These notes summarize the material in §3.2 presented in lecture.

Our goal for is to prove analytic continuation and a functional equation for the L-functions attached to cusp forms, as originally proved by Hecke. While this is a classical result, it is of critical importance for two reasons:

1. It makes modularity a much more powerful tool than it would otherwise be. Knowing that an elliptic curve $E/\mathbb{Q}$ has the same L-function as a modular form is interesting precisely because of the things we know about L-functions of modular forms. The LHS of the BSD formula makes sense only because we know both the modularity theorem and that L-functions of modular forms admit an analytic continuation and functional equation; modularity alone is not enough.
2. It is a model for proving facts about L-functions of more general automorphic forms.

In the spirit of the second point, as a warmup we recall the proof of the analytic continuation and functional equation for the Riemann zeta function, proved by Riemann in 1859, which was the basis for Hecke's results.

13.1 The Riemann zeta function

Given a sequence of complex numbers $a_1, a_2, a_3, \ldots$ we can define the Dirichlet series

$$\sum_{n\geq 1} a_n n^{-s}.$$

In order for this Dirichlet series to converge, we need bounds on the a_n. Note that $|n^{-s}| = n^{-\operatorname{Re}(s)}$, so if we have $|a_n| = O(n^\sigma)$ as $n \to \infty$ then the Dirichlet series above converges absolutely and uniformly to a holomorphic function on $\operatorname{Re}(s) > \sigma + 1$. This function then uniquely determines the Dirichlet coefficients a_n (no matter what σ is); they are intrinsic to the function.

Lemma 13.1. *Suppose $\sum_{n\geq 1} a_n n^{-s}$ and $\sum_{n\geq 1} b_n n^{-s}$ converge absolutely at $s = \sigma > 0$. If $\sum_{n\geq 1} a_n n^{-s} = \sum_{n\geq 1} b_n n^{-s}$ on $\operatorname{Re}(s) \geq \sigma$ then $a_n = b_n$ for all $n \geq 1$.*

Proof. This is Lemma 3.2.1 in [1]. □

The Riemann zeta function $\zeta(s)$ is defined by the Dirichlet series with $1 = a_1 = a_2 = \cdots$:

$$\zeta(s) := \sum n^{-s},$$

which converges absolutely and uniformly to a holomorphic function on $\operatorname{Re}(s) > 1$, as does any Dirichlet series with bounded coefficients, since we can take $\sigma = 0$.

Definition 13.2. Euler's gamma function is defined by the integral

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt.$$

It is holomorphic on $\operatorname{Re}(s) > 0$ and satisfies the functional equation $\Gamma(s+1) = s\Gamma(s)$, which allows us to extend it to a meromorphic function on $\mathbb{C}$ whose only poles are simple poles at $s = -n$ for $n \in \mathbb{Z}_{\geq 0}$, with residue $(-1)^n/n!$. The gamma function has no zeros.

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The gamma function can also be defined as the Mellin transform of e^{-t}. Recall that the Mellin transform of a function $h\colon \mathbb{R}_{>0} \to \mathbb{C}$ is defined by

$$(\mathcal{M}h)(s) := \int_0^\infty h(t)t^{s-1}dt,$$

and is holomorphic on vertical strips $\mathrm{Re}(s) \in (a,b)$ in which $\int_0^\infty |h(t)|t^{\sigma-1}dt$ converges for all $\sigma \in (a,b)$. The Mellin transform is related to the Fourier transform, which we will also need.

Recall that a Schwarz function $f(x)$ is a C^∞-function for which

$$\sup_x \left| x^n \frac{d^m f(x)}{dx^m} \right| < \infty.$$

The Fourier transform of a Schwarz function $f(x)$ is defined by

$$(\mathcal{F}f)(y) := \int_{\mathbb{R}} f(x)e^{-2\pi i x y}dx,$$

and is also Schwarz function. We also have the inverse Fourier transform

$$(\mathcal{F}^{-1}f)(x) := \int_{\mathbb{R}} f(y)e^{+2\pi i x y}dy,$$

with $\mathcal{F}^{-1}\mathcal{F}f = \mathcal{F}\mathcal{F}^{-1}f = f$ for all Schwarz functions f. We will need the following fact about the Fourier function of the Guassian function.

Lemma 13.3. *For any $a > 0$ the Fourier transform of $e^{-\pi a x^2}$ is $\frac{1}{\sqrt{a}}e^{-\pi y^2/a}$.*

Proof. See any textbook on Fourier analysis. □

The Mellin transform is related to the Fourier transform via

$$(\mathcal{M}h)(s) = (\mathcal{F}h(e^{-x}))(\tfrac{s}{2\pi i}),$$

provided that $h(e^{-x})$ is a Schwarz function. We also have the inverse Mellin transform

$$(\mathcal{M}^{-1}f)(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}f(s)ds,$$

defined for $f(s)$ holomorphic on a vertical strip $\mathrm{Re}(s) \in (a,b)$ with $c \in (a,b)$. We than have $\mathcal{M}^{-1}\mathcal{M}h = h$ and $\mathcal{M}\mathcal{M}^{-1}f = f$ for suitable $h\colon \mathbb{R}_{>0} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$.

Theorem 13.4 (Poisson summation). *Let f be a Schwarz function with Fourier transform $\hat{f}$. Then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Proof. Both f and $\hat{f}$ are Schwarz functions, which decay rapidly as $|n| \to \infty$, so the sums converge. Let $F(x) := \sum_{n\in\mathbb{Z}} f(x+n)$. Then F is a periodic C^∞-function with Fourier expansion

$$F(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n z},$$

whose Fourier coefficients are given by

$$c_n = \int_0^1 F(t)e^{-2\pi int}dt = \int_0^1 \sum_{m\in\mathbb{Z}} f(t+m)e^{-2\pi int}dt = \int_{\mathbb{R}} f(t)e^{-2\pi int}dt = \hat{f}(n),$$

and we have

$$\sum_{n\in\mathbb{Z}} f(n) = F(0) = \sum_{n\in\mathbb{Z}} c_n = \sum_{n\in\mathbb{Z}} \hat{f}(n). \qquad \square$$

We now prove Riemann's theorem for $\zeta(s)$, carefully spelling out each step (this is our model for the next section where we won't be as pedantic).

Theorem 13.5 (Riemann). *Let $\Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$. Then $\Lambda(s)$ has a meromorphic continuation to $\mathbb{C}$ and $\Lambda(s) + \frac{1}{s} + \frac{1}{1-s}$ is holomorphic on $\mathbb{C}$ and satisfies $\Lambda(s) = \Lambda(1-s)$.*

Proof. For $\mathrm{Re}(s) > 1$ we have

$$\begin{aligned}\Lambda(2s) &= \sum_{n=1}^{\infty}(\pi n^2)^{-s}\int_0^{\infty} e^{-t}t^{s-1}dt\\ &= \sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t}t^{s-1}dt\\ &= \int_0^{\infty}\left(\sum_{n=1}^{\infty} e^{-\pi n^2 t}\right)t^{s-1}dt,\end{aligned}$$

with a change of variable $t \mapsto \pi n^2 t$ in the second line and $\sum_{n=1}^{\infty} e^{-\pi n^2 t}|t^{s-1}| < \infty$ justifying the third line (via Fubini). Applying Poisson summation and Lemma 13.3 to $e^{-\pi x^2 t}$ yields

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t},$$

thus $g(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$ satisfies the functional equation $g(t) = \frac{1}{\sqrt{t}}g(\frac{1}{t})$, and we have

$$\begin{aligned}\Lambda(2s) &= \frac{1}{2}\int_0^{\infty}(g(t)-1)t^{s-1}dt\\ &= \frac{1}{2}\left(\int_0^1\left(\tfrac{1}{\sqrt{t}}g(\tfrac{1}{t})-1\right)t^{s-1}dt + \int_1^{\infty}(g(t)-1)\,t^{s-1}dt\right)\\ &= \frac{1}{2}\left(\int_1^{\infty}\left(\sqrt{t}g(t)-1\right)t^{-1-s}dt + \int_1^{\infty}(g(t)-1)\,t^{s-1}dt\right)\\ &= \frac{1}{2}\left(\int_1^{\infty} t^{1/2}g(t)t^{-1-s}dt - \frac{1}{s} - \int_1^{\infty} t^{1/2-s}t^{-1}dt + \int_1^{\infty} t^{-1/2-s}dt + \int_1^{\infty}(g(t)-1)t^{s-1}dt\right)\\ &= \frac{1}{2}\int_1^{\infty}\left(t^{1/2-s}+t^s\right)(g(t)-1)\,t^{-1}dt - \frac{1}{2s} - \frac{1}{1-2s}.\end{aligned}$$

with a change of variable $t \mapsto \frac{1}{t}$ to get the third line. The last integral converges uniformly on any compact subset and defines a holomorphic function on $\mathbb{C}$, hence a meromorphic continuation of $\Lambda(s)$ to $\mathbb{C}$ for which $\Lambda(s) + \frac{1}{s} + \frac{1}{1-s}$ is holomorphic. The last line is invariant under the transformation $s \mapsto \frac{1}{2} - s$, which implies $\Lambda(s) = \Lambda(1-s)$. $\square$

Corollary 13.6. *The function $\zeta(s) + \frac{1}{1-s}$ is holomorphic on $\mathbb{C}$.*

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.