These notes summarize the material in §4.3 and §4.5 of [2] covered in lecture.

12.1 Hecke algebras for congruence subgroups

Let \( N \) be a positive integer. Recall the semigroups \( \Delta_0(N), \Delta_0^*(N) \subseteq \text{GL}_2^+(\mathbb{Q}) \):

\[
\Delta_0(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \mod N, a \perp N, ad - bc > 0 \},
\]

\[
\Delta_0^*(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \mod N, d \perp N, ad - bc > 0 \},
\]

and the corresponding Hecke algebras:

\[
\mathcal{T}(N) := \mathbb{Z}[\Gamma_0(N) \backslash \Delta_0(N)/\Gamma_0(N)],
\]

\[
\mathcal{T}^*(N) := \mathbb{Z}[\Gamma_0(N) \backslash \Delta_0^*(N)/\Gamma_0(N)],
\]

which act on \( M_k(N, \chi) \) and \( S_k(N, \chi) \) for all \( N \geq 1 \) and Dirichlet characters \( \chi \) of modulus \( N \) via

\[
f|_k \Gamma_0(N) \alpha \Gamma_0(N) = \det(\alpha)^{k/2-1} \sum_{i=1}^{r} \overline{\chi}(a_i)f|_k \alpha_i,
\]

\[
= \det(\alpha)^{k-1} \sum_{i=1}^{r} \overline{\chi}(a_i)j(\alpha_i, z)^{-k} f(\alpha_i z),
\]

where \( \Gamma_0(N)^* \alpha \Gamma_0(N) = \prod_{i=1}^{r} \Gamma_0(N) a_i \) and we extend \( \chi \) to \( (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0(N) \) via \( \chi(\alpha) := \overline{\chi}(a) \) and to \( (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0^*(N) \) via \( \chi^*(\alpha) := \chi(d) \). It follows from [2, Lemma 4.52] that

\[
\Gamma_0(N) \Delta_0(N)/\Gamma_0(N) = \{ \Gamma_0(N) \left( \begin{array}{cc} l & 0 \\ 0 & m \end{array} \right) \Gamma_0(N) : l, m \in \mathbb{Z}_{\geq 1} \text{ with } N \perp l|m \},
\]

\[
\Gamma_0(N) \Delta_0^*(N)/\Gamma_0(N) = \{ \Gamma_0(N) \left( \begin{array}{cc} m & 0 \\ 0 & l \end{array} \right) \Gamma_0(N) : l, m \in \mathbb{Z}_{\geq 1} \text{ with } N \perp l|m \},
\]

and distinct pairs \( (l, m) \) yield distinct double cosets on the RHS in each case. For \( m \perp N \) we have

\[
\Gamma_0(N) \left( \begin{array}{cc} l & 0 \\ 0 & m \end{array} \right) \Gamma_0(N) = \Gamma_0(N) \left( \begin{array}{cc} m & 0 \\ 0 & l \end{array} \right) \Gamma_0(N),
\]

(1)

but this does not contradict the statement above, we are just choosing a different representative for this double coset depending on whether we are working with \( \Delta_0(N) \) or \( \Delta_0^*(N) \).

For positive integers \( l, m, n \) with \( l|m \) and \( l \perp N \) we define the following elements of \( \mathcal{T}(N), \mathcal{T}^*(N) \):

\[
T(l, m) := \Gamma_0(N) \left( \begin{array}{cc} l & 0 \\ 0 & m \end{array} \right) \Gamma_0(N) \quad \text{and} \quad T^*(m, l) := \Gamma_0(N) \left( \begin{array}{cc} m & 0 \\ 0 & l \end{array} \right) \Gamma_0(N),
\]

\[
T(n) := \sum_{lm=n} T(l, m) \quad \text{and} \quad T^*(n) := \sum_{lm=n} T^*(m, l),
\]

where the sums are taken over \( l, m \in \mathbb{Z}_{\geq 1} \) with \( l|m \) and \( l \perp N \) as above, and we note that

\[
T(p) = T(1, p) \quad \text{and} \quad T^*(p) = T^*(p, 1).
\]

For \( n \perp N \) the double coset \( T(n, n) \) is also a right coset (since \( \left( \begin{array}{cc} n & 0 \\ 0 & n \end{array} \right) \) is scalar), which implies

\[
T(n, n) T(l, m) = T(nl, nm) \quad \text{and} \quad T^*(n, n) T^*(m, l) = T^*(nm, nl).
\]

**Lemma 12.1.** For any \( f \in M_k(N, \chi) \) and positive integers \( l, m, n \) with \( l|m \) and \( lmn \perp N \) we have

\[
f|_k T^*(m, l) = \overline{\chi}(lm)f|_k T(l, m) \quad \text{and} \quad f|_k T^*(n) = \overline{\chi}(n)f|_k T(n).
\]
Proof. We have $m \perp N$ and (1) implies that we can choose right coset representatives $\alpha_1, \ldots, \alpha_r$ for $T(l, m) = T^*(m, l)$ with $\alpha_i \in \Delta(N) = \Delta_0^\circ(N)$ so that

$$\chi^+(\alpha_i) = \chi(\det \alpha_i) \chi(\alpha_i) = \chi(lm) \chi(\alpha_i),$$

for $1 \leq i \leq r$, since for $\alpha_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$\chi^+(\alpha_i)/\chi(\alpha_i) = \chi(d)/\chi(a) = \chi(d) = \chi(ad - \alpha_i) = \chi(\det \alpha) = \chi(lm),$$

because $c \equiv 0$ mod $N$ and $\chi$ is a Dirichlet character of modulus $N$. We then have

$$f|_k T^*(m, l) = (lm)^{k/2-1} \sum_i \overline{\chi^+(\alpha_i)} f|_k \alpha_i$$

$$= \overline{\chi}(lm)(lm)^{k/2-1} \sum_i \overline{\chi(\alpha_i)} f|_k \alpha_i$$

$$= \overline{\chi}(lm)f|_k T(l, m),$$

and

$$f|_k T^*(n) = \sum_{lm=n} f_k|T^*(m, l) = \sum_{lm=n} \overline{\chi}f|_k T(l, m) = \overline{\chi}(n)f|_k T(n).$$

\[\square\]

12.2 Quick linear algebra review

Let $V$ be a complex vector space. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies

- $\langle u + \lambda v, w \rangle = \langle u, w \rangle + \lambda \langle v, w \rangle$,
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$ (which then implies $\langle u, v + \lambda w \rangle = \langle u, v \rangle + \lambda \langle u, w \rangle$),
- $\langle u, u \rangle \geq 0$ with $\langle u, u \rangle = 0$ only if $u = 0$,

for all $\lambda \in \mathbb{C}$ and $u, v, w \in V$ is a (positive definite) **Hermitian inner product**. For every $N \in \mathbb{Z}_{\geq 1}$, $k \in \mathbb{Z}$ and Dirichlet character $\chi$ of modulus $N$, the Petersson inner product is a Hermitian inner product on the $\mathbb{C}$-vector space $S_k(N, \chi)$, making it a (positive definite) **Hermitian space**.

If $T$ is a linear transformation of a Hermitian space $V$, a linear operator $T^*$ that satisfies $\langle Tu, v \rangle = \langle u, T^*v \rangle$ for all $u, v \in V$ is the **adjoint operator** of $T$; it is necessarily unique and guaranteed to exist when $V$ is finite dimensional (take the linear operator defined by the conjugate transpose of a matrix representing $T$ with respect to a $(\cdot, \cdot)$-orthonormal basis for $V$).

For all $u, v \in V$ and linear transformations $S, T$ we have

- $\langle T^*v, u \rangle = \langle v, T^*u \rangle = \overline{\langle v, (T^*)^*u \rangle} = \overline{\langle v, (T^*)^*u \rangle}$,
- $\langle (S + T)u, v \rangle = \langle Su, v \rangle + \langle Tu, v \rangle = \langle u, S^*v \rangle + \langle u, T^*v \rangle = \langle u, (S^* + T^*)v \rangle$,

thus $T$ is the adjoint of its adjoint and the adjoint of the sum is the sum of the adjoints.

Linear operators on a Hermitian space may be classified as

- **Hermitian**: $T = T^*$,
- **unitary**: $TT^* = 1$,
- **normal**: $TT^* = T^*T$. 

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If $V$ is finite dimensional and we fix a $\langle \cdot , \cdot \rangle$-orthonormal basis, then the matrix $M$ representing $T$ is Hermitian ($M = M^*$), unitary ($MM^* = 1$), normal ($MM^* = M^*M$) if and only if $T$ is. Hermitian and unitary operators are normal, but normal operators need not be Hermitian/unitary. The essential property of normal operators is that their eigenspaces are orthogonal, in fact this property uniquely characterizes normal operators.

**Theorem 12.2** (Spectral theorem). Let $T$ be a linear operator on a Hermitian space $V$ of finite dimension. Then $T$ is normal if and only if $V$ is the orthogonal sum of the eigenspaces of $T$.

**Proof.** This is Theorem 7.31 in [1].

If $\mathcal{F} = \{T_1, T_2, T_3, \ldots \}$ is a (possibly infinite) family of normal operators on a finite dimensional Hermitian space $V$ that pairwise commute, then $V$ has an orthonormal basis whose elements are simultaneous eigenvectors of every $T \in \mathcal{F}$; this follows from the fact that diagonalizable operators that commute can be simultaneously diagonalized; see [1, Theorem 5.76].

### 12.3 Adjoint Hecke operators

We now want to apply the spectral theorem to Hecke operators in $T(N)$ acting on the Hermitian space $S_k(N, \chi)$. We continue in the setting of §12.1, with $\chi$ a Dirichlet character of modulus $N$ extended to $\Delta_0(N)$ and $\Delta_0(N)$ as above, and Hecke operators $T(l, m)$ and $T(n)$ defined as above for positive integers with $l|m$ and $l \perp N$.

**Theorem 12.3.** $T(l, m)$ and $T^*(m, l)$ are adjoint operators on the Hermitian space $S_k(N, \chi)$ with the Petersson inner product, as are $T(n)$ and $T^*(n)$.

**Proof.** For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R})$ let $\alpha' := \det(\alpha)\alpha^{-1} = \begin{pmatrix} d & -b \\ c & a \end{pmatrix}$. The map $\alpha \mapsto \alpha'$ is an anti-isomorphism form $\Delta_0(N)$ to $\Delta_0(N)$, which allows us to choose coset representatives

$$\Gamma_0(N)\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma_0(N) = \prod_i \Gamma_0(N)\alpha_i$$

so that

$$\Gamma_0(N)\begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix} \Gamma_0(N) = \prod_i \Gamma_0(N)\alpha'_i,$$

and we have $\chi(\alpha_i) = \overline{\chi}(\alpha'_i)$. For $g \in S_k(N, \chi)$ and any $\alpha = \alpha_i$ we have

$$\langle f |_k \alpha, g \rangle = \nu(\Gamma_0(N)\backslash H)^{-1} \int_{\Gamma_0(n) \backslash H} \det(\alpha)^{k/2} j(\alpha, z)^{-k} f(\alpha z) \overline{g(z)} \text{Im}(z)^k \text{d}\nu(z)$$

$$= \nu(\Gamma_0(N)\backslash H)^{-1} \int_{\alpha \Gamma_0(n) \alpha^{-1} \backslash H} \det(\alpha)^{k/2} j(\alpha, \alpha^{-1} z)^{-k} f(z) \overline{g(\alpha^{-1} z)} \text{Im}(\alpha^{-1} z)^k \text{d}\nu(\alpha^{-1} z).$$

We have $\alpha^{-1} z = \alpha' z$ for any $z \in H$ and $\alpha \alpha' = \det(\alpha) = \det(\alpha')$. For $\gamma, \delta \in \text{GL}_2^+(\mathbb{R})$ we have $j(\gamma, \delta z) = j(\gamma \delta, z)/j(\delta, z)$ and $\text{Im}(\delta z) = \text{det}(\delta)|j(\delta, z)|^{-2} \text{Im}(z)$. Taking $\gamma = \alpha$ and $\delta = \alpha'$ yields

$$\det(\alpha)^{k/2} j(\alpha, \alpha^{-1} z)^{-k} \text{Im}(\alpha^{-1} z)^k = \det(\alpha')^{k/2} j(\alpha', \alpha' z)^{-k} \text{Im}(\alpha' z)^k$$

$$= \det(\alpha')^{k/2}(\det(\alpha')/j(\alpha', z))^{-k}(\det(\alpha')|j(\alpha', z)|^{-2} \text{Im}(z))^k$$

$$= \det(\alpha')^{k} j(\alpha', z)^{-k} \text{Im}(z)^k.$$
Now \( \nu \) is the invariant measure, so
\[
\nu(\alpha \Gamma_0(N) \alpha^{-1} \backslash \mathcal{H}) = \nu(\Gamma_0(N) \backslash \mathcal{H})
\]
and
\[
d\nu(\alpha^{-1}z) = d\nu(z),
\]
and
\[
\langle f |_{k} \alpha, g \rangle = \nu(\alpha \Gamma_0(N) \alpha^{-1} \backslash \mathcal{H})^{-1} \int_{\alpha \Gamma_0(n) \alpha^{-1} \backslash \mathcal{H}} f(z) \overline{\det(\alpha')^{k/2}j(\alpha', z)^{-k}} g(\alpha'z) \Im(z)^k \, d\nu(z)
\]
\[
= \langle f, g |_{k} \alpha' \rangle.
\]
Left linearity and right sesquilinearity of the Petersson inner product then implies
\[
\langle f |_{k} T(l, m), g \rangle = \langle \sum_{i=1}^{r} \det(\alpha_i)^{k/2-1} \overline{\chi(\alpha_i)} f |_{k} \alpha_i, g \rangle
\]
\[
= \sum_{i=1}^{r} \det(\alpha_i)^{k/2-1} \overline{\chi(\alpha_i)} \langle f |_{k} \alpha_i, g \rangle
\]
\[
= \sum_{i=1}^{r} \det(\alpha_i')^{k/2-1} \chi'(\alpha_i') \langle f, g |_{k} \alpha_i' \rangle
\]
\[
= \langle f, \sum_{i=1}^{r} \det(\alpha_i')^{k/2-1} \overline{\chi'(\alpha_i')} g |_{k} \alpha_i' \rangle = \langle f, g |_{k} T^{*}(m, l) \rangle,
\]
proving that \( T(l, m) \) and \( T^{*}(m, l) \) are adjoint operators as claimed, as are \( T(n) = \sum_{l,m} T(l, m) \) and \( T^{*}(n) = \sum_{l,m} T^{*}(m, l) \) \( \blacksquare \)

**Corollary 12.4.** For \( l|n \) and \( lmn \perp N \) the Hecke operators \( T(l, m) \) and \( T(n) \) are normal operators on the Hermitian space \( \mathcal{S}_k(N, \chi) \).

**Proof.** By Lemma 12.1 we have
\[
T(l, m)T^{*}(m, l) = T(l, m)\overline{\chi(lm)}T(l, m) = \overline{\chi}(T(l, m))T(l, m)T(l, m) = T^{*}(m, l)T(l, m),
\]
which shows that \( T(l, m) \) commutes with its adjoint \( T^{*}(m, l) \), by Theorem 12.3. So \( T(l, m) \) is a normal operator, and the same argument applies to \( T(n) \). \( \blacksquare \)

**Corollary 12.5.** The \( \mathbb{C} \)-vector space \( \mathcal{S}_k(N, \chi) \) has a basis of common eigenfunctions for the Hecke operators \( T(n) \) and \( T(l, m) \) with \( l|m \) and \( lmn \perp N \).

**Proof.** The Hecke algebra \( T(N) \) is commutative, by Theorem 10.3, and the spectral theorem implies that \( \mathcal{S}_k(N, \chi) \) admits a basis of simultaneous eigenvectors for the operators \( T(l, m) \) and \( T(n) \) when they are normal, which holds for \( lmn \perp N \), by the previous corollary. \( \blacksquare \)

**Proposition 12.6.** The isomorphisms \( M_k(N, \chi) \cong M_k(N, \overline{\chi}) \) and \( S_k(N, \chi) \cong S_k(N, \overline{\chi}) \) induced by the map \( f \mapsto f |_{k} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \) commute with the Hecke operators \( T(l, m) \) and \( T(n) \) for \( l|m \) with \( l, n \perp N \).

**Proof.** This is Theorem 4.5.5 in [2]. \( \blacksquare \)

**Lemma 12.7.** For all prime numbers \( p \) and \( e \geq 1 \) we have
\[
T(p)(T(1, p^e) = T(1, p^{e+1}) + \begin{cases} (p + 1)T(p, p) & \text{if } p \nmid N \text{ and } e = 1, \\
pT(p, p)T(1, p^{e-1}) & \text{if } p \nmid N \text{ and } e > 1, \\
0 & \text{otherwise}. \end{cases}
\]

\(^1\)Note that a sum of normal operators need not be normal, so one really does need to apply the argument to \( T(n) \).
• \( T(p)T(p^e) = \begin{cases} T(p^{e+1}) + pT(p, pT(p^{e-1}) & \text{if } p \nmid N, \\ T(p^{e+1}) & \text{otherwise.} \end{cases} \)

**Proof.** This is Lemma 4.5.7 in [2]. □

**Lemma 12.8.** For positive integers \( l|m' \) with \( lm \perp l'm' \), and \( m \perp n \) we have

• \( T(l, m)T(l', m') = T(ll', mm') \),
• \( T(m)T(n) = T(mn) \).

**Proof.** This is Lemma 4.5.8 in [2]. □

**Theorem 12.9.** The Hecke algebra \( T(N) \) is equal to the polynomial ring over \( \mathbb{Z} \) generated by the Hecke operators \( T(p) \), \( T(p, p) \), \( T(q) \) for prime numbers \( p \nmid N \) and \( q|N \).

**Proof.** That \( T(p), T(p, p), T(q) \) generate \( T(N) \) follows from the commutativity of \( T(N) \) and the preceding lemmas, and one can verify that they are algebraically independent over \( \mathbb{Q} \). □

**Lemma 12.10.** Let \( p \) be prime, \( e \geq 1 \). If \( p \nmid N \) then \( \Gamma_0(N) \left( \frac{1}{0} p^e \right) \Gamma_0(N) \) contains \( p^e + p^{e-1} \) distinct right cosets \( \Gamma_0(N)\alpha \) with

\[
\alpha \in \left\{ \left( \frac{m}{e} p^f \right) : 0 \leq f \leq e, 0 \leq m < p^f, \gcd(m, p^f, p^{e-f}) = 1 \right\},
\]

and if \( p|N \) then \( \Gamma_0(N) \left( \frac{1}{0} p^e \right) \Gamma_0(N) \) contains \( p^e \) distinct right cosets \( \Gamma_0(N)\alpha \) with

\[
\alpha \in \left\{ \left( \frac{m}{e} p^f \right) : 0 \leq m < p^e \right\}.
\]

**Proof.** See Lemma 4.5.6 in [2]. □

Let \( l|m \) be positive integers with \( l \perp N \), and let \( m/l = p_1^{e_1} \cdots p_r^{e_r} \). Then

\[
T(l, m) = T(l, l)T(1, m/l) = T(l, l) \prod_{i=1}^{r} T(1, p_i^{e_i}),
\]

and we obtain right coset decompositions

\[
T(l, m) = \bigsqcup \Gamma_0(N) \left( \frac{a}{0} d \right) \quad (ad = lm, 0 \leq b < d, \gcd(a, b, d) = l)
\]

\[
T(n) = \bigsqcup \Gamma_0(N) \left( \frac{a}{0} d \right) \quad (ad = n, 0 \leq b < d, a \perp N).
\]

It follows that for \( f \in M_k(N, \chi) \) we have

\[
(f|_k T(n))(z) = n^{k-1} \sum_{ad=n \atop 0 \leq b < d} \chi(a)d^{-k} f\left( \frac{a+b}{d} \right)
\]

and for \( l \perp N \) we have

\[
(f|_k T(l, l))(z) = l^{k-2} \chi(l)f(z).
\]

**References**
