These notes summarize the material in §6.1-6.3 of [1] presented in lecture.

18.1 The Jacobian of a compact Riemann surface

Let X be a compact Riemann surface of genus g with holomorphic atlas $\{\psi_i\colon U_i \xrightarrow{\sim} V_i \subseteq \mathbb{C}\}$ and transition maps $\psi_{ij}\colon \psi_j \circ \psi_i^{-1} : V_{ij} \to V_j$ for all $U_i \cap U_j \neq \emptyset$, where $V_{ij} \coloneqq \psi_i(U_i \cap U_j)$. In Lecture 7 we considered the space $\Omega^1(X)$ of meromorphic differentials ω of degree n on X defined by a collection of meromorphic functions $\{\omega_i\colon V_i \to \mathbb{C}\}$ satisfying

$$\omega_i = (\omega_j \circ \psi_{ij}\left(\frac{d\psi_i}{d\psi_j}\right)^n$$

on V_{ij} (modulo equivalence $\{\omega_i\} = \omega \simeq \theta = \{\theta_i\}$ if $\{\omega_i\} \cup \{\theta_i\}$ is a meromorphic differential). We associated to each ω a divisor $\operatorname{div}(\omega) = \sum_P \operatorname{ord}_P(\omega_i \circ \psi_i)$ in the divisor group $\operatorname{Div}(X)$ (the free abelian group generated by $P \in X(\mathbb{C})$), and call a meromorphic differential ω holomorphic if $\omega = 0$ or $\operatorname{div}(\omega) \geq 0$ (meaning every coefficient of $\operatorname{div}(\omega)$ is nonnegative). We showed that the space of holomorphic differentials $\Omega^1_0(X)$ is a complex vector space of dimension g.

For a lattice $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$, we showed how to associate to each automorphic $f \in A_{2n}(\Gamma)$ a meromorphic differential ω_f of degree n, and we showed that ω_f is holomorphic precisely when f is a cusp form. For cusp forms of weight 2 we get a holomorphic differential of degree 1 and the map $f \mapsto \omega_f$ induces an isomorphism $S_2(\Gamma) \simeq \Omega^1_0(X_\Gamma)$, where $X_\Gamma \coloneqq \Gamma\backslash\mathbf{H}^*$.

The homology group of $H_1(X,\mathbb{Z})$ formed by equivalence classes of closed loops on X (1-cycles modulo boundaries) is a free abelian group of rank $2g$. Each $\gamma \in H_1(X,\mathbb{Z})$ determines a linear form on $\Omega^1_0(X)$ via the map

$$\omega \mapsto \int_\gamma \omega.$$

This induces an injective map $\mathbb{Z}^{2g} \simeq H_1(X,\mathbb{Z}) \hookrightarrow \Omega^1_0(X)^\vee = \operatorname{Hom}(H^1_0(X),\mathbb{C}) \simeq \mathbb{C}^g$ that allows us to view $H_1(X,\mathbb{Z})$ as a lattice $\Lambda = H_1(X,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$ in the vector space $V = \Omega^1_0(X)^\vee \simeq \mathbb{C}^g$. The quotient $H^1_0(X)^\vee/H_1(X,\mathbb{Z})$ is a complex torus V/Λ that can be equipped with a Riemann form (a positive definite Hermitian form $V \times V \to \mathbb{C}$ that maps $\Lambda \times \Lambda$ to $\mathbb{Z}$) that induces a polarization (a homomorphism to the dual torus) which makes it an abelian variety called the Jacobian of X, denoted $\operatorname{Jac}(X)$.

It turns out that because our complex torus V/Λ arose from a Riemann surface, this polarization is actually an isomorphism and $\operatorname{Jac}(X)$ is a principally polarized abelian variety.

Recall that compact Riemann surfaces are also algebraic curves (varieties of dimension 1) that are smooth projective curves over $\mathbb{C}$. The Jacobian of a compact Riemann surfaces is also an algebraic variety, and in fact a smooth projective variety of dimension g. Moreover, as a complex torus, the Jacobian is naturally equipped with a group operation (add vectors in V and reduce modulo Λ), and this induces a group structure on the corresponding projective variety that is defined by morphisms, making it an algebraic group. This group is obviously abelian (since V/Λ is), but in fact it any smooth projective variety that is also an algebraic group, including those that are not necessarily Jacobians, has an abelian group structure; thus all projective algebraic groups are called abelian varieties (note that linear algebraic groups like GL_n are not projective).

The moduli space $\mathscr{A}_g$ of principally polarized abelian varieties of dimension g has dimension $g(g+1)/2$, while the moduli space $\mathscr{M}_g$ of smooth projective curves of genus $g \geq 2$ has dimension $3g-3$. Thus for large g almost all principally polarized abelian varieties are not Jacobians. But for $g = 1$ every abelian variety is a Jacobian (in fact an elliptic curve), and for $g = 2,3$

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the dimensions of $\mathscr{A}_g$ and $\mathscr{M}_g$ coincide, meaning that almost all principally polarized abelian varieties of dimension $g \leq 3$ are Jacobians.

Recall the divisor class group $\mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/\mathrm{Prin}(X)$ of divisors of degree zero modulo principal divisors those associated to elements of the function field $\mathbb{C}(X)$. We should note that our identification of $\mathrm{Pic}^0(X)$ with the divisor class group is valid for smooth projective curves over an algebraically closed field, but $\mathrm{Pic}^0(X)$ also has an intrinsic definition that may differ from the divisor class group in other settings.

There is a canonical isomorphism $\mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Jac}(X)$ defined as follows. Given $D \in \mathrm{Div}^0(X)$ we write D in the form

$$D = \sum_i (P_i - Q_i)$$

(this is always possible for $X/\mathbb{C}$) and associate to D the linear form

$$\omega \mapsto \sum_i \int_{P_i}^{Q_i} \omega$$

in $\Omega_0^1(X)^\vee$. Reducing modulo the lattice $H_1(X, \mathbb{C})$ yields an element of $\mathrm{Jac}(X)$. This is the Abel-Jacobi map, which can be viewed as an extension of the map $X \to \mathrm{Pic}^0(X)$ that sends $x \in X(\mathbb{C})$ to the divisor $x - x_0$, where $x_0 \in X(\mathbb{C})$ is a fixed rational point.

If $\phi : X \to Y$ is a nonconstant holomorphic map of compact Riemann surfaces, we have a pullback map $\phi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ defined by $\phi^* f = f \circ \phi$, which extends to a linear map of holomorphic differentials

$$\phi^* : \Omega_0^1(Y) \to \Omega_0^1(X).$$

Taking duals induces the pushforward map on Jacobians $h_* : \mathrm{Jac}(X) \to \mathrm{Jac}(Y)$. There is also a pushforward map $h_* :\ \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ induced by the pushforward map $\mathrm{Div}^0(X) \to \mathrm{Div}^0(Y)$ that sends $P \in \mathbb{C}(X)$ to $h(P) \in \mathbb{C}(Y)$, and the two maps are compatible with the isomorphisms $\mathrm{Jac}(X) \simeq \mathrm{Pic}^0(X)$ and $\mathrm{Jac}(Y) \simeq \mathrm{Pic}^0(Y)$.

There is also a pullback map $h^* :\ \mathrm{Pic}_0(Y) \to \mathrm{Pic}_0(X)$ induced by the map $\mathrm{Div}^0(Y) \to \mathrm{Div}^0(X)$ that sends $Q \in Y(\mathbb{C})$ to the sum of the points $P \in \phi^{-1}(Q)$, and a corresponding pullback map $\mathrm{Jac}(Y) \to \mathrm{Jac}(X)$ that can be defined using the trace map $\Omega_0^1(X) \to \Omega_0^1(Y)$. These pullback maps are also compatible with the isomorphisms $\mathrm{Jac}(X) \simeq \mathrm{Pic}^0(X)$ and $\mathrm{Jac}(Y) \simeq \mathrm{Pic}^0(Y)$.

18.2 Modular Jacobians and Hecke operators

Let Γ_1 and Γ_2 be congruence subgroups, $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$. If we decompose $\Gamma_3\backslash\Gamma_2$ as $\coprod_i \Gamma_3\gamma_i$ and $\Gamma_1\alpha\Gamma_2$ as $\coprod_j \Gamma_1\beta_j$ and let $\Gamma_3' := \alpha\Gamma_3\alpha^{-1} = \Gamma_1 \cap \alpha\Gamma_2\alpha^{-1}$ then

$$\Gamma_2 \longleftarrow \Gamma_3 \xrightarrow{\sim} \Gamma_3' \longrightarrow \Gamma_1$$

where the isomorphism is $\gamma \mapsto \alpha\gamma\alpha^{-1}$ and the other arrows are inclusions. This induces morphism of modular curves

$$X_{\Gamma_2} \xleftarrow{\pi_2} X_{\Gamma_3} \simeq X_3' \xrightarrow{\pi_1} X_{\Gamma_1}$$

Each point of X_{Γ_2} is mapped via $\pi_1 \circ \alpha \circ \pi_2^{-1}$ to a multiset of points of X_{Γ_1}. Viewing points on X_{Γ_1} and X_{Γ_2} as Γ_i-orbits in $\mathbf{H}^*$ we have

$$\Gamma_2 z \overset{\pi_2^{-1}}{\mapsto} \{\Gamma_3\gamma_i z\} \overset{\alpha}{\to} \{\Gamma_3'\beta_j z\} \overset{\pi_1}{\to} \{\Gamma_1\beta_j z\}$$

We have a corresponding map $\Gamma_1 \alpha \Gamma_2 \colon \operatorname{Div}(X_{\Gamma_2}) \to \operatorname{Div}(X_{\Gamma_1})$, which induces a map

that we can view as map of Jacobians of modular curves: $\operatorname{Jac}(X_{\Gamma_2}) \to \operatorname{Jac}(X_{\Gamma_1})$.

Applying $S_2(\Gamma) \simeq \Omega_0^1(X_\Gamma)$ and taking the corresponding isomorphism of the dual space yields an isomorphism

$$\operatorname{Jac}(X_\Gamma) \simeq S_2(\Gamma)^\vee / H_1(X_\Gamma, \mathbb{Z})$$

that is compatible with pushforward and pullback maps. The map

$$\Gamma_1 \alpha \Gamma_2 \colon S_2(\Gamma_1) \to S_2(\Gamma_2)$$

induces a map on the dual spaces, and a map on Jacobians

$$\Gamma_1 \alpha \Gamma_2 \colon \operatorname{Jac}(X_2) \to \operatorname{Jac}(X_1),$$

which defines an action of the Hecke algebra $\mathrm{T} = \mathbb{Z}[\Gamma \alpha \Gamma]$ on the Jacobian $\operatorname{Jac}(X_\Gamma)$.

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Now let $f \in S_2(\Gamma_1(N))$ be a newform, hence an eigenform of $\mathbb{T}(N) = \mathbb{Z}[\Gamma_1(N)\Delta(N)/\Gamma_1(N)]$, and consider the eigenvalue map

$$\lambda_f : \mathbb{T}(N) \to \mathbb{C}$$

defined by $f|_2 T = \lambda_f(T) f$. Let $I_f := \ker \lambda_f := \{T \in \mathbb{T}(N) : f|_2 T = 0\}$. Since $\mathbb{T}(N)$ acts on $\operatorname{Jac}(X_1(N))$, we can view $I_f \operatorname{Jac}(X_1(N))$ as a subgroup of the divisor class group $\operatorname{Pic}^0(X_1(N))$ corresponding to an abelian subvariety of $\operatorname{Jac}_1(X_1(N))$.

Definition 18.1. The modular abelian variety A_f associated to the newform $f \in S_2(\Gamma_1(N))$ is the quotient $\operatorname{Jac}(X_1(N))/I_f \operatorname{Jac}(X_1(N))$.

The eigenvalues $\lambda_f(T)$ and the Fourier coefficients $a_n(f)$ are algebraic integers that generate a number field $\mathbb{Q}(f)$. By letting $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act on the coefficients of f, we obtain a Galois conjugate newform f^σ associated to each $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (equivalently, we can view the coefficients of f as abstract elements of the number field $\mathbb{Q}(f)$ and consider the different embedding of $\mathbb{Q}(f)$ into $\mathbb{C}$.

The set of all Galois conjugates of f spans a subspace of $S_2(\Gamma_1(N))$ of dimension $[\mathbb{Q}(f) : \mathbb{Q}]$, and this is equal to the dimension of A_f; indeed A_f is invariant under the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and can be viewed as an abelian variety over $\mathbb{Q}$. Replacing f with a Galois conjugate newform does not change A_f.

If we let V_f denote the subspace of $S_2(\Gamma_1(N))$ spanned by the Galois conjugates of f and consider $H_1(X_1(N), \mathbb{Z}) \hookrightarrow \Omega_0^1(X_1(N)) \simeq S_2(\Gamma_1(N))$, restricting $H_1(X_1(N), \mathbb{Z})$ to its action on elements of $V_f \subseteq S_2(\Gamma_1(N))$ yields a lattice $\Lambda_f \subseteq V_f$, and we can alternatively define A_f as the abelian variety corresponding to the complex torus V_f/Λ_f (which also admits a polarization).

References

[1] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Springer, 2005.

[2] Toshitsune Miyake, *Modular forms*, Springer, 2006.