These notes summarize the material in §6.1 of [1] presented in lecture.

16.1 Hilbert spaces of functions on the complex upper half plane

Fix a weight $k \in \mathbb{Z}_{\geq 0}$. Let $p \in \mathbb{R}_{\geq 1} \cup \infty$ be an exponent. For $f \in \mathbf{H} \to \mathbb{C}$ we define

$$\|f\|_k^p := \begin{cases} \left(\int_{\mathbf{H}} |f(z)\,\mathrm{Im}(z)^{k/2} d\nu(z)\right)^{1/p} & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{z\in\mathbf{H}} |f(z)\,\mathrm{Im}(z)|^{k/2} & p = \infty, \end{cases}$$

where $d\nu(z) = \frac{dx\,dy}{y^2}$ is the invariant measure on $x + iy = z \in \mathbf{H}$ and "ess sup" denotes the essential supremum, the infimum of the set of essential upper bounds.

If $f : \Omega \to \mathbb{R}$ is a real-valued function on a topological space Ω equipped with a measure, we call a real number a an essential upper bound of f if the set $\{z \in \Omega : f(z) \geq a\}$ has measure zero. Thus if $\|f\|_\infty = a \in \mathbb{R}$ then the set $\{z \in \mathbf{H} : |f(z)\,\mathrm{Im}(z)^{k/2} > a\}$ has measure 0 but the sets $\{z \in \mathbf{H} : |f(z)\,\mathrm{Im}(z)^{k/2}| : a - \epsilon\}$ have nonzero measure for every $\epsilon > 0$.

We use $L_k^p(\mathbf{H})$ to denote the Banach space (complete normed vector space) of all measurable functions $f : \mathbf{H} \to \mathbb{C}$ satisfying $\|f\|_p < \infty$ (recall that a function is measurable if pre-images of measurable sets are measurable). We call exponents $p, q \in \mathbb{R}_{\geq 1} \cup \infty$ conjugate if

$$\frac{1}{p} + \frac{1}{q} = 1,$$

where $\frac{1}{0} := \infty$ and $\frac{1}{\infty} = 0$. For conjugate exponents p, q and functions $f \in L_k^p(\mathbf{H})$ and $g \in L_k^q(\mathbf{H})$ we define the pairing

$$\langle f, g\rangle := \int_{\mathbf{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^k d\nu(z),$$

which allows us to identify $g \in L_k^q(\mathbf{H})$ with an element of the dual space of $L_k^p(\mathbf{H})$, continuous linear functions on $L_k^p(\mathbf{H})$, and for $p \neq \infty$ we can view $L_k^q(\mathbf{H})$ as the dual space of $L_k^p(\mathbf{H})$. When $p = q = 2$ this defines an inner product on $L_k^2(\mathbf{H})$, making it a Hilbert space (a complete metric space with distance given by the inner product). We use $H_k^p(\mathbf{H})$ to denote the closed subspace of $L_k^p(\mathbf{H})$ consisting of holomorphic functions.

For any Hilbert space H of functions $f : X \to \mathbb{C}$ with inner product $\langle,\cdot,\cdot\rangle$ we call a function $K : X \times X \to \mathbb{C}$ a kernel function of H if the following hold:

- for every $y \in X$ the function $K_y : X \to \mathbb{C}$ defined by $x \mapsto K(x, y)$ lies in H;
- for every $f \in H$ we have $f(y) = \langle f(x), K_y(x)\rangle$ for all $y \in X$.

If K is a kernel function then it is conjugate symmetric:

$$K(x,y) = K_y(x) = \langle K_y, K_x\rangle = \overline{\langle K_x, K_y\rangle} = \overline{K_x(y)} = \overline{K(y,x)}.$$

Kernel functions need not exist but are uniquely determined when the do (in which case H is sometimes called a reproducing kernel Hilbert space or RKHS; we won't use this terminology). Indeed, if K and K' are both kernel functions of H then for every $x, y \in X$ we have

$$K'(x,y) = K'_y(x) = \langle K'_y, K_x\rangle = \overline{\langle K_x, K'_y\rangle} = \overline{K_x(y)} = \overline{K(y,x)} = K(x,y).$$

If H is finite dimensional then it has the kernel function

$$K(x,y) := \sum_i f_i(x)\overline{f_i(y)},$$

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where $f_1,\ldots,f_r$ is any orthonormal basis of H. Indeed, for $y\in X$ the function $K_y=\sum_i \overline{f_i(y)}f_i$ is an element of H, and if we write $f\in H$ as $f=\sum_i c_if_i$, then

$$\langle f,K_y\rangle=\left\langle \sum_i c_if_i,\sum_i \overline{f_i(y)}f_i\right\rangle=\sum_i c_i\overline{\overline{f_i(y)}}=\sum_i c_if_i(y)=f(y).$$

We now want to compute the kernel function of the Hilbert space $H_k^2(\mathbf{H})$. It follows from [1, Corollary 2.62] that for any fixed $k\in\mathbb{Z}_{\ge 0}$ and $z_0\in\mathbf{H}$ there is a constant c such that

$$|f(z_0)|\le c\|f\|_k^2$$

for all $f\in H_k^2(\mathbf{H})$. Thus for any fixed $z_0\in\mathbf{H}$ the map $f\mapsto f(z_0)$ is a continuous linear functional on $H_k^2(\mathbf{H})$. Since $H_k^2(\mathbf{H})$ is a Hilbert space (so self-dual), there is therefor a unique $g_{z_0}\in H_k^2(\mathbf{H})$ for which

$$f(z_0)=\langle f,g_{z_0}\rangle=\int_{\mathbf{H}} f(z)\overline{g_{z_0}(z)}\,\mathrm{Im}(z)^k dv(z),$$

for all $f\in H_k^2(\mathbf{H})$. We thus may take

$$K(w,z):=g_z(w)$$

as the kernel function of $H_k^2(\mathbf{H})$. For any $f\in H_k^2(\mathbf{H})$ we have $f(z_1)=\langle f,g_{z_1}\rangle=\langle f,K_{z_1}\rangle$ and

$$f(z_1)=\langle f,K_{z_1}\rangle=\int_{\mathbf{H}} f(z_2)\overline{K(z_2,z_1)}\,\mathrm{Im}(z_2)^k dv(z_2)=\int_{\mathbf{H}} K(z_1,z_2)f(z_2)\,\mathrm{Im}(z_2)^k dv(z_2).$$

We now want to consider the behavior of $K(z_1,z_2)$ under the action of $\alpha\in\mathrm{SL}_2(\mathbb{R})$ on $\mathbf{H}$ (via linear fractional transformations). Recall that for $\alpha=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ we define $j(\alpha,z):=cz+d$, which satisfies the identities

$$j(\alpha\alpha',z)=j(\alpha,\alpha'z)j(\alpha',z)$$

and we have

$$\mathrm{Im}(\alpha z)=\frac{\mathrm{Im}(z)}{|j(\alpha,z)|^2}=\frac{\mathrm{Im}(z)}{j(\alpha,z)\overline{j(\alpha,z)}}.$$

For any $\alpha\in\mathrm{SL}_2(\mathbb{R})$ and $f\in H_k^2(\mathbf{H})$, the function $f(\alpha^{-1}z)j(\alpha,\alpha^{-1}z)^{-k}=f(\alpha^{-1}z)j(\alpha^{-1},z)^k$ lies in $H_k^2(\mathbf{H})$, and we have $dv(\alpha^{-1}z)=dv(z)$ and $\mathrm{Im}(\alpha^{-1}z)\overline{j(\alpha^{-1},z)}=\mathrm{Im}(z)j(\alpha^{-1},z)^{-1}$, thus

$$\begin{aligned}
&\int_H K(\alpha z_1,\alpha z_2)j(\alpha,z_1)^{-k}\overline{j(\alpha,z_2)^{-k}}f(z_2)\,\mathrm{Im}(z_2)^k dv(z_2)\\
&=j(\alpha,z_1)^{-k}\int_{\mathbf{H}} K(\alpha z_1,z_2)f(\alpha^{-1}z_2)\overline{j(\alpha,\alpha^{-1}z_2)^{-k}}\,\mathrm{Im}(\alpha^{-1}z_2)^k dv(\alpha^{-1}z_2)\\
&=j(\alpha,z_1)^{-k}\int_{\mathbf{H}} K(\alpha z_1,z_2)f(\alpha^{-1}z_2)j(\alpha,\alpha^{-1}z_2)^k\,\mathrm{Im}(z_2)^k dv(z_2)\\
&=j(\alpha,z_1)^{-k}f(\alpha^{-1}(\alpha z_1))j(\alpha,\alpha^{-1}(\alpha z_1))^k\\
&=j(\alpha,z_1)^{-k}f(z_1)j(\alpha,z_2)^k\\
&=f(z_1)\\
&=\int_{\mathbf{H}} K(z_1,z_2)f(z_2)\,\mathrm{Im}(z_2)^k dv(z_2).
\end{aligned}$$

It follows from the uniqueness of kernel functions that for any $\alpha \in \mathrm{SL}_2(\mathbb{R})$ we have

$$K(\alpha z_1, \alpha z_2) = K(z_1, z_2) j(\alpha, z_1)^k \overline{j(\alpha, z_2)}^k$$

and

$$K(\alpha z_1, z_2) j(\alpha, z_1)^{-k} = K(z_1, \alpha^{-1} z_2) \overline{j(\alpha^{-1}, z_2)^{-k}}.$$

Applying this to $\alpha = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ yields

$$K(z_1 + b, z_2 + b) = K(z_1, z_2)$$

for all $b \in \mathbb{R}$. If we let $M := \left\{(z_1, z_2) \in \mathbb{C}^2 : z_1 \in \mathbf{H}, \overline{z_1 - z_2} \in \mathbf{H}\right\}$ and define

$$h(z_1, z_2) := K(z_1, \overline{z_1 - z_2}),$$

then $h(z_1, z_2)$ is holomorphic function on M. Now $h(z_1 + b, z_2) = h(z_1, z_2)$ for $b \in \mathbb{R}$, so $h(z_1, z_2)$ is actually independent of z_1. For any $z \in \mathbf{H}$ we can pick $z_1 \in \mathbf{H}$ so $(z_1, z) \in \mathbf{H}$ and define

$$P(z) := h(z_1, z) = K(z_1, \overline{z_1 - z}).$$

Then $P(z)$ is holomorphic on $\mathbf{H}$ and we have

$$K(z_1, z_2) = P(z_1 - \bar{z}_2)$$

for all $z_1, z_2 \in \mathbf{H}$. If we now consider $\alpha = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we have $P(a^2 z) = a^{-2k} P(z)$ and

$$P(iy) = y^{-k} P(i)$$

for all $y > 0$. Since $P(z)$ is holomorphic on $\mathbf{H}$ we must have

$$P(z) = c_k \left(\tfrac{z}{2i}\right)^{-k}$$

for some constant $c_k > 0$. This yields the following theorem.

Theorem 16.1. *$H_k^2(\mathbf{H})$ has kernel function $K(z_1, z_2) = c_k \left(\frac{z_1 - \bar{z}_2}{2i}\right)^{-k}$ for some $c_k \in \mathbb{R}_{>0}$.*

Corollary 16.2. *$H_k^2(\mathbf{H}) \subseteq H_k^\infty(\mathbf{H})$.*

Proof. For any $f \in H_k^2(\mathbf{H})$ and $z_0 \in \mathbf{H}$ we have

$$|f(z_0)|^2 = |\langle f(z), K(z, z_0)\rangle|^2 \le \|f\|_2^2 \|K(z, z_0)\|_2^2 = \|f\|_2^2 K(z_0, z_0) = c_k \operatorname{Im}(z_0)^{-k} \|f\|_2^2$$

so $|f(z_0) \operatorname{Im}(z_0)^{k/2}| \le \sqrt{c_k} \|f\|_2$ for any $z_0 \in \mathbf{H}$, thus $f \in H_k^\infty(\mathbf{H})$. □

We now want to compute the constant c_k in Theorem 16.1, and for this we need to define the Fourier transform of $f \in H_k^2(\mathbf{H})$. For any $y \in \mathbb{R}_{>0}$ we define the function $f_y :— \mathbb{R} \to \mathbb{R}$ via

$$f_y(x) := f(x + iy).$$

Since $\|f\|_2^2 = \int_{\mathbf{H}} |f(x + iy)|^2 y^{k-2} dx\, dy < \infty$, we have $f_y \in L^2(\mathbb{R})$ for all y outside a set of measure zero; let $\hat{f}_y(u) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i u x} dx$ of f_y for all such y.

Theorem 16.3. *For each $f \in H_k^2(\mathbf{H})$ there is a function $\hat{f}: \mathbb{R} \to \mathbb{R}$ such that $\hat{f}_y(h) = \hat{f}(u)e^{-2\pi uy}$ for all y outside a set of measure zero, and $\hat{f}(u)$ vanishes almost everywhere on $\mathbb{R}$.*

Corollary 16.4. *If $k \le 1$ then $H_k^2(\mathbf{H}) = \{0\}$.*

We call the function $\hat{f}: \mathbb{R} \to \mathbb{R}$ of Theorem 16.3 the Fourier transform of $f \in H_k^2(\mathbf{H})$. We henceforth assume $k > 1$, since $H_k^2(\mathbf{H})$ has dimension zero otherwise; see [1, Corollary 6.1.7].

We now define the function

$$G_k(u) := \int_0^\infty y^{k-2} e^{-\pi uy} dy = (\pi u)^{1-k}\Gamma(k-1),$$

for $u > 0$, and let $G_k(u) := 0$ for $u \le 0$. Let $\hat{H}_k^2$ denote the space of measurable functions $\phi: \mathbb{R} \to \mathbb{C}$ for which

- ϕ vanishes almost everywhere on $\mathbb{R}_{<0}$;
- $\int_{-\infty}^{\infty} |\phi(u)|^2 G_k(4u) du < \infty$.

Then $hatH_k^2$ is a Hilbert space with inner product

$$\langle \phi, \varphi \rangle := \int_0^\infty \phi(u)\overline{\varphi(u)} G_k(4u) du,$$

and for $f \in H_k^2(\mathbf{H})$ we have

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |\hat{f}(u)|^2 G_k(4u) du = \langle \hat{f}, \hat{f} \rangle.$$

Thus for $f \in H_k^2(\mathbf{H})$ we have $\hat{f} \in \hat{H}_k^2$, and in fact this defines an isomorphism of Hilbert spaces.

Theorem 16.5. *The map $f \mapsto \hat{f}$ is an isomorphism from $H_k^2(\mathbf{H})$ to $\hat{H}_k^2$.*

Proof. This is Theorem 6.1.6 in [1]. □

For $\phi \in \hat{H}_k^2$ we define

$$\hat{\phi}(z) := \int_{-\infty}^{\infty} \phi(u) e^{2\pi i u z} du$$

Let $\hat{K}(u,z)$ denote the Fourier transform of $K(z_1, z)$ as a function of z_1, for any fixed z. For any $\phi \in \hat{H}_k^2$ we have

$$\langle \phi(u), \hat{K}(u,z) \rangle = \langle \hat{\phi}(z_1), K(z_1, z) \rangle = \hat{\phi}(z) = \int_0^\infty \phi(u) e^{2\pi i u z} dz,$$

and we also have

$$\langle \phi(u, \hat{K}(u,z)) \rangle = \int_0^\infty \phi(u) \overline{\hat{K}(u,z)} G_k(4u) du,$$

which implies

$$\hat{K}(u,x) = G_k(4u)^{-1} e^{-2\pi i u \bar{z}}$$

for all $u \in \mathbb{R}$ and $z \in \mathbf{H}$. Taking the inverse transform of $\hat{K}(u,z)$ as a function of u yields

$$K(z_1,z_2) = \int_0^\infty G_k(4u)^{-1} e^{2\pi i u(z_1 - \bar{z}_2)} du,$$

and therefore

$$c_k \left(\tfrac{z}{2i}\right)^{-k} = \int_0^\infty G_k(4u)^{-1} e^{2\pi i u z} du.$$

Applying this with $z = 2i$ yields

$$c_k = \frac{k-1}{4\pi}$$

and the following theorem.

Theorem 16.6. *For $k \in \mathbb{Z}_{>1}$ the kernel function of $H_k^2(\mathbf{H})$ is*

$$K(z_1,z_2) = \frac{k-1}{4\pi}\left(\frac{z_1 - \bar{z}_2}{2i}\right)^{-k}.$$

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.