These notes summarize the material in §6.1-6.3 of [1] presented in lecture.

18.1 The Jacobian of a compact Riemann surface

Let $X$ be a compact Riemann surface of genus $g$ with holomorphic atlas $\{\psi_i: U_i \rightarrow V_i \subseteq \mathbb{C}\}$ and transition maps $\psi_{ij}: \psi_j \circ \psi_i^{-1}: V_i \rightarrow V_j$ for all $U_i \cap U_j \neq \emptyset$, where $V_{ij} := \psi_i(U_i \cap U_j)$. In Lecture 7 we considered the space $\Omega^1(X)$ of meromorphic differentials $\omega$ of degree $n$ on $X$ defined by a collection of meromorphic functions $\{\omega_i: V_i \rightarrow \mathbb{C}\}$ satisfying

$$\omega_i = (\omega_j \circ \psi_{ij} \left( \frac{d\psi_i}{d\psi_j} \right))^n$$

on $V_{ij}$ (modulo equivalence $\{\omega_i\} = \omega \cong \theta = \{\theta_i\}$ if $\{\omega_i\} \cup \{\theta_i\}$ is a meromorphic differential). We associated to each $\omega$ a divisor $\text{div}(\omega) = \sum \text{ord}_p(\omega_i \circ \psi_i)$ in the divisor group $\text{Div}(X)$ (the free abelian group generated by $P \in X(\mathbb{C})$), and call a meromorphic differential $\omega$ holomorphic if $\omega = 0$ or $\text{div}(\omega) \geq 0$ (meaning every coefficient of $\text{div}(\omega)$ is nonnegative). We showed that the space of holomorphic differentials $\Omega^1_0(X)$ is a complex vector space of dimension $g$.

For a lattice $\Gamma \leq \text{SL}_2(\mathbb{R})$, we showed how to associate to each automorphic $f \in A_{2g}(\Gamma)$ a meromorphic differential $\omega_f$ of degree $n$, and we showed that $\omega_f$ is holomorphic precisely when $f$ is a cusp form. For cusp forms of weight 2 we get a holomorphic differential of degree 1 and the map $f \mapsto \omega_f$ induces an isomorphism $\text{SL}_2(\Gamma) \simeq \Omega^1_0(X_F)$, where $X_F := \Gamma \backslash \text{H}^*$. The homology group of $H_1(X, \mathbb{Z})$ formed by equivalence classes of closed loops on $X$ (1-cycles modulo boundaries) is a free abelian group of rank $2g$. Each $\gamma \in H_1(X, \mathbb{Z})$ determines a linear form on $\Omega^1_0(X)$ via the map

$$\omega \mapsto \int \gamma \omega.$$

This induces an injective map $\mathbb{Z}^{2g} \simeq H_1(X, \mathbb{Z}) \hookrightarrow \Omega^1_0(X)^\vee = \text{Hom}(\Omega^1_0(X), \mathbb{C}) \simeq \mathbb{C}^g$ that allows us to view $H_1(X, \mathbb{Z})$ as a lattice $\Lambda = H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ in the vector space $V = \Omega^1_0(X)^\vee \simeq \mathbb{C}^g$. The quotient $\Omega^1_0(X)^\vee / H_1(X, \mathbb{Z})$ is a complex torus $V / \Lambda$ that can be equipped with a Riemann form (a positive definite Hermitian form $V \times V \rightarrow \mathbb{C}$ that maps $\Lambda \times \Lambda$ to $\mathbb{Z}$) that induces a polarization (a homomorphism to the dual torus) which makes it an abelian variety called the Jacobian of $X$, denoted $\text{Jac}(X)$.

It turns out that because our complex torus $V / \Lambda$ arose from a Riemann surface, this polarization is actually an isomorphism and $\text{Jac}(X)$ is a principally polarized abelian variety.

Recall that compact Riemann surfaces are also algebraic curves (varieties of dimension 1) that are smooth projective curves over $\mathbb{C}$. The Jacobian of a compact Riemann surfaces is also an algebraic variety, and in fact a smooth projective variety of dimension $g$. Moreover, as a complex torus, the Jacobian is naturally equipped with a group operation (add vectors in $V$ and reduce modulo $\Lambda$), and this induces a group structure on the corresponding projective variety that is defined by morphisms, making it an algebraic group. This group is obviously abelian (since $V / \Lambda$ is), but in fact it any smooth projective variety that is also an algebraic group, including those that are not necessarily Jacobians, has an abelian group structure; thus all projective algebraic groups are called abelian varieties (note that linear algebraic groups like $\text{GL}_n$ are not projective).

The moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension $g$ has dimension $g(g+1)/2$, while the moduli space $\mathcal{M}_g$ of smooth projective curves of genus $g \geq 2$ has dimension $3g - 3$. Thus for large $g$ almost all principally polarized abelian varieties are not Jacobians. But for $g = 1$ every abelian variety is a Jacobian (in fact an elliptic curve), and for $g = 2, 3$
the dimensions of $\mathcal{A}_g$ and $\mathcal{M}_g$ coincide, meaning that almost all principally polarized abelian varieties of dimension $g \leq 3$ are Jacobians.

Recall the divisor class group $\text{Pic}^0(X) = \text{Div}^0(X)/\text{Prin}(X)$ of divisors of degree zero modulo principal divisors those associated to elements of the function field $\mathbb{C}(X)$. We should note that our identification of $\text{Pic}^0(X)$ with the divisor class group is valid for smooth projective curves over an algebraically closed field, but $\text{Pic}^0(X)$ also has an intrinsic definition that may differ from the divisor class group in other settings.

There is a canonical isomorphism $\text{Pic}^0(X) \cong \text{Jac}(X)$ defined as follows. Given $D \in \text{Div}^0(X)$ we write $D$ in the form

$$D = \sum_i (P_i - Q_i)$$

(this is always possible for $X/\mathbb{C}$) and associate to $D$ the linear form

$$\omega \mapsto \sum_i \int_{P_i}^{Q_i} \omega$$

in $\Omega^1_0(X)^\vee$. Reducing modulo the lattice $H_1(X, \mathbb{Z})$ yields an element of $\text{Jac}(X)$. This is the Abel-Jacobi map, which can be viewed as an extension of the map $X \to \text{Pic}^0(X)$ that sends $x \in X(\mathbb{C})$ to the divisor $x - x_0$, where $x_0 \in X(\mathbb{C})$ is a fixed rational point.

If $\phi: X \to Y$ is a nonconstant holomorphic map of compact Riemann surfaces, we have a pullback map $\phi^*: \Omega^1_Y \to \Omega^1_X$ defined by $\phi^*f = f \circ \phi$, which extends to a linear map of holomorphic differentials

$$\phi^*: \Omega^1_0(Y) \to \Omega^1_0(X).$$

Taking duals induces the pushforward map on Jacobians $h_*: \text{Jac}(X) \to \text{Jac}(Y)$. There is also a pushforward map $h_*: \text{Pic}^0(X) \to \text{Pic}^0(Y)$ induced by the pushforward map $\text{Div}^0(X) \to \text{Div}^0(Y)$ that sends $P \in \mathbb{C}(X)$ to $h(P) \in \mathbb{C}(Y)$, and the two maps are compatible with the isomorphisms $\text{Jac}(X) \cong \text{Pic}^0(X)$ and $\text{Jac}(Y) \cong \text{Pic}^0(Y)$.

There is also a pullback map $h^*: \text{Pic}^0(Y) \to \text{Pic}^0(X)$ induced by the map $\text{Div}^0(Y) \to \text{Div}^0(X)$ that sends $Q \in Y(\mathbb{C})$ to the sum of the points $P \in \phi^{-1}(Q)$, and a corresponding pullback map $\text{Jac}(Y) \to \text{Jac}(X)$ that can be defined using the trace map $\Omega^1_0(X) \to \Omega^1_0(Y)$. These pullback maps are also compatible with the isomorphisms $\text{Jac}(X) \cong \text{Pic}^0(X)$ and $\text{Jac}(Y) \cong \text{Pic}^0(Y)$.

### 18.2 Modular Jacobians and Hecke operators

Let $\Gamma_1$ and $\Gamma_2$ be congruence subgroups, $\alpha \in \text{GL}_2^+(\mathbb{Q})$ and $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$. If we decompose $\Gamma_3 \backslash \Gamma_2$ as $\bigsqcup_i \Gamma_3 \gamma_i$ and $\Gamma_1 \alpha \Gamma_2$ as $\bigsqcup_j \Gamma_1 \beta_j$ and let $\Gamma_3' := \alpha \Gamma_3 \alpha^{-1} = \Gamma_1 \cap \alpha \Gamma_2 \alpha^{-1}$ then

$$\Gamma_2 \leftarrow \Gamma_3 \cong \Gamma_3' \rightarrow \Gamma_1$$

where the isomorphism is $\gamma \mapsto \alpha \gamma \alpha^{-1}$ and the other arrows are inclusions. This induces morphism of modular curves

$$X_{\Gamma_2} \xleftarrow{\pi_2} X_{\Gamma_1} \cong X_{\Gamma_3'} \xrightarrow{\pi_1} X_{\Gamma_1}$$

Each point of $X_{\Gamma_2}$ is mapped via $\pi_1 \circ \alpha \circ \pi_2^{-1}$ to a multiset of points of $X_{\Gamma_1}$. Viewing points on $X_{\Gamma_1}$ and $X_{\Gamma_2}$ as $\Gamma_i$-orbits in $\mathbb{H}$ we have

$$\Gamma_2 \xrightarrow{\pi_2^{-1}} \{\Gamma_3 \gamma_i z\} \xrightarrow{\alpha} \{\Gamma_3' \beta_j z\} \xrightarrow{\pi_1} \{\Gamma_1 \beta_j z\}$$

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We have a corresponding map $\Gamma_1\alpha\Gamma_2 : \text{Div}(X_{\Gamma_2}) \to \text{Div}(X_{\Gamma_1})$, which induces a map that we can view as map of Jacobians of modular curves: $\text{Jac}(X_{\Gamma_2}) \to \text{Jac}(X_{\Gamma_1})$.

Applying $S_2(\Gamma) \cong \Omega_0^1(X)\Gamma$ and taking the corresponding isomorphism of the dual space yields an isomorphism

$$\text{Jac}(X) \cong S_2(\Gamma)^\vee / H_1(X, \mathbb{Z})$$

that is compatible with pushforward and pullback maps. The map

$$\Gamma_1\alpha\Gamma_2 : S_2(\Gamma_1) \to S_2(\Gamma_2)$$

induces a map on the dual spaces, and a map on Jacobians

$$\Gamma_1\alpha\Gamma_2 : \text{Jac}(X_2) \to \text{Jac}(X_1),$$

which defines an action of the Hecke algebra $T = \mathbb{Z}[\Gamma\alpha\Gamma]$ on the Jacobian $\text{Jac}(X)$. 

### 18.3 Eichler-Shimura

Now let $f \in S_2(\Gamma_1(N))$ be a newform, hence an eigenform of $\mathbb{T}(N) = \mathbb{Z}[\Gamma_1(N)\Delta(N)/\Gamma_1(N)]$, and consider the eigenvalue map

$$\lambda_f : \mathbb{T}(N) \to \mathbb{C}$$

defined by $f|_2 T = \lambda_f(T)f$. Let $I_f := \ker \lambda_f := \{T \in \mathbb{T}(N) : f|_2 T = 0\}$. Since $\mathbb{T}(N)$ acts on $\text{Jac}(X_1(N))$, we can view $I_f\text{Jac}(X_1(N))$ as a subgroup of the divisor class group $\text{Pic}^0(X_1(N))$ corresponding to an abelian subvariety of $\text{Jac}(X_1(N))$.

**Definition 18.1.** The **modular abelian variety** $A_f$ associated to the newform $f \in S_2(\Gamma_1(N))$ is the quotient $\text{Jac}(X_1(N))/I_f\text{Jac}(X_1(N))$.

The eigenvalues $\lambda_f(T)$ and the Fourier coefficients $a_n(f)$ are algebraic integers that generate a number field $\mathbb{Q}(f)$. By letting $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act on the coefficients of $f$, we obtain a Galois conjugate newform $f^\sigma$ associated to each $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (equivalently, we can view the coefficients of $f$ as abstract elements of the number field $\mathbb{Q}(f)$ and consider the different embedding of $\mathbb{Q}(f)$ into $\mathbb{C}$.

The set of all Galois conjugates of $f$ spans a subspace of $S_2(\Gamma_1(N))$ of dimension $[\mathbb{Q}(f) : \mathbb{Q}]$, and this is equal to the dimension of $A_f$; indeed $A_f$ is invariant under the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and can be viewed as an abelian variety over $\mathbb{Q}$. Replacing $f$ with a Galois conjugate newform does not change $A_f$.

If we let $V_f$ denote the subspace of $S_2(\Gamma_1(N))$ spanned by the Galois conjugates of $f$ and consider $H_1(X_1(N), \mathbb{Z}) \to \Omega_0^1(X_1(N)) \cong S_2(\Gamma_1(N))$, restricting $H_1(X_1(N), \mathbb{Z})$ to its action on elements of $V_f \subseteq S_2(\Gamma_1(N))$ yields a lattice $\Lambda_f \subseteq V_f$, and we can alternatively define $A_f$ as the abelian variety corresponding to the complex torus $V_f/\Lambda_f$ (which also admits a polarization).

### References
