These notes summarize the material in §4.3 of [1] presented in lecture.

14.1 Nice holomorphic functions

We now recall some facts from complex analysis about the growth rate of the coefficients a_n that can appear in the Fourier expansions $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ of holomorphic functions $f : \mathbf{H} \to \mathbb{C}$. We omit the proofs, which can be found in [1].

For $z \in \mathbb{C}$ we put $z = x + iy$, with $x, y \in \mathbb{R}$ (so x and y are implicitly functions of z).

Lemma 14.1. *Fix $\sigma \in \mathbb{R}$. If $f(z)$ is a holomorphic function on $\mathbf{H}$ whose Fourier expansion $\sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ converges absolutely and uniformly on $\mathbf{H}$ with $f(z) = O(y^{-\sigma})$ as $y \to 0$, then $|a_n| = O(n^{\sigma})$ as $n \to \infty$.*

Proof. See Corollary 2.1.6 in [1]. □

Lemma 14.2. *Let $\{a_n\}_{n\geq 0}$ be a sequence of complex numbers, let*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

If $|a_n| = O(n^{\sigma})$ as $n \to \infty$, for some $\sigma > 0$, then the following hold:

- *the sum defining $f(z)$ converges absolutely and uniformly on every compact subset of $\mathbf{H}$;*
- *the function $f(z)$ is holomorphic on H;*
- *$|f(z)| = O(y^{-\sigma-1})$ as $y \to 0$;*
- *$|f(z) - a_0| = O(e^{-2\pi y})$ as $y \to \infty$.*

Proof. This is Lemma 4.3.3 in [1]. □

Finally, we note the following lemma, which allows us to extend bounds on a holomorphic function that hold at the edges of a vertical strip to the interior of the strip.

Lemma 14.3. *Let $\sigma_1, \sigma_2, a, c \in \mathbb{R}$ with $a, c > 0$, and let $\phi(z)|$ be a function that is holomorphic on an open set containing a vertical strip S bounded by the lines $x = \sigma_1$ and $x = \sigma_2$.*

Suppose $|\phi(z)| = O\big(e^{|y|^a}\big)$ uniformly on S, with $|\phi(z)| = O(|y|^c)$ on the lines $x = \sigma_1$ and $x = \sigma_2$, as $|y| \to \infty$. Then $|\phi(z)| = O(|y|^c)$ uniformly on S.

Proof. This is Lemma 4.3.4 in [1]. □

Now suppose f is a holomorphic on $\mathbf{H}$ with a Fourier expansion $f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$ that converges absolutely and uniformly on $\mathbf{H}$, such that $|f(z)| = O(y^{-\sigma})$ as $y \to 0$, for some $\sigma > 0$. Then we also have $a_n = O(n^{\sigma})$ and $|f(z) - a_0| = O(e^{-2\pi y})$, by the first two lemmas above. Let us call such $f(z)$ nice holomorphic functions.

To each nice holomorphic function $f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$ we associate the Dirichlet series

$$L(f, s) := \sum_{n=1}^{\infty} a_n n^{-s},$$

which converges absolutely and uniformly on $\mathrm{Re}(s) > \sigma + 1$ and is holomorphic on this right half plane. For each positive integer N we define the completed L-function

$$\Lambda_N(f, s) := N^{s/2} (2\pi)^{-s} \Gamma(s) L(f, s).$$

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The motivation for this definition is that when $a_0 = 0$ (as when f is a cusp form, for example), then $N^{-s/2}\Lambda_N(f,s)$ is the Mellin transform of the function $f(it)\colon \mathbb{R}_{>0} \to \mathbb{C}$, which we expect to lead to a functional equation as in the proof for the Riemann zeta function.

Theorem 14.4 (Hecke). *Let $f = \sum_{n\ge 0} a_n e^{2\pi i n z}$ and $g(z) = \sum_{n\ge 0} b_n e^{2\pi n z}$ be nice holomorphic functions, and let k and N be positive integers. The following are equivalent:*

(A) $g(z) = (-i\sqrt{N}z)^{-k} f(-\frac{1}{Nz})$.

(B) *The completed L-functions $\Lambda_N(f,s)$ and $\Lambda_N(g,s)$ can be analytically continued to $\mathbb{C}$, satisfy*

$$\Lambda_N(f,s) = \Lambda_N(g,k-s),$$

with $\Lambda_N(f,s) + \frac{a_0}{s} + \frac{a_0}{k-s}$ and $\Lambda_N(s;g) + \frac{b_0}{s} + \frac{b_0}{k-s}$ holomorphic and bounded on vertical strips.

Proof. (A)$\Rightarrow$(B): We have $|a_n| = O(n^\sigma)$ and $|b_n| = O(n^\sigma)$ for some $\sigma > 0$, which implies that

$$\sum_{n\ge 1} |a_n| e^{-2\pi n t/\sqrt{N}}$$

converges for $t > 0$, and that $\sum_{n\ge 1} \int_0^\infty |a_n| t^c e^{-2\pi n t/\sqrt{N}} t^{-1} dt$ converges on $c > \sigma + 1$, and similarly for the b_n. Thus for $\mathrm{Re}(s) > \sigma + 1$ we have

$$\begin{aligned}
\Lambda_N(f,s) &= \sum_{n\ge 1} a_n (2\pi n/\sqrt{N})^{-s} \int_0^\infty e^{-t} t^{s-1} dt \\
&= \sum_{n\ge 1} \int_0^\infty a_n e^{-2\pi n t/\sqrt{N}} t^{s-1} dt \\
&= \int_0^\infty \left(\sum_{n\ge 1} a_n e^{-2\pi n t/\sqrt{N}} \right) t^{s-1} dt \\
&= \int_0^\infty \left(f(it/\sqrt{N}) - a_0 \right) t^{s-1} dt
\end{aligned}$$

where the second line uses the change of variable $t \mapsto (2\pi n/\sqrt{N})t$, the sum-integral swap in the third line is justified by the convergence noted above, and the fourth line simply applies the definition of $f(z) = \sum_{n\ge 0} a_n e^{2\pi i n z}$. This implies

$$\begin{aligned}
\Lambda_N(f,s) &= -\frac{a_0}{s} + \int_1^\infty f(\tfrac{i}{t\sqrt{N}}) t^{-s-1} dt + \int_1^\infty \left(f(\tfrac{it}{\sqrt{N}}) - a_0 \right) t^{s-1} dt \\
&= -\frac{a_0}{s} - \frac{b_0}{k-s} + \int_1^\infty \left(g(it/\sqrt{N}) - b_0 \right) t^{k-s-1} dt + \int_1^\infty \left(f(it/\sqrt{N}) - a_0 \right) t^{s-1} dt,
\end{aligned}$$

where the change of variable $t \mapsto t^{-1}$ gives the middle term in the first line, and (A) implies

$$g(it/\sqrt{N}) = (-i\sqrt{N}(it/\sqrt{N}))^{-k} f(-\tfrac{1}{it\sqrt{N}}) = t^{-k} f(\tfrac{i}{t\sqrt{N}}),$$

which we used in the second line. The fact that f and g are nice implies $|f(it) - a_0| = O(e^{-2\pi t})$ and $|g(it) - b_0| = O(e^{-2\pi t})$ as $t \to \infty$, so the two integrals in the last displayed equation converge absolutely and uniformly on any vertical strip, and are therefore holomorphic on $\mathbb{C}$. It follows that $\Lambda_N(f,s)$ has a meromorphic continuation to $\mathbb{C}$ and that

$$\Lambda_N(f,s) + \frac{a_0}{s} + \frac{b_0}{k-s}$$

is holomorphic on $\mathbb{C}$ and bounded on every vertical strip. Applying the exact same calculation to $\Lambda_N(g,k-s)$ yields the desired identity $\Lambda_N(f,s) = \Lambda_N(g,k-s)$ of meromorphic functions.

(B)$\Rightarrow$ (A): Applying the inverse Mellin transform to the Mellin transform $\Gamma(s)$ of e^{-t} yields

$$e^{-t} = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\alpha} \Gamma(s)t^{-s}ds,$$

valid for any $\alpha > 0$. Applying this to all but the first term of $f(iy) = \sum_{n\geq 0} a_n e^{-2\pi n y}$ yields

$$f(iy) = \frac{1}{2\pi i}\sum_{n\geq 1} a_n \int_{\mathrm{Re}(s)=\alpha} \Gamma(s)(2\pi n y)^{-s}ds + a_0$$

For $\alpha > \sigma + 1$ the function $L(f,s) = \sum_{n\geq 1} a_n n^{-s}$ is absolutely convergent and bounded on $\mathrm{Re}(s) = \alpha$, which implies we can swap the order of summation/integration to obtain

$$f(iy) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\alpha} \left(\sqrt{N}y\right)^{-s} \Lambda_N(f,s)ds + a_0. \tag{1}$$

Since $L(f,s)$ is bounded on $\mathrm{Re}(s) = \alpha$, for any $c > 0$ we have

$$|\Lambda_N(f,s)| = O(|\operatorname{Im}(s)|^{-c}) \tag{2}$$

as $|y| \to \infty$ on $x = \alpha$, and applying a similar argument using $k - \beta > \sigma + 1$ yields

$$|\Lambda_N(f,s)| = |\Lambda_N(g,k-s)| = O(|\operatorname{Im}(s)|^{-c})$$

as $|\operatorname{Im}(s)| \to \infty$ on $\mathrm{Re}(s) = \beta$. Now (B) implies $\Lambda_N(f,s)+\frac{a_0}{s}+\frac{b_0}{k-s}$ is bounded on $\beta \leq \mathrm{Re}(s) \leq \alpha$, so Lemma 14.3 implies that (2) holds uniformly on the vertical strip $\beta \leq \mathrm{Re}(s) \leq \alpha$. We may choose $\alpha > k$ and $\beta < 0$. By (B), the function $(\sqrt{N}y)^{-s}\Lambda_N(f,s)$ has simple poles at $s = 0$ and $s = k$ with residues $-a_0$ and $(\sqrt{N}y)^{-k}b_0$, respectively, which lie in the strip $\beta \leq \mathrm{Re}(s) \leq \alpha$, so if we shift the path of integration from $\mathrm{Re}(s) = \alpha$ to $\mathrm{Re}(s) = \beta$ in (1) we obtain

$$f(iy) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\beta} \left(\sqrt{N}y\right)^{-s} \Lambda_N(f,s)ds + (\sqrt{N}y)^{-k}b_0.$$

Now the functional equation given by (B) implies

$$\begin{aligned} f(iy) &= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\beta} \left(\sqrt{N}y\right)^{-s} \Lambda_N(k-s;g)ds + (\sqrt{N}y)^{-k}b_0 \\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=k-\beta} \left(\sqrt{N}y\right)^{s-k} \Lambda_N(s;g)ds + (\sqrt{N}y)^{-k}b_0 \\ &= (\sqrt{N}y)^{-k} g\left(\tfrac{-1}{Niy}\right). \end{aligned}$$

The functions $f(z)$ and $g(z)$ are holomorphic on $\mathbf{H}$, so this implies $f(z) = (\sqrt{N}z/i)^{-k}g\left(\frac{-1}{Nz}\right)$, which is equivalent to (A): $g(z) = (-i\sqrt{N}z)^{-k}f\left(\frac{-1}{Nz}\right)$. $\square$

14.2 Analytic continuation and functional equation for cusp forms

Let N be a positive integer and χ a Dirichlet character of modulus N. As in Lecture 11 (see Lemma 11.6), if we define $\omega(N) := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ with $N \in \mathbb{Z}_{\geq 1}$, then the map $f \mapsto f|_k\omega(N)$ gives an isomorphism $S_k(N,\chi) \simeq S_k(N,\overline{\chi})$.

We now observe that any cusp form $f \in S_k(N,\chi)$ is a nice holomorphic function, as is $g = f|_k\omega(N) \in S_k(N,\overline{\chi})$. Applying Theorem 14.4 to f and g, with $a_0 = b_0 = 0$ yields the following corollary.

Corollary 14.5. *For any cusp form $f \in S_k(N\chi)$ the function $\Lambda_N(f,s)$ is entire and satisfies*

$$\Lambda_N(f,s) = i^k\Lambda_N(f|_k\omega(N), k-s).$$

When $N = 1$ the character χ is trivial, and $f|_k\omega(1) = f$, since $\omega(1) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$, so the functional equation becomes

$$\Lambda_N(f,s) = i^k\Lambda_N(f, k-s).$$

For $N > 1$, if χ is trivial (in which case we should assume k is even), for $f \in S_k(N)$ we still have $f|_k\omega(N) \in S_k(N)$, but we do not quite have $f|_k\omega(N) = f$, since $\omega(N) \notin \Gamma_0(N)$. But in fact, up to a sign the functional equation above still holds provided that f is an eigenfunction for the Fricke involution ω_N, the linear operator defined by $f \mapsto f|_k\omega(N)$. Note that

$$(f|_k\omega_N)(z) = N^{-k/2}z^{-k}f\left(\tfrac{-1}{Nz}\right),$$

so

$$((f|_k\omega_N)|_k\omega_N)(z) = N^{-k/2}z^{-k}N^{-k/2}(\tfrac{-1}{Nz})^{-k}f(z) = (-1)^kf(z) = f(z),$$

where the last equality follows from the fact that for $f \in S_k(N)$, either k is even or $f = 0$. Thus ω_N is an involution, and its eigenvalues are ± 1. We will see later that if $f \in S_k(N)$ is an eigenform for all the Hecke operators, then it is automatically an eigenfunction for the Fricke involution, but in any case, for any $f \in S_k(N)$ that is an eigenfunction for ω_N we have the functional equation

$$\Lambda_N(f,s) = \pm i^k\Lambda_N(f, k-s).$$

More generally, if $f \in S_k(N,\chi)$ is an eigenform for all the Hecke operators, we will always have a functional equation of the form

$$\Lambda_N(f,s) = \varepsilon i^k\overline{\Lambda}_N(f, k-s),$$

where ε is a root of unity.

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.