These notes summarize the material in §6.3-4 presented in lecture. Recall the weight-k slash operator associated to $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ on functions $f : \mathbf{H} \to \mathbb{C}$ is defined by

$$(f|_k\alpha)(z) := (\det\alpha)^k j(\alpha,z)^{-k} f(\alpha z),$$

where $j(\alpha,z) := cz+d$ for $\alpha = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and $z \in \mathbf{H}$, and we have $\mathrm{Im}(\alpha z) = \frac{\det(\alpha)\,\mathrm{Im}(z)}{|j(\alpha,z)|^2}$.

17.1 Function spaces of automorphic forms

Let $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ be a lattice (which we recall is a discrete cofinite subgroup of $\mathrm{SL}_2(\mathbb{R})$, equivalently, a finitely generated Fuchsian of the first kind; see Lecture 5). Let $k \le in\mathbb{Z}_{>2}$, and let χ be a finite order character of Γ with $\chi(-1) = (-1)^k$ if $-1 \in \Gamma$.

For any measurable function $f : \mathbf{H} \to \mathbb{C}$ that satisfies

$$(f|_k\gamma) = \chi(\gamma)f(z) \qquad (\text{for all } \gamma \in \Gamma) \tag{1}$$

we define

$$\|f\|_{\Gamma,p} := \begin{cases} \left(\int_{\Gamma\backslash\mathbf{H}} |f(z)\,\mathrm{im}(z)^{k/2}|^p\,d\nu(z)\right)^{1/p} & 1 \le p < \infty \\ \operatorname{ess\,sup}_{z\in\mathbf{H}} |f(z)\,\mathrm{Im}(z)^{k/2}| & p = \infty. \end{cases}$$

For $\gamma \in \Gamma$ we have

$$|f(\gamma z)\,\mathrm{Im}(\gamma z)^{k/2}| = |(f|k\gamma)(z) j(\gamma,z)^{-k}\,\mathrm{Im}(\gamma z)^{k/2}| = |f(z)\,\mathrm{Im}(z)^{k/2}|,$$

which ensures that $\|f\|_{\Gamma,p}$ is well defined. We now let $L_k^p(\Gamma,\chi)$ denote the set of measurable functions $f : \mathbf{H} \to \mathbb{C}$ that satisfy (1) with $\|f\|_{\Gamma,p} < \infty$, and use $H_k^p(\Gamma,\chi)$ to denote the subspace of holomorphic $f \in L_k^p(\Gamma,\chi)$. Then $L_k^2(\Gamma,\chi)$ is a Hilbert space with inner product

$$\langle f, g\rangle := \int_{\Gamma\backslash\mathbf{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^k\,d\nu(z).$$

Now $\Gamma\backslash\mathbf{H}$ has finite volume (Γ is a lattice), which implies $L_k^\infty(\Gamma,\chi) \subseteq L_k^p(\Gamma,\chi)$ and $H_k^\infty(\Gamma,\chi) \subseteq H_k^p(\Gamma,\chi)$ for all $p \in \mathbb{R}_{\ge 1}$. Moreover, we have $H_k^\infty(\Gamma,\chi) = S_k(\Gamma,\chi)$ (see [1, Theorem 2.1.5]) and the restriction of the inner product of $L_k^2(\Gamma,\chi)$ to $H_k^\infty(\Gamma,\chi)$ is just a rescaled version of the Petersson inner product that omits the leading factor $\nu(\Gamma\backslash\mathbf{H})^{-1}$.

Theorem 17.1. $H_k^2(\Gamma,\chi) = H_k^\infty(\Gamma,\chi)$

Proof. This is immediate when Γ has no cusps, since then $\Gamma\backslash\mathbf{H}$ is compact. So let x_0 is a cusp of Γ and $f \in H_k^2(\Gamma,\chi)$. By replacing Γ with a finite index subgroup we can assume χ is trivial. Pick $\sigma \in \mathrm{SL}_2(\mathbb{R})$ so that $\sigma x = \infty$ and put $\pm\sigma^{-1}\Gamma_{x_0}\sigma = \pm\left\langle\left(\begin{smallmatrix} 1 & h\\ 0 & 1\end{smallmatrix}\right)\right\rangle$, with $h \in \mathbb{R}_{>0}$. If we pick a neighborhood $U_l = \{z \in \mathbf{H} \mid \mathrm{Im}(z) > l\}$ of ∞ and let $\sum a_n e^{\pi i n z/h}$ denote the Fourier espansion of $f|_k\sigma$, then

$$\begin{aligned}
\infty &> \int_{\Gamma\backslash\mathbf{H}} |f(z)|^2\,\mathrm{Im}(z)^k\,d\nu(z) \\
&\ge \frac{1}{2}\iint_{\substack{0\le \mathrm{Re}(z)\le 2h\\ l\le \mathrm{Im}(z)<\infty}} |f(\sigma z) j(\sigma,z)^{-k}|^2\,\mathrm{Im}(z)^k\,d\nu(z) \\
&= \frac{1}{2}\int_l^\infty\int_0^{2h} \sum_{m,n\in\mathbb{Z}} a_m\bar{a}_n e^{-\pi y(m+n)/h} e^{\pi i x(m-n)/h} y^{k-2}\,dx\,dy \\
&\ge h|a_n|^2 \int_l^\infty e^{-2\pi y n/h} y^{k-2}\,dy.
\end{aligned}$$

for any $n \in \mathbb{Z}$. For $k > 2$ this implies $a_n = 0$ if $n \le 0$, so $f \in S_k(\Gamma,\chi) = H_k^\infty(\Gamma,\chi)$. □

For $f \in L_k^1(\mathbf{H})$ we define

$$f^\Gamma(z) := \tfrac{1}{\#Z(\Gamma)} \sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} f(\gamma z) j(\gamma,z)^{-z}$$

and

$$K^\Gamma(z_1,z_2) := \tfrac{1}{\#Z(\Gamma)} \sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} K(\gamma z_1,z_2) j(\gamma,z_1)^{-k},$$

where $\#Z(\Gamma) \in \{1,2\}$ counts the trivial elements of Γ. In the previous lecture we proved

$$K(z_1,z_2) = \frac{k-1}{4\pi}\left(\frac{z_1-\bar{z}_2}{2i}\right)^{-k}, \tag{2}$$

and we note that $\overline{K(z_2,z_1)} = K(z_1,z_2)$ implies $\overline{K^\Gamma(z_2,z_1)} = K^\Gamma(z_1,z_2)$, for all $z_1,z_2 \in \mathbf{H}$.

Theorem 17.2. *If $f \in L_k^1(\mathbf{H})$ then $f^\Gamma \in L_k^1(\Gamma,\chi)$, and if $f \in H_k^1(\mathbf{H})$ then $f^\Gamma \in H_k^1(\Gamma,\chi)$.*

Proof. See Theorem 6.3.2 in [1]. □

Theorem 17.3. *For $f \in L_k^1(\mathbf{H})$ the sum in f^Γ converges absolutely on $\mathbf{H}$ and f^Γ lies in $L_k^1(\Gamma,\chi)$, and in $H_k^1(\Gamma,\chi)$ if f is holomorphic. In particular, $K^\Gamma(z,z_2) \in H_k^1(\Gamma,\chi)$ for every fixed $z_2 \in \mathbf{H}$.*

Proof. See [1, Theorem 6.4.2]. □

Theorem 17.4. *$K^\Gamma(z_1,z_2)$ is the kernel function of $H_k^2(\Gamma,\chi)$.*

Proof. This is [1, Theorem 6.3.3], but modulo issues of convergence, for $f \in H_k^2(\Gamma,\chi)$ we have

$$\begin{aligned}
\langle f, K_{z_1}^\Gamma\rangle &= \int_{\Gamma\backslash\mathbf{H}} f(z_2)\overline{K^\Gamma(z_2,z_1)}\,\mathrm{Im}(z_2)^k dv(z_2) = \int_{\Gamma\backslash\mathbf{H}} K^\Gamma(z_1,z_2) f(z_2)\,\mathrm{Im}(z_2)^k dv(z_2)\\
&= \tfrac{1}{\#Z(\Gamma)} \sum_{\gamma\in\Gamma} \int_{\gamma^{-1}F} K(z_1,z_2) f(z_2)\,\mathrm{Im}(z_2)^k dv(z_2)\\
&= \int_{\mathbf{H}} K(z_1,z_2) f(z_2)\,\mathrm{Im}(z_2)^k dv(z_2)\\
&= f(z_1),
\end{aligned}$$

since $f \in H_k^2(\Gamma,\chi) \subseteq H_k^2(\mathbf{H})$ and $K(z_1,z_2)$ is the kernel function of $H_k^2(\mathbf{H})$. and this implies that $K^\Gamma(z_1,z_2)$ is the kernel function of $H_k^2(\Gamma,\chi)$, provided $K_{z_1}^\Gamma \in H_k^2$, which one can show follows from $K_{z_1} \in H_k^2(\mathbf{H})$. □

This gives us a new way to compute the dimension of $S_k(\Gamma,\chi)$.

Corollary 17.5. *For any lattice $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ and integer $k > 2$ we have the dimension formula*

$$\dim S_k(\Gamma,\chi) = \int_{\Gamma\backslash\mathbf{H}} K^\Gamma(z,z)\,\mathrm{Im}(z)^k dv(z).$$

Proof. Let $f_1,\dots,f_r$ be an orthonormal basis for $S_k(\Gamma,\chi)$ with respect to the inner product on $H_k^2(\Gamma,\chi)=H_k^\infty(\Gamma,\chi)=S_k(\Gamma,\chi)$. Then

$$K^\Gamma(z_1,z_2)=\sum_{i=1}^r f_i(z_1)\overline{f_i(z_2)},$$

since as noted in Lecture 16, the RHS is the kernel of any finite dimensional Hilbert space with orthonormal basis $f_1,\dots,f_r$, and we have

$$\dim S_k(\Gamma,\chi)=r=\sum_{i=1}^r\langle f_i,f_i\rangle=\sum_{i=1}^r\int_{\Gamma\backslash\mathbf{H}} f_i(z)\overline{f_i(z)}\,\mathrm{Im}(z)^k d\nu(z)=\int_{\Gamma\backslash\mathbf{H}} K^\Gamma(z,z)\,\mathrm{Im}(z)^k d\nu(z).\quad\square$$

17.2 Traces of Hecke operators

Let $\Gamma\le \mathrm{SL}_2(\mathbb{R})$ be a lattice and let $\Delta\supseteq$ be a semigroup contained in the commensurator $\tilde\Gamma$ of Γ in $\mathrm{GL}_2^+(\mathbb{R})$ (so for $\alpha\in\Delta$ the group $\alpha\Gamma\alpha^{-1}$ is commensurable with Γ). Let χ be a finite order character of Γ with $\chi(-1)=(-1)^k$ if $-1\in\Gamma$ as above, extending to a homomorphism $\Delta\to\mathbb{C}$ with

$$\chi(\alpha\gamma\alpha^{-1})=\chi(\gamma)$$

for all $\gamma\in\Gamma$ and $\alpha\in\Delta$.

Fix $k\in\mathbb{Z}_{>2}$. Recall that the Hecke algebra $\mathbb{Z}[\Gamma\backslash\Delta/\Gamma]$ acts on $f\in S_k(\Gamma,\chi)$ via

$$(f|_k\Gamma\alpha\Gamma)(z):=\det(\alpha)^{k-1}\sum_{i=1}^r\overline{\chi}(\alpha_i)j(\alpha_i,z)^{-k}f(\alpha_i z),$$

where $\Gamma\alpha\Gamma=\coprod_{i=1}^r\Gamma\alpha_i$ is any right coset decomposition and the action of an integer linear combination of double cosets is computed by extending $\mathbb{Z}$-linearly. For $\alpha\in\Delta$ we define

$$\kappa(\alpha,z):=\det(\alpha)^{k-1}\overline{\chi}(\alpha)K(\alpha z,a)j(\alpha,z)^{-k}\,\mathrm{Im}(z)^k.$$

If $T\in\mathbb{Z}[\Gamma\backslash\Delta/\Gamma]$ is a sum of disjoint double cosets (a single double coset, for example) then the union of the double coset summands of T is a subset of Δ that admits a finite decomposition into right cosets $T=\coprod_{i=1}^r\Gamma\alpha_i$; the Hecke operator $T(n)\in\mathbb{T}(N)=\mathbb{Z}[\Gamma_0(N)\backslash\Delta_0(N)/\Gamma_0(N)]$ is a notable example; it corresponds to the set $\{\alpha\in\Delta_0(N)|\det(\alpha)=n\}\subseteq\Delta_0(N)$. We will regard such Hecke operators as elements of $\mathbb{Z}[\Gamma\backslash\Delta/\Gamma]$ that are also subsets of Δ we can sum over.

Theorem 17.6. *Let $T\in\mathbb{Z}[\Gamma\backslash\Delta/\Gamma]$ be a sum of double cosets. The trace of T acting on $S_k(\Gamma,\chi)$ is*

$$\mathrm{tr}(T)=\mathrm{tr}(T|S_k(\Gamma,\chi))=\frac{1}{\#Z(\Gamma)}\int_{\Gamma\backslash\mathbf{H}}\sum_{\alpha\in T}\kappa(\alpha,z)d\nu(z)$$

Proof. Let $f_1,\dots,f_r$ be an orthonormal basis of $S_k(\Gamma,\chi)$ and let $T=\coprod_{i=1}^n \Gamma\alpha_i$. Then

$$\begin{aligned}
\operatorname{tr}(\Gamma\alpha\Gamma) &= \sum_{l=1}^r \langle f_l|\Gamma\alpha\Gamma, f_l\rangle \\
&= \sum_{l=1}^r \int_{\Gamma\backslash\mathbf{H}} \sum_{i=1}^n \det(\alpha_i)^{k-1}\overline{\chi}(\alpha_i) j(\alpha_i,z)^{-k} f_l(\alpha_i z)\overline{f_l(z)}\operatorname{Im}(z)^k d\nu(z) \\
&= \int_{\Gamma\backslash\mathbf{H}} \sum_{i=1}^n \det(\alpha_i)^{k-1}\overline{\chi}(\alpha_i) j(\alpha_i,z)^{-k} K^\Gamma(\alpha_i z_1, z_2)\operatorname{Im}(z)^k d\nu(z) \\
&= \frac{1}{\#Z(\Gamma)}\int_{\Gamma\backslash\mathbf{H}} \sum_{\alpha\in T} \det(\alpha)^{k-1}\overline{\chi}(\alpha) K(\alpha z, z) j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\nu(z) \\
&= \frac{1}{\#Z(\Gamma)}\int_{\Gamma\backslash\mathbf{H}} \sum_{\alpha\in T} \kappa(\alpha,z) d\nu(z).
\end{aligned}$$

□

In order to compute the integral in Theorem 17.6 we need to treat cusps of Γ separately. For $x\in\mathbb{P}^1(\mathbb{R})$ a cusp of Γ, and a Hecke operator $T\subseteq\Delta$, let $T_x := \{\alpha\in T : \alpha x = x\}$. If U_x is a neighborhood of x that is stable under the action of $\Gamma_x := \{\gamma\in\Gamma : \gamma x = x\}$ then

$$\int_{\Gamma_x\backslash U_x}\sum_{\alpha\in T}\kappa(\alpha,z)d\nu(z) = \int_{\Gamma_x\backslash U_x}\sum_{\alpha\in T-T_x}\kappa(\alpha,z)d\nu(z) + \int_{\Gamma_x\backslash U_x}\sum_{\alpha\in T_x}\kappa(\alpha,z)d\nu(z)$$

Lemma 17.7. *For any cusp x of Γ with Γ_x-stable neighborhood U_x and Hecke operator $T\in\mathbb{Z}[\Gamma\backslash\Delta/\Gamma]$ that is a sum of double cosets we have*

$$\int_{\Gamma_x\backslash U_x}\sum_{\alpha\in T-T_x}\kappa(\alpha,z)d\nu(z) = \sum_{\alpha\in T-T_x}\int_{\Gamma_x\backslash U_x}\kappa(\alpha,z)d\nu(z)$$

and $[T_x:\Gamma_x]<\infty$.

Proof. See [1, Theorem 6.4.5] and [1, Lemma 6.4.6]. □

Theorem 17.8. *Let x be a cusp of Γ with $\sigma x=\infty$ for some $\sigma\in\mathrm{SL}_2(\mathbb{R})$. then*

$$\int_{\Gamma_x\backslash U_x}\sum_{\alpha\in T_x}\kappa(\alpha,z)d\nu(z) = \lim_{s\to 0^+}\sum_{\alpha\in T_x}\int_{\Gamma_x\backslash U_x}\kappa(\alpha,z)\operatorname{Im}(z)^{-s}|j(\sigma,z)|^{2s}d\nu(z).$$

Proof. This is [1, Theorem 6.4.7]. □

Let $P_\Gamma\subseteq\mathbb{P}^1(\mathbb{R})$ denote the set of cusps of Γ and choose neighborhoods U_x of $x\in P_\Gamma$ so that $U_{\gamma x}=\gamma U_x$ and $U_x\cap U_{x'}=\emptyset$ for $x\neq x'$, and choose $\sigma_x\in\mathrm{SL}_2(\mathbb{R})$ so that $\sigma_x x=\infty$ and $\operatorname{Im}(\sigma_{\gamma x}\gamma z)=\operatorname{Im}(\sigma_x z)$ for $\gamma\in\Gamma$ and $z\in\mathbf{H}$. Let $T\subseteq\Delta$ be a union of double Γ-cosets, put $Z(T):=T\cap\mathbb{R}^\times$ and $T_\infty:=\cup_{x\in P_\Gamma}(T_x - Z(T))$. We then have

$$\operatorname{tr}(T) = \frac{1}{\#Z(\Gamma)}\left(\sum_{\alpha\in T-T_\infty}\int_{\Gamma\backslash\mathbf{H}}\kappa(\alpha,z)d\nu(z) + \lim_{s\to 0^+}\sum_{\alpha\in T^\infty}\int_{\Gamma\backslash\mathbf{H}}\kappa(\alpha,z,s)d\nu(z)\right),$$

where $\kappa(\alpha,z,s):=\kappa(\alpha,z)\operatorname{Im}(z)^{-s}|j(\sigma_x,z)|^{2s}$ for $z\in\bigcup_{\alpha x=x}U_x$.

For $\alpha \in T$ let $\Gamma(\alpha) := \{\gamma \in \Gamma : \alpha\gamma = \gamma\alpha\}$, and for any union of Γ-conjugacy classes $S \subseteq T$ let $\operatorname{conj}_\Gamma(S)$ denote a set of Γ-conjugacy class representatives. Then

$$\operatorname{tr}(T) = \frac{1}{\#Z(\Gamma)}\left(\sum_{\alpha\in\operatorname{conj}_\Gamma(T-T_\infty)}\int_{\Gamma(\alpha)\backslash\mathbf{H}}\kappa(\alpha,z)d\nu(z) + \lim_{s\to 0^+}\sum_{\alpha\in\operatorname{conj}_\Gamma(T_\infty)}\int_{\Gamma(\alpha)\backslash\mathbf{H}}\kappa(\alpha,z,s)d\nu(z)\right).$$

We now consider the integrals $\int_{\Gamma(\alpha)\backslash\mathbf{H}}\kappa(\alpha,z)d\nu(z)$. There are five different cases:

1. α is scalar;
2. α is elliptic ($\operatorname{tr}(\alpha) < 4\det(\alpha)$);
3. α is hyperbolic ($\operatorname{tr}(\alpha) > 4\det(\alpha)$) with fixed points that are not cusps of Γ;
4. α is hyperbolic ($\operatorname{tr}(\alpha) > 4\det(\alpha)$) with a fixed point that is a cusp of Γ;
5. α is parabolic ($\operatorname{tr}(\alpha) = 4\det(\alpha)$.

17.2.1 Scalar case

We have $\alpha = \left(\begin{smallmatrix} a & 0\\ 0 & a\end{smallmatrix}\right)$ with $\Gamma(\alpha) = \Gamma$ and $\alpha z = z$, so

$$\begin{aligned}\int_{\Gamma(\alpha)\backslash\mathbf{H}}\kappa(\alpha,z)d\nu(z) &= \det(\alpha)^{k-1}\overline{\chi}(\alpha)\int_{\Gamma\backslash\mathbf{H}}K(\alpha z,z)j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\nu(z)\\ &= a^{2k-2}\overline{\chi}(\alpha)\int_{\Gamma\backslash\mathbf{H}}\frac{k-1}{4\pi}\left(\frac{z-\bar z}{2i}\right)^{-k}a^{-k}\operatorname{Im}(z)^k d\nu(z)\\ &= \frac{k-1}{4\pi}a^{k-2}\overline{\chi}(\alpha)\nu(\Gamma\backslash\mathbf{H}).\end{aligned}$$

17.2.2 Elliptic case

If $z_0 \in \mathbf{H}$ is the fixed point of α and we put $\rho := \left(\begin{smallmatrix} 1 & -z_0\\ 1 & -\bar z_0\end{smallmatrix}\right)$ then $\rho\alpha\rho^{-1} = \left(\begin{smallmatrix} \eta & 0\\ 0 & \zeta\end{smallmatrix}\right)$, where η,ζ are the eignevalues of α, and if we put $w = \rho z = re^{i\theta}$ then one finds that

$$\kappa(\alpha,z) = \frac{k-1}{4\pi}(\eta\zeta)^{k-1}\overline{\chi}(\alpha)\zeta^{-k}\left(\frac{1-r^2}{1-(\eta/\zeta)r^2}\right)^k$$

and using $d\nu(w) = 4r(1-r^2)^{-2}drd\theta$ we obtain

$$\begin{aligned}\int_{\Gamma(\alpha)\backslash\mathbf{H}}\kappa(\alpha,z)d\nu(z) &= \frac{k-1}{4\pi\zeta}\eta^{k-1}\overline{\chi}(\alpha)\int_0^1\int_0^\pi 4r(1-r^2)^{k-2}(1-(\eta/\zeta)r^2)^{-k}drd\theta\\ &= \frac{(k-1)\eta^{k-1}\overline{\chi}(\alpha)}{[\Gamma(\alpha):Z(\Gamma)]\zeta}\int_0^1(1-t)^{k-2}(1-(\eta/\zeta)t)^{-k}dt\\ &= \frac{(k-1)\eta^{k-1}\overline{\chi}(\alpha)}{[\Gamma(\alpha):Z(\Gamma)]\zeta}((k-1)(1-\eta/\zeta))^{-1}\\ &= \frac{\overline{\chi}(\alpha)}{[\Gamma(\alpha):Z(\Gamma)]}\frac{\eta^{k-1}}{(\zeta-\eta)}\end{aligned}$$

17.2.3 Hyperbolic case with no fixed cusps

In this case one finds that

$$\int_{\Gamma(\alpha)\backslash H} \kappa(\alpha,z)d\nu(z) = 0.$$

17.2.4 Hyperbolic case with a fixed cusp

In this case one finds that

$$\int_{\Gamma(\alpha)\backslash H} \kappa(\alpha,z)d\nu(z) = -\overline{\chi}(\alpha\frac{\min(|\eta|,|\zeta|)^{k-1}}{|\zeta-\eta|}\operatorname{sgn}(\zeta)^k.$$

17.2.5 Parabolic case

Let x be the fixed point of α; then x is a cusp of Γ and $\Gamma(\alpha) = \Gamma_z$; see the argument in [1]. Choose $\sigma \in \mathrm{SL}_2(\mathbb{R})$ so that $\sigma x = \infty$, and let $\sigma\alpha\sigma = \left(\begin{smallmatrix}\zeta & \lambda\\ 0 & \zeta\end{smallmatrix}\right)$. Let T^p be the set of parabolic elements in T, and for each cusp x of Γ let $T_x^p \coloneqq T^p \cap T_x$. Then $\bigcup_{x\in\Gamma\backslash P_\Gamma} T_x^p$ is a complete set of representatives for $\operatorname{conj}_\Gamma(T^p)$. Let $\pm\sigma\Gamma_x\sigma^{-1} = \langle\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)\rangle$ with $h > 0$, and for $\alpha \in T_x^p$, put $h(\alpha) \coloneqq \lambda/\zeta$ and let $\operatorname{sgn}(\alpha) = \operatorname{sgn}(\zeta)$. Then

$$\lim_{s\to 0^+}\sum_{\alpha\in T_x^p}\int_{\Gamma(\alpha)\backslash \mathbf{H}} \kappa(\alpha,z,x)d\nu(s) = \lim_{s\to 0^+}\frac{1}{2\pi}\sum_{\alpha\in T_x^p}\overline{\chi}(\alpha)\operatorname{sgn}(\alpha)^k\det(\alpha)^{k/2-1}(ih/h(\alpha))^{1+s}.$$

17.3 The trace formula

Let $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ be a lattice, let Δ be a semigroup with $\Gamma \le \Delta \le \tilde{\Gamma}$, let χ be a finite order character of Γ with $\chi(-1) = (-1)^k$ if $-1 \in \Gamma$, extended to Δ so that $\chi(\alpha\gamma\alpha^{-1}) = \chi(\gamma)$ for $\alpha \in \Delta$ and $\gamma \in \Gamma$, and let $T \in \mathbb{Z}[\Gamma\backslash\Delta/\Gamma]$ be a sum of Γ-double cosets viewed as a subset of Δ.

Define $Z(T) = T \cap \mathbb{R}^\times$, let T^e be the subset of elliptic $\alpha \in T$, let T^h be the subset of hyperbolic $\alpha \in T$ whose fixed points are cusps of Γ, let $T^{h'}$ be the subset of hyperbolic $\alpha \in T$ whose fixed points are not cusps of Γ, and let T^p be the subset of parabolic $\alpha \in T$ whose fixed points are cusps of Γ, so that

$$T = Z(T) \sqcup T^e \sqcup T^h \sqcup T^{h'} \sqcup T^p.$$

For $\alpha \in T$ let η_α and ζ_α denote the eigenvalues of α, with $\eta_\alpha = re^{i\theta}$ and $\zeta_\alpha = re^{-i\theta}$ if α is elliptic with $\sigma\alpha\sigma^{-1} = r\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$. For non-elliptic α let $\operatorname{sgn}(\alpha) \coloneqq \operatorname{sgn}(\zeta_\alpha)$, and for parabolic α let $m(\alpha) = \lambda/(h\zeta_\alpha)$ where $\pm\sigma\Gamma(\alpha)\sigma^{-1} = \pm\left\langle\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)\right\rangle$.

Theorem 17.9. *The trace of T acting on $S_k(\Gamma,\chi)$ is*

$$\operatorname{tr}(T) = t_0 + t_e + t_h + t_p,$$

where

$$t_0 = \frac{(k-1)\nu(\Gamma\backslash\mathbf{H})}{4\pi\#Z(\Gamma)} \sum_{\alpha\in Z(T)} \overline{\chi}(\alpha)\,\mathrm{sgn}(\alpha)^k \det(\alpha)^{k/2-1},$$

$$t_e = -\sum_{\alpha\in\mathrm{conj}_\Gamma(T^e)} \frac{\overline{\chi}(\alpha)\eta_\alpha^{k-1}}{|\Gamma(\alpha)|(\eta_\alpha-\zeta_\alpha)},$$

$$t_h = -\frac{1}{\#Z(\Gamma)} \sum_{\alpha\in\mathrm{conj}_\Gamma(T^h)} \overline{\chi}(\alpha)\,\mathrm{sgn}(\alpha)^k \frac{\min(|\zeta_\alpha|,|\eta_\alpha|)^{k-1}}{|\zeta_\alpha-\eta_\alpha|},$$

$$t_p = \lim_{s\to 0^+} \frac{1}{2\pi\#Z(\Gamma)} \sum_{\alpha\in\mathrm{conj}_\Gamma(T^p)} \overline{\chi}(\alpha)\,\mathrm{sgn}(\alpha)^k \det(\alpha)^{k/2-1}(i/m(\alpha))^{1+s}.$$

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.