These notes summarize the material in §4.3, §4.6 of [3] and §11 of [1] presented in lecture.

In Lecture 11 we reduced the study of modular forms for congruence subgroups Γ to the study of modular forms for $\Gamma_0(N)$ with character $\chi \in X(N)$, where $X(N)$ denotes the group of Dirichlet characters of modulus N. We did this via two standard reductions/decompositions that can be applied to any weight $k \in \mathbb{Z}$ and level $N \in \mathbb{Z}_{\geq 1}$:

- $M_k(\Gamma) \subseteq M_k(\Gamma_1(N^2))$, since $\Gamma_1(N^2) \subseteq \delta_N^{-1}\Gamma(N)\delta_N$, where $\delta_N := \left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)$,
- $M_k(\Gamma_1(N)) = \bigoplus_{\chi \in X(N)} M_k(\Gamma_0(N), \chi)$,

both of which preserve cusp forms. To ease notation we define $M_k(N,\chi) := M_k(\Gamma_0(N),\chi)$ and let $S_k(N,\chi) \subseteq M_k(N,\chi)$ denote the subspace of cusp forms.

In Lecture 11 we also defined the diamond operator $\langle d \rangle$ defined by $f \mapsto f|_k\alpha$, where $\alpha = \left(\begin{smallmatrix} a & b \\ c & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$ satisfies $\delta \equiv d \bmod N$, and noted that $M_k(N,\chi)$ and $S_k(N,\chi)$ are χ-eigenspaces of diamond operators, they are the subspaces of $M_k(\Gamma_1(N)$ and $S_k(\Gamma_1(N)$ for which $\langle d \rangle f = \chi(d) f$ for all $d \in (\mathbb{Z}/N\mathbb{Z})^\times$.

In Lecture 12 we showed that the Hecke algebra $\mathbb{T}(N) := \mathbb{Z}[\Gamma_0(N)\backslash\Delta_0(N)/\Gamma_0(N)$, where

$$\Delta_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) : c \equiv 0 \bmod N, a \perp N, ad - bc > 0 \right\},$$

is generated by the Hecke operators

$$T(l,m) := \Gamma_0(N) \left(\begin{smallmatrix} l & 0 \\ 0 & m \end{smallmatrix}\right) \Gamma_0(N)$$

with $N \perp l|m$, and we defined the Hecke operator

$$T(n) := \sum_{lm=n} T(l,m).$$

Recall that each double coset $\Gamma_0(N)\alpha\Gamma_0(N)$ acts on $f \in M_k(\mathrm{N},\chi)$ via

$$\begin{aligned} f|_k\Gamma_0(N)\alpha\Gamma_0(N) &= \det(\alpha)^{k/2-1}\sum_{i=1}^r \overline{\chi}(\alpha_i) f|_k\alpha_i, \\ &= \det(\alpha)^{k-1}\sum_{i=i}^r \overline{\chi}(\alpha_i) j(\alpha_i,z)^{-k} f(\alpha_i z), \end{aligned}$$

for any $\alpha \in \Delta_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) : c \equiv 0 \bmod N, a \perp N, ad - bc > 0 \right\}$. We then proved that that every Hecke operator in $\mathbb{T}(N)$ is a polynomial in $T(p)$, $T(p,p)$, and $T(q)$, where $p \nmid N$ and $q|N$ vary over primes.

In Lecture 12 we proved that $S_k(N,\chi)$ has a basis of common eigenfunctions for $T(n)$ and $T(l,m)$ with $l|m$ and $lmn \perp N$, equivalently, for $T(p)$ and $T(p,p)$ with $p \nmid N$. This naturally leads to the question of whether there is a basis of common eigenfunctions for all the Hecke operators, including $T(q)$ for primes $q|N$.

This is not always true, but it is true if we restrict our attention to the new subspace of $S_k(N,\chi)$.

15.1 Old and new modular forms

For all positive integers $N|M$ the inclusion $\Gamma_0(M) \subseteq \Gamma_0(N)$ implies $M_k(N,\chi) \subseteq M_k(M,\chi)$ for any $\chi \in X(N)$. But there are many other ways to embed $M_k(N,\chi)$ into $M_k(M,\chi)$. For every divisor d of M/N we have

$$f(dz) = d^{-k/2} f|_k\delta_n \in M_k(dN,\chi),$$

where $\delta_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which induces a map $\delta_d \colon M_k(N,\chi) \to M_k(M,\chi)$ that restricts to a map on cusp forms.

Lemma 15.1. *For any $l,n,N \in \mathbb{Z}_{\geq 1}$, if $n \perp lN$ then the map δ_l commutes with the action of $T(n)$ on the spaces $M_k(N,\chi)$ and $M_k(lN,\chi)$.*

Proof. It suffices to consider the case where n is a prime $p \perp lN$, and one can check this directly using explicit coset representatives, see [3, Lemma 4.6.2]. □

The lemma below will be used to prove a key lemma due to Hecke that appears as Lemma 4.6.3 in [3] (the proof in [3] omits some details that are filled in by this lemma).

Lemma 15.2. *Let $f \in M_k(N,\chi)$ with $k > 0$, let $h > 1$ be an integer prime to N. Suppose $cf(z) = f(z/h)$ for some nonzero $c \in \mathbb{C}$ (as functions on the upper half plane). Then $f = 0$.*

Proof. Observe that $f \in M_k(N,\chi)$ implies $f(z+1) = f(z)$, and therefore

$$f((z+1)/h) = cf(z+1) = cf(z) = f(z/h).$$

If $f(z) = \sum_{n\geq 0} a(n)e^{2\pi i n z}$ is the Fourier expansion of f at ∞, comparing coefficients of $e^{2\pi i n z/h}$ in the Fourier expansions of both sides of the equality above yields

$$a(n)e^{2\pi i n/h} = a(n),$$

for all $n \geq 0$. This implies $a(n) = 0$ whenever h does not divide n (because $e^{2\pi i n/h} \neq 1$).

We now observe that $c^2 f(z) = cf(z/h) = f(z/h^2)$, and the same argument shows that $a(n) = 0$ whenever h^2 does not divide n. Repeating this argument *ad infinitum* shows that $a(n) = 0$ for all $n > 0$. For $k > 0$ there are no nonzero constant functions in $M_k(N,\chi)$, so we also have $a(0) = 0$, Thus $f = 0$ as claimed. □

Lemma 15.3 (Hecke)**.** *Let $f \in M_k(N,\chi)$ and let $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(N)$ satisfy $N \perp \det(\alpha) > 1$ and $\gcd(a,b,c,d) = 1$. If $f|_k\alpha \in M_k(N,\chi)$ then $f = 0$.*

Proof. Choose $\gamma_1, \gamma_2 \in \Gamma_0(N)$ so that $\gamma_1\alpha\gamma_2 = \left(\begin{smallmatrix} l & 0 \\ 0 & m \end{smallmatrix}\right)$ for some with $l|m$ and $l,m > 0$. Now

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & 1/m \\ 0 & 1 \end{smallmatrix}\right) \notin \Gamma_0(N)$$

so $\alpha\Gamma_0(N)\alpha^{-1} \notin \Gamma_0(N)$, and we can choose $\gamma \in \Gamma_0(N)$ so that $\alpha\gamma\alpha^{-1} \notin \Gamma_0(N)$. We have $\det(\alpha)\alpha^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \in \Delta_0(N)$, so $\det(\alpha)\alpha\gamma\alpha^{-1} \in \Delta_0(N)$, and for some $\gamma_3, \gamma_4 \in \Gamma_0(N)$ we have

$$\det(\alpha)\gamma_3\alpha\gamma\alpha^{-1}\gamma_4 = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right),$$

with $u|v$ and $u,v \in \mathbb{Z}_{>0}$. We have $uv = \det(\alpha)^2$ with $u \neq v$, otherwise $\alpha\gamma\alpha^{-1} = \gamma_3^{-1}\gamma_4^{-1} \in \Gamma_0(N)$. Therefore $h = u/v \in \mathbb{Z}_{>1}$. Now let $c := h^{k/2}\chi(\gamma_3)\chi(\gamma)\chi(\gamma_4)$, and suppose that $f|_k\alpha \in M_k(N,\chi)$. Then $f \in M_k(\alpha\Gamma_0(n)\alpha^{-1},\chi)$, and $f \in M_k(\gamma_4^{-1}\alpha\Gamma_0(N)\alpha^{-1}\gamma_4)$, so we have

$$\begin{aligned}
cf(z) &= h^{k/2}\chi(\gamma_3)\chi(\gamma)\chi(\gamma_4)f(z) \\
&= h^{k/2}\chi(\gamma_3)\chi(\gamma_4)(f|_k\gamma_4^{-1}\alpha\gamma\alpha^{-1}\gamma_4)(z) \\
&= h^{k/2}\chi(\gamma_3)(f|_k\alpha\gamma\alpha^{-1}\gamma_4)(z) \\
&= h^{k/2}\chi(\gamma_3)(f|_k\gamma_3^{-1}\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right))(z) \\
&= h^{k/2}(f|_k\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right))(z) \\
&= h^{k/2}(uv)^{k/2}v^{-k}f(z/h) = f(z/h).
\end{aligned}$$

Thus $cf(z) = f(z/h)$, with $c \neq 0$, and Lemma 15.2 then implies $f = 0$. □

Definition 15.4. Let χ be a Dirichlet character of modulus N and conductor m. We define the space of old cusp forms $S_k^{\text{old}}(N,\chi)$ to be the subspace of $S_k(N,\chi)$ spanned by the set

$$\bigcup_M \bigcup_l \{f(lz) | f(z) \in S_k(M,\chi)\}$$

where M ranges over proper divisors of N divisible by m and l ranges over divisors of N/M. We define the space of new cusp forms $S_k^{\text{new}}(N,\chi)$ to be the orthogonal complement of $S_k^{\text{old}}(N,\chi)$ with respect to the Petersson inner product.

Lemma 15.5. *The spaces $S_k^{\text{old}}(N,\chi)$ and $S_k^{\text{new}}(N,\chi)$ are stable under the action of $T(n)$ for all $n \perp N$.*

The lemma implies that the old and new subspaces of $S_k(N\chi)$ each have bases of common eigenfunctions for the Hecke operators $T(n)$ with $n \perp N$. Moreover, each eigenfunction generates a one-dimensional subspace that is uniquely determined by the eigenvalues of the Hecke operators $T(n)$ for all $n \perp N$, or more generally for all $n \perp L$, for any integer L.

Theorem 15.6. *Fix $L \in \mathbb{Z}$. If $f \in S_k^{\text{new}}(N,\chi)$ and $g \in S_k(N,\chi)$ are common eigenfunctions of $T(n)$ with the same eigenvalue for $n \perp L$ then g is a multiple of f, and in particular, $g \in S_k^{\text{new}}(N,\chi)$.*

Proof. This is Theorem 4.6.12 in [3]. □

For each proper divisor N_i of N and divisor d of N_i the map $\delta_d : S_k(N_i,\chi) \to S_k(N,\chi)$ defined by $f(z) \mapsto f(dz)$ sends the new subspace of $S_k(N_i,\chi)$ to the old subspace of $S_k(N,\chi)$. The images of these maps give us a complete decomposition of $S_k(N,\chi)$ into new subspaces

$$S_k(N,\chi) \simeq \bigoplus_{\text{cond}(\chi)|N_i|N} S_k^{\text{new}}(N_i,\chi)^{\oplus m_i}$$

where m_i is the number of divisors of N/N_i. As usual when we write $S_k(M,\chi)$ for an integer M divisible by $\text{cond}(\chi)$ we understand χ to denote the unique Dirichlet character of modulus M induced by the primitive character χ_0 of modulus $\text{cond}(\chi)$ that induces χ.

Definition 15.7. We call $f \in S_k^{\text{new}}(N,\chi)$ a newform if f is a common eigenfunction of $T(n)$ for all $n \perp N$ with $a_1(f) = 1$.

Theorem 15.8. *The newforms in $S_k^{\text{new}}(N,\chi)$ are common eigenfunctions of $\mathbb{T}(N) \cup \mathbb{T}^*(N)$ and form a basis for $S_k^{\text{new}}(N,\chi)$.*

Proof. We have $f|_k T(n) = a_n f$ for all $n \perp N$. Consider $T \in \mathbb{T}(N)$ and $T^* \in \mathbb{T}^*(N)$. Now $\mathbb{T}(N)$ is commutative, so T commutes with all the $T(n)$, and similarly, T^* commutes with all the $T^*(n)$. We have

$$f|_k T(n) = \chi(N) f|_k T^*(n)$$

to T^* also commutes with $T(n)$ for all $n \perp N$, and $f|_k T$ and $f|_k T^*$ are common eigenfunctions of $T(n)$ with the same eigenvalue a_n for all $n \perp N$. It follows that $f|_k T$ and $f|_k T^*$ must be multiples of f, thus f is en eigenfunctions for T and T^*.

We have already shown that $S_k(N,\chi)$ has a basis of common eigenfunctions for $T(n)$ with $n \perp N$, a subset of which form a basis for $S_k^{\text{new}}(N,\chi)$, and it follows that if we normalize these so that $a_1(f) = 1$ we obtain a basis of newforms. □

Corollary 15.9. *If $f \in S_k(N,\chi)$ is an eigenfunction of $T(n)$ with eigenvalue $a_n(f)$ for all $n \perp N$ there is a divisors $M|N$ divisible by the conductor of χ and a newform $g \in S_k^{\text{new}}(M,\chi)$ for which $g|_k T(n) = a_n(f)g$ for all $n \perp N$. Moreover, if $f \notin S_k^{\text{new}}(N,\chi)$ then $M < N$.*

It follows from the theorem that for newforms $f \in S_k^{\text{new}}(N,\chi)$ we have $f|_k T(n) = a_n f$ for all n, not just for $n \perp N$, and the adjointness of $T(n)$ and $T^*(n)$ implies that $f|_k T^*(n) = \bar{a}_n(f) f$. Moreover, if we put $\omega_N := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ then

$$(f|_k\omega_N)|_k T(n) = \bar{a}_n(f|_k\omega_N)$$
$$(f|_k\omega_N)|_k T^*(n) = a_n(f|_k\omega_N)$$

for all $n \in \mathbb{Z}$, and this implies the following theorem.

Theorem 15.10. *The action of ω_N induces an isomorphism between $S_k^{\text{new}}(N,\chi)$ and $S_k^{\text{new}}(N,\overline{\chi})$, and also between $S_k^{\text{old}}(N,\chi)$ and $S_k^{\text{old}}(N,\overline{\chi})$.*

We conclude this section with two important theorems about newforms. The first is a strong multiplicity one result, which implies that any newform f is uniquely determined by any subset of its Fourier coefficients (equivalently, Hecke eigenvalues) $a_n(f)$ that includes all n coprime to some integer L that we are free to choose.

Theorem 15.11. *Fix $L \in \mathbb{Z}$, let $f \in S_k^{\text{new}}(N,\chi)$ be a newform and let $g \in S_k(M,\psi)$. If g is a common eigenfunction of $\mathbb{T}(M) \cup \mathbb{T}^*(M)$ with $a_n(g) = a_n(f)$ for all $n \perp L$ then $M = N$ and $g = f$.*

Proof. See Theorem 4.6.19 in [3]. □

The second theorem characterizes the multiplicative relations between the Fourier coefficients of a newform, which together with the theorem above imply that every newform f is uniquely determined by its Fourier coefficients $a_p(f)$ at almost all primes p.

Theorem 15.12. *Let $f \in M_k(N,\chi)$. Then f is a newform if and only if the following hold:*

- *$a_1(f) = 1$;*
- *$a_{p^r}(f) = a_p(f)a_{p^{r-1}}(f) - \chi(p)p^{k-1}a_{p^{r-2}}(f)$ for all primes p and $r \geq 2$;*
- *$a_{mn}(f) = a_m(f)a_n(f)$ for all integers $m \perp n$.*

Proof. See Proposition 5.8.5 in [2]. □

15.2 Twisting newforms

Definition 15.13. For a modular form $f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z} \in M_k(N,\chi)$ and a Dirichlet character ψ we define the twist of $f \otimes \psi$ by the Dirichlet series

$$(f \otimes \psi)(z) := \sum_{n\geq 0} \psi(n) a_n e^{2\pi n z}.$$

Lemma 15.14. *Let $f \in M_k(N,\chi)$ and let ψ be a Dirichlet character. Then $f \otimes \psi \in M_k(M,\chi\psi^2)$, where $M = \operatorname{lcm}(N, \operatorname{cond}(\psi)^2, \operatorname{cond}(\psi)\operatorname{cond}(\chi))$, and if f is a cusp form, so is $f \otimes \psi$.*

Proof. See Lemma 4.3.10 in [3]. □

For newforms $f \in S_k^{\text{new}}(N,\chi)$ one can take $M = \operatorname{lcm}(N, \operatorname{cond}(\psi)\operatorname{cond}(\chi\psi))$ in the lemma above; see [1, Lemma 11.2.1].

Definition 15.15. Let $f \in S_k^{\text{new}}(N,\chi)$ be a newform, and let ψ be a Dirichlet character. If $f \otimes \psi = f$, we say that $f \otimes \psi$ is a self-twist of ψ.

The set $\operatorname{SelfTw}(f) := \{\psi : f \otimes \psi = f\}$ is a group (under multiplication of Dirichlet characters). For $\psi \in \operatorname{SelfTw}(f)$ we must have $\operatorname{cond}(\psi)|N$ and $\psi^2 = 1$, so it is a finite elementary abelian 2-group. The theory of complex multiplication implies that for $k \geq 2$ the group $\operatorname{SelfTw}(f)$ is cyclic, and if it is nontrivial it is generated by the Kronecker symbol of an imaginary quadratic field; see [4, Thm. 4.5]. In this case we say that f has complex multiplication (CM).

For $k = 1$, the group $\operatorname{SelfTw}(f)$ is isomorphic to a subgroup of $(\mathbb{Z}/2\mathbb{Z})^2$ and may contain Kronecker symbols of both real and imaginary fields.

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