These notes summarize the material in §4.3 and §4.5 of [2] covered in lecture

12.1 Hecke algebras for congruence subgroups

Let N be a positive integer. Recall the semigroups $\Delta_0(N), \Delta_0^*(N) \subseteq \mathrm{GL}_2^+(\mathbb{Q})$:

$$\Delta_0(N) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) : c \equiv 0 \bmod N, a \perp N, ad - bc > 0\right\},$$
$$\Delta_0^*(N) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) : c \equiv 0 \bmod N, d \perp N, ad - bc > 0\right\},$$

and the corresponding Hecke algebras:

$$\mathbb{T}(N) := \mathbb{Z}[\Gamma_0(N)\backslash\Delta_0(N)/\Gamma_0(N)],$$
$$\mathbb{T}^*(N) := \mathbb{Z}[\Gamma_0(N)\backslash\Delta_0^*(N)/\Gamma_0(N)],$$

which act on $M_k(N,\chi)$ and $S_k(N,\chi)$ for all $N \geq 1$ and Dirichlet characters χ of modulus N via

$$f|_k\Gamma_0(N)\alpha\Gamma_0(N) = \det(\alpha)^{k/2-1}\sum_{i=1}^{r}\overline{\chi}(\alpha_i)f|_k\alpha_i,$$
$$= \det(\alpha)^{k-1}\sum_{i=i}^{r}\overline{\chi}(\alpha_i)j(\alpha_i,z)^{-k}f(\alpha_i z),$$

where $\Gamma_0(N)\alpha\Gamma_0(N) = \coprod_{i=1}^r \Gamma_0(N)\alpha_i$ and we extend χ to $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(N)$ via $\chi(\alpha) := \overline{\chi}(a)$ and to $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0^*(N)$ via $\chi^*(\alpha) := \chi(d)$. It follows from [2, Lemma 4.52] that

$$\Gamma_0(N)\backslash\Delta_0(N)/\Gamma_0(N) = \left\{\Gamma_0(N)\left(\begin{smallmatrix} l & 0 \\ 0 & m \end{smallmatrix}\right)\Gamma_0(N) : l,m \in \mathbb{Z}_{\geq 1} \text{ with } N \perp l|m\right\},$$
$$\Gamma_0(N)\backslash\Delta_0^*(N)/\Gamma_0(N) = \left\{\Gamma_0(N)\left(\begin{smallmatrix} m & 0 \\ 0 & l \end{smallmatrix}\right)\Gamma_0(N) : l,m \in \mathbb{Z}_{\geq 1} \text{ with } N \perp l|m\right\},$$

and distinct pairs (l,m) yield distinct double cosets on the RHS in each case. For $m \perp N$ we have

$$\Gamma_0(N)\left(\begin{smallmatrix} l & 0 \\ 0 & m \end{smallmatrix}\right)\Gamma_0(N) = \Gamma_0(N)\left(\begin{smallmatrix} m & 0 \\ 0 & l \end{smallmatrix}\right)\Gamma_0(N), \tag{1}$$

but this does not contradict the statement above, we are just choosing a different representative for this double coset depending on whether we are working with $\Delta_0(N)$ or $\Delta_0^*(N)$.

For positive integers l,m,n with $l|m$ and $l \perp N$ we define the following elements of $\mathbb{T}(N), \mathbb{T}^*(N)$:

$$T(l,m) := \Gamma_0(N)\left(\begin{smallmatrix} l & 0 \\ 0 & m \end{smallmatrix}\right)\Gamma_0(N) \qquad \text{and} \qquad T^*(m,l) := \Gamma_0(N)\left(\begin{smallmatrix} m & 0 \\ 0 & l \end{smallmatrix}\right)\Gamma_0(N),$$
$$T(n) := \sum_{lm=n} T(l,m) \qquad \text{and} \qquad T^*(n) := \sum_{lm=n} T^*(m,l),$$

where the sums are taken over $l,m \in \mathbb{Z}_{\geq 1}$ with $l|m$ and $l \perp N$ as above, and we note that

$$T(p) = T(1,p) \qquad \text{and} \qquad T^*(p) = T^*(p,1).$$

For $n \perp N$ the double coset $T(n,n)$ is also a right coset (since $\left(\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix}\right)$ is scalar), which implies

$$T(n,n)T(l,m) = T(nl,nm) \qquad \text{and} \qquad T^*(n,n)T^*(m,l) = T^*(nm,nl).$$

Lemma 12.1. *For any $f \in M_k(N,\chi)$ and positive integers l,m,n with $l|m$ and $lmn \perp N$ we have*

$$f|_kT^*(m,l) = \overline{\chi}(lm)f|_kT(l,m) \qquad \textit{and} \qquad f|_kT^*(n) = \overline{\chi}(n)f|_kT(n).$$

Proof. We have $m \perp N$ and (1) implies that we can choose right coset representatives $\alpha_1, \ldots, \alpha_r$ for $T(l,m) = T^*(m,l)$ with $\alpha_i \in \Delta_0(N) \cap \Delta_0^*(N)$ so that

$$\chi^*(\alpha_i) = \chi(\det \alpha_i)\chi(\alpha_i) = \chi(lm)\chi(\alpha_i),$$

for $1 \le i \le r$, since for $\alpha_i = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we have

$$\chi^*(\alpha_i)/\chi(\alpha_i) = \chi(d)/\overline{\chi}(a) = \chi(d)\chi(a) = \chi(ad) = \chi(ad-bc) = \chi(\det \alpha) = \chi(lm),$$

because $c \equiv 0 \bmod N$ and χ is a Dirichlet character of modulus N. We then have

$$\begin{aligned} f|_k T^*(m,l) &= (lm)^{k/2-1} \sum_i \overline{\chi^*}(\alpha_i) f|_k \alpha_i \\ &= \overline{\chi}(lm)(lm)^{k/2-1} \sum_i \overline{\chi}(\alpha_i) f|_k \alpha_i \\ &= \overline{\chi}(lm) f|_k T(l,m), \end{aligned}$$

and

$$f|_k T^*(n) = \sum_{lm=n} f_k | T^*(m,l) = \sum_{lm=n} \overline{\chi} f|_k T(lm) = \overline{\chi}(n) f|_k T(n). \qquad \square$$

12.2 Quick linear algebra review

Let V be a complex vector space. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies

- $\langle u + \lambda v, w \rangle = \langle u, w \rangle + \lambda \langle v, w \rangle$,
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$ (which then implies $\langle u, v + \lambda w \rangle = \langle u, v \rangle + \bar{\lambda} \langle u, w \rangle$),
- $\langle u, u \rangle \ge 0$ with $\langle u, u \rangle = 0$ only if $u = 0$,

for all $\lambda \in \mathbb{C}$ and $u, v, w \in V$ is a (positive definite) Hermitian inner product. For every $N \in \mathbb{Z}_{\ge 1}$, $k \in \mathbb{Z}$ and Dirichlet character χ of modulus N, the Petersson inner product is a Hermitian inner product on the $\mathbb{C}$-vector space $S_k(N, \chi)$, making it a (positive definite) Hermitian space.

If T is a linear transformation of a Hermitian space V, a linear operator T^* that satisfies $\langle Tu, v \rangle = \langle u, T^* v \rangle$ for all $u, v \in V$ is the adjoint operator of T; it is necessarily unique and guaranteed to exist when V is finite dimensional (take the linear operator defined by the conjugate transpose of a matrix representing T with respect to a $\langle \cdot, \cdot \rangle$-orthonormal basis for V).

For all $u, v \in V$ and linear transformations S, T we have

- $\langle Tu, v \rangle = \langle u, T^* v \rangle = \overline{\langle T^* v, u \rangle} = \overline{\langle v, (T^*) * u \rangle} = \langle (T^*)^* u, v \rangle$,
- $\langle (S+T)u, v \rangle = \langle Su, v \rangle + \langle Tu, v \rangle = \langle u, S^* v \rangle + \langle u, T^* v \rangle = \langle u, (S^* + T^*) v \rangle$,

thus T is the adjoint of its adjoint and the adjoint of the sum is the sum of the adjoints.

Linear operators on a Hermitian space may be classified as

- Hermitian: $T = T^*$,
- unitary: $TT^* = 1$,
- normal: $TT^* = T^*T$.

If V is finite dimensional and we fix a $\langle\cdot,\cdot\rangle$-orthonormal basis, then the matrix M representing T is Hermitian ($M = M^*$), unitary ($MM^* = 1$), normal ($MM^* = M^*M$) if and only if T is. Hermitian and unitary operators are normal, but normal operators need not be Hermitian/unitary. The essential property of normal operators is that their eigenspaces are orthogonal, in fact this property uniquely characterizes normal operators.

Theorem 12.2 (Spectral theorem). *Let T be a linear operator on a Hermitian space V of finite dimension. Then T is normal if and only if V is the orthogonal sum of the eigenspaces of T.*

Proof. This is Theorem 7.31 in [1]. □

If $\mathscr{T} = \{T_1, T_2, T_3, \ldots\}$ is a (possibly infinite) family of normal operators on a finite dimensional Hermitian space V that pairwise commute, then V has an orthonormal basis whose elements are simultaneous eigenvectors of every $T \in \mathscr{T}$; this follows from the fact that diagonalizable operators that commute can be simultaneously diagonalized; see [1, Theorem 5.76].

12.3 Adjoint Hecke operators

We now want to apply the spectral theorem to Hecke operators in $\mathrm{T}(N)$ acting on the Hermitian space $S_k(N,\chi)$. We continue in the setting of §12.1, with χ a Dirichlet character of modulus N extended to $\Delta_0(N)$ and $\Delta_0^*(N)$ as above, and Hecke operators $T(l,m)$ and $T(n)$ defined as above for positive integers with $l|m$ and $l \perp N$.

Theorem 12.3. *$T(l,m)$ and $T^*(m,l)$ are adjoint operators on the Hermitian space $S_k(N,\chi)$ with the Petersson inner product, as are $T(n)$ and $T^*(n)$.*

Proof. For $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ let $\alpha' := \det(\alpha)\alpha^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. The map $\alpha \mapsto \alpha'$ is an anti-isomorphism form $\Delta_0(N)$ to $\Delta_0^*(N)$, which allows us to choose coset representatives

$$\Gamma_0(N)\left(\begin{smallmatrix} l & 0 \\ 0 & m \end{smallmatrix}\right)\Gamma_0(N) = \coprod_i^r \Gamma_0(N)\alpha_i$$

so that

$$\Gamma_0(N)\left(\begin{smallmatrix} m & 0 \\ 0 & l \end{smallmatrix}\right)\Gamma_0(N) = \coprod_i^r \Gamma_0(N)\alpha_i',$$

and we have $\chi(\alpha_i) = \overline{\chi^*}(\alpha_i')$. For $g \in S_k(N,\chi)$ and any $\alpha = \alpha_i$ we have

$$\begin{aligned}\left\langle f|_k\alpha, g\right\rangle &= \nu(\Gamma_0(N)\backslash\mathbf{H})^{-1}\int_{\Gamma_0(n)\backslash H} \det(\alpha)^{k/2} j(\alpha,z)^{-k} f(\alpha z)\overline{g(z)}\,\mathrm{Im}(z)^k d\nu(z)\\ &= \nu(\Gamma_0(N)\backslash\mathbf{H})^{-1}\int_{\alpha\Gamma_0(n)\alpha^{-1}\backslash H} \det(\alpha)^{k/2} j(\alpha,\alpha^{-1}z)^{-k} f(z)\overline{g(\alpha^{-1}z)}\,\mathrm{Im}(\alpha^{-1}z)^k d\nu(\alpha^{-1}z).\end{aligned}$$

We have $\alpha^{-1}z = \alpha' z$ for any $z \in \mathbf{H}$ and $\alpha\alpha' = \det(\alpha) = \det(\alpha')$. For $\gamma,\delta \in \mathrm{GL}_2^+(\mathbb{R})$ we have $j(\gamma,\delta z) = j(\gamma\delta,z)/j(\delta,z)$ and $\mathrm{Im}(\delta z) = \det(\delta)|j(\delta,z)|^{-2}\,\mathrm{Im}(z)$. Taking $\gamma = \alpha$ and $\delta = \alpha'$ yields

$$\begin{aligned}\det(\alpha)^{k/2} j(\alpha,\alpha^{-1}z)^{-k}\,\mathrm{Im}(\alpha^{-1}z)^k &= \det(\alpha')^{k/2} j(\alpha,\alpha' z)^{-k}\,\mathrm{Im}(\alpha' z)^k\\ &= \det(\alpha')^{k/2}(\det(\alpha')/j(\alpha',z))^{-k}(\det(\alpha')|j(\alpha',z)|^{-2}\,\mathrm{Im}(z))^k\\ &= \det(\alpha')^k\,\overline{j(\alpha',z)^{-k}}\,\mathrm{Im}(z)^k.\end{aligned}$$

Now ν is the invariant measure, so $\nu(\alpha\Gamma_0(N)\alpha^{-1}\backslash\mathbf{H}) = \nu(\Gamma_0(N)\backslash\mathbf{H})$ and $d\nu(\alpha^{-1}z) = d\nu(z)$, and

$$\begin{aligned}\langle f|_k\alpha, g\rangle &= \nu(\alpha\Gamma_0(N)\alpha^{-1}\backslash\mathbf{H})^{-1}\int_{\alpha\Gamma_0(n)\alpha^{-1}\backslash H} f(z)\overline{\det(\alpha')^{k/2}j(\alpha',z)^{-k}g(\alpha'z)}\,\mathrm{Im}(z)^k d\nu(z)\\ &= \langle f, g|_k\alpha'\rangle.\end{aligned}$$

Left linearity and right sesquilinearity of the Petersson inner product then implies

$$\begin{aligned}\langle f|_kT(l,m), g\rangle &= \left\langle \sum_{i=1}^r \det(\alpha_i)^{k/2-1}\overline{\chi}(\alpha_i)f|_k\alpha_i,\ g\right\rangle\\ &= \sum_{i=1}^r \det(\alpha_i)^{k/2-1}\overline{\chi}(\alpha_i')\langle f|_k\alpha_i, g\rangle\\ &= \sum_{i=1}^r \det(\alpha_i')^{k/2-1}\chi^*(\alpha_i')\langle f, g|_k\alpha_i'\rangle\\ &= \left\langle f, \sum_{i=1}^r \det(\alpha_i')^{k/2-1}\overline{\chi}^*(\alpha_i')g|_k\alpha_i'\right\rangle = \langle f, g|_kT^*(m,l)\rangle,\end{aligned}$$

proving that $T(l,m)$ and $T^*(m,l)$ are adjoint operators as claimed, as are $T(n) = \sum_{lm=n} T(l,m)$ and $T^*(n) = \sum_{lm=n} T^*(m,l)$ □

Corollary 12.4. *For $l|m$ and $lmn \perp N$ the Hecke operators $T(l,m)$ and $T(n)$ are normal operators on the Hermitian space $S_k(N,\chi)$.*

Proof. By Lemma 12.1 we have

$$T(l,m)T^*(m,l) = T(l,m)\overline{\chi}(lm)T(l,m) = \overline{\chi}T(l,m)T(l,m) = T^*(m,l)T(l,m),$$

which shows that $T(l,m)$ commutes with its adjoint $T^*(m,l)$, by Theorem 12.3. So $T(l,m)$ is a normal operator, and the same argument applies to $T(n)$.[1] □

Corollary 12.5. *The $\mathbb{C}$-vector space $S_k(N,\chi)$ has a basis of common eigenfunctions for the Hecke operators $T(n)$ and $T(l,m)$ with $l|m$ and $lmn \perp N$.*

Proof. The Hecke algebra $\mathrm{T}(N)$ is commutative, by Theorem 10.3, and the spectral theorem implies that $S_k(N,\chi)$ admits a basis of simultaneous eigenvectors for the operators $T(l,m)$ and $T(n)$ when they are normal, which holds for $lmn \perp N$, by the previous corollary. □

Proposition 12.6. *The isomorphisms $M_k(N,\chi) \simeq M_k(N,\overline{\chi})$ and $S_k(N,\chi) \simeq S_k(N,\overline{\chi})$ induced by the map $f \mapsto f|_k\left(\begin{smallmatrix}0 & -1\\ N & 0\end{smallmatrix}\right)$ commute with the Hecke operators $T(l,m)$ and $T(n)$ for $l|m$ with $l,n \perp n$.*

Proof. This is Theorem 4.5.5 in [2]. □

Lemma 12.7. *For all prime numbers p and $e \geq 1$ we have*

- $T(p)(T(1,p^e) = T(1,p^{e+1}) + \begin{cases}(p+1)T(p,p) & \text{if } p \nmid N \text{ and } e = 1,\\ pT(p,p)T(1,p^{e-1}) & \text{if } p \nmid N \text{ and } e > 1,\\ 0 & \text{otherwise.}\end{cases}$

[1] Note that a sum of normal operators need not be normal, so one really does need to apply the argument to $T(n)$.

- $T(p)T(p^e) = \begin{cases} T(p^{e+1}) + pT(p, pT(p^{e-1}) & \textit{if } p \nmid N, \\ T(p^{e+1}) & \textit{otherwise}. \end{cases}$

Proof. This is Lemma 4.5.7 in [2]. □

Lemma 12.8. *For positive integers $l'|m'$ with $lm \perp l'm'$, and $m \perp n$ we have*

- $T(l,m)T(l',m') = T(ll',mm')$,
- $T(m)T(n) = T(mn)$.

Proof. This is Lemma 4.5.8 in [2]. □

Theorem 12.9. *The Hecke algebra $\mathbb{T}(N)$ is equal to the polynomial ring over $\mathbb{Z}$ generated by the Hecke operators $T(p)$, $T(p,p)$, $T(q)$ for prime numbers $p \nmid N$ and $q|N$.*

Proof. That $T(p), T(p,p), T(q)$ generate $\mathrm{T}(N)$ follows from the commutativity of $\mathrm{T}(N)$ and the preceeding lemmas, and one can verify that they are algebraically independent over $\mathbb{Q}$. □

Lemma 12.10. *Let p be prime, $e \geq 1$. If $p \nmid N$ then $\Gamma_0(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^e \end{smallmatrix}\right)\Gamma_0(N)$ contains $p^e + p^{e-1}$ distinct right cosets $\Gamma_0(N)\alpha$ with*

$$\alpha \in \left\{ \left(\begin{smallmatrix} p^{e-r} & m \\ e & p^f \end{smallmatrix}\right) : 0 \leq f \leq e, 0 \leq m < p^f, \gcd(m, p^f, p^{e-f}) = 1 \right\},$$

and if $p|N$ then $\Gamma_0(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^e \end{smallmatrix}\right)\Gamma_0(N)$ contains p^e distinct right cosets $\Gamma_0(N)\alpha$ with

$$\alpha \in \left\{ \left(\begin{smallmatrix} 1 & m \\ e & p^e \end{smallmatrix}\right) : 0 \leq m < p^e \right\},$$

Proof. See Lemma 4.5.6 in [2]. □

Let $l|m$ be positive integers with $l \perp N$, and let $m/l = p_1^{e_1} \cdots p_r^{e_r}$. Then

$$T(l,m) = T(l,l)T(1,m/l) = T(l,l)\prod_{i=1}^{r} T(1,p_i^{e_i}),$$

and we obtain right coset decompositions

$$\begin{aligned} T(l,m) &= \coprod \Gamma_0(N)\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \qquad (ad = lm, 0 \leq b < d, \gcd(a,b,d) = l) \\ T(n) &= \coprod \Gamma_0(N)\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \qquad (ad = n, 0 \leq b < d, a \perp N). \end{aligned}$$

It follows that for $f \in M_k(N,\chi)$ we have

$$(f|_k T(n))(z) = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} \chi(a) d^{-k} f\left(\tfrac{az+b}{d}\right)$$

and for $l \perp N$ we have

$$(f_k|T(l,l))(z) = l^{k-2}\chi(l)f(z).$$

References

[1] Sheldon Axler, *Linear algebra done right*, Fourth Edition, Springer, 2023.

[2] Toshitsune Miyake, *Modular forms*, Springer, 2006.