that a continuous map $f : X \to Y$ is quasiproper if inverse images of compact sets are compact, and if it is also a closed map then f is proper.

Lemma 2.11. *Every quasiproper continuous map of locally compact Hausdorff spaces is proper.*

Proof. See [3, Lemma 34.4.5]. □

Definition 2.12. A topological group G acts properly on a topological space X if the action map $\lambda\colon G \times X \to X \times X$ defined by $(g, x) \mapsto (x, gx)$ is proper.

Proposition 2.13. *Let G be a Hausdorff group acting properly on a Hausdorff space X. Then the following hold:*

(i) *$G\backslash X$ is Hausdorff;*

(ii) *Every G-orbit Gx is closed;*

(iii) *The map $G/G_x \to Gx$ defined by $g \mapsto gx$ is a homeomorphism;*

(iv) *Every stabilizer G_x is compact.*

Proof. (i)–(iii) follow from Lemma 2.10, since $\lambda(G \times X)$ is closed, and (iv) follows from the fact that the singleton set $(x, x) \in X \times X$ is compact, so $G_x \simeq G_x \times \{x\} = \lambda^{-1}(x, x)$ is too. □

Theorem 2.14. *Let G be a Hausdorff group acting on a locally compact Hausdorff space X. The following are equivalent:*

(i) *G is discrete and the G-action is proper;*

(ii) *For every compact $K \subset X$ the set $\{g \in G : K \cap gK \neq \emptyset\}$ is finite;*

(iii) *For every compact $K, L \subset X$ the set $\{g \in G : K \cap gL \neq \emptyset\}$ is finite;*

(iv) *For all $x, y \in X$ there exist open $U \ni x$, $V \ni y$ for which $\{g \in G : U \cap gV \neq \emptyset\}$ is finite.*

These four equivalent conditions imply the following two:

(v) *The G-action is wandering;*

(vi) *Every orbit Gx is discrete and every stabilizer G_x is finite.*

When X is metrizable and G acts via isometries, all six conditions are equivalent.

Proof. This is [3, Theorem 34.5.1]. □

2.4 Fuchsian groups

Proposition 2.15. *Let* Γ *be a subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *or* $\mathrm{PSL}_2(\mathbb{R})$*. The action of* Γ *on* $\mathbf{H}$ *is wandering if and only if* Γ *is discrete.*

Proof. The action of $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ is is the same as that of its image in $\mathrm{PSL}_2(\mathbb{R})$, and Γ is discrete if and only if its image in $\mathrm{PSL}_2(\mathbb{R})$ is discrete. For $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ see [3, Proposition 34.7.2]. □

Definition 2.16. A Fuchsian group is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{R})$.

References

[1] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Springer, 1999.

[2] Joe Diestel and Angela Spalsbury, *The Joys of Haar Measure*, Amer. Math. Society, 2014.

[3] John Voight, *Quaternion algebras*, Springer, 2021.