These notes summarize the material in §4.3 and §4.5 of [1] covered in lecture

11.1 Modular forms for congruence subgroups

**Definition 11.1.** For each positive integer \( N \) the principal congruence subgroup of level \( N \) is

\[
\Gamma(N) := \{ \gamma \in \text{SL}_2(\mathbb{Z}) : \gamma \equiv 1 \text{ mod } N \}.
\]

A subgroup \( \Gamma \subset \Gamma(1) \) that contains \( \Gamma(N) \) for some \( N \geq 1 \) is a congruence subgroup. The least such \( N \geq 1 \) is the level of \( \Gamma \). The set of congruence subgroups of level \( N \) includes the groups

\[
\begin{align*}
\Gamma_0(N) &:= \{ \gamma \in \Gamma(1) : \gamma \equiv (0,1) \text{ mod } N \}, \\
\Gamma_1(N) &:= \{ \gamma \in \Gamma(1) : \gamma \equiv (1,0) \text{ mod } N \}.
\end{align*}
\]

Congruence subgroups are lattices in \( \text{SL}_2(\mathbb{R}) \), so Fuchsian groups of the first kind.

If \( \Gamma \) is a congruence subgroup containing \( \Gamma(N) \), then \( M_k(\Gamma) \subset M_k(\Gamma(N)) \) for any \( k \in \mathbb{Z} \). Moreover, for \( \delta_N := \left( \begin{array}{cc} N & 0 \\ 0 & 1 \end{array} \right) \in \text{GL}_2^+(\mathbb{Q}) \) we have

\[
\delta_N^{-1} \Gamma(N) \delta_N = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \text{ mod } N^2, a \equiv d \equiv 1 \text{ mod } N \right\} \supseteq \Gamma_1(N^2).
\]

Thus for any \( f \in M_k(\Gamma(N)) \) we have

\[
f(Nz) = N^{-k/2}(f|_k \delta_N)(z) \in M_k(\Gamma_1(N^2)),
\]

and if \( f(z) = \sum_{n \geq 0} a_n q^{n/N} \) is the \( q \)-expansion of \( f \) at \( \infty \) (with \( q(z) := e^{2\pi i nz} \)), then

\[
f(Nz) = \sum_{n \geq 0} a_n q^n,
\]

and we can read off the Fourier coefficients of \( f(Nz) \) from those of \( f(z) \). It follows that provided we are willing to square the level when necessary, the study of modular forms for congruence subgroups reduces to the study of modular forms for \( \Gamma_1(N) \).

Now let \( \chi \) be a Dirichlet character of modulus \( N \); this means that \( \chi : \mathbb{Z} \to \mathbb{C} \) is a periodic multiplicative function that is the extension by zero of a group homomorphism \((\mathbb{Z}/NZ)\times \to \mathbb{C}\). The set of all such \( \chi \) form a group \( X(N) \) that can be identified with the character group of \((\mathbb{Z}/NZ)\times\), which is abstractly isomorphic to its dual \((\mathbb{Z}/NZ)\times\), since \((\mathbb{Z}/NZ)\times\) is abelian.

We define a character \( \chi : \Gamma_0(N) \to \mathbb{C} \) via

\[
\chi \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \right) := \chi(d).
\]

**Lemma 11.2.** For each positive integer \( N \) we have \( \mathbb{C} \)-vector space decompositions

\[
M_k(\Gamma_1(N)) = \bigoplus_{\chi \in X(N)} M_k(\Gamma_0(N), \chi) \quad \text{and} \quad S_k(\Gamma_1(N)) = \bigoplus_{\chi \in X(N)} S_k(\Gamma_0(N), \chi).
\]

**Proof.** We have \( \Gamma_1(N) \leq \Gamma_0(N) \) and an action of \( \Gamma_0 \) on \( M_k(\Gamma_1(N)) \) and \( S_k(\Gamma_1(N)) \) via \( f \to f|_k \gamma \) in which elements of \( \Gamma_1(N) \) act trivially. This induces a representation of \( \Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/NZ)\times \). Now \((\mathbb{Z}/NZ)\times\) is abelian, with character group \( X(N) \), so every such representation is induced by a Dirichlet character \( \chi \in X(N) \), and the decompositions above are simply decompositions into irreducible representations. \(\square\)
The lemma implies that we can restrict our study of modular forms for congruence subgroups to the spaces $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$, and to simplify the notation we define

$$M_k(N, \chi) := M_k(\Gamma_0(N), \chi) \quad \text{and} \quad S_k(N, \chi) := S_k(\Gamma_0(N), \chi),$$

and just write $M_k(N)$ and $S_k(N)$ when the character $\chi$ is the trivial character of modulus $N$. For any multiple $M$ of $N$, each $\chi \in X(N)$ induces a character $\chi \in X(M)$, and we have

$$M_k(N, \chi) \subseteq M_k(M, \chi) \quad \text{and} \quad S_k(N, \chi) \subseteq S_k(M, \chi).$$

**Definition 11.3.** A Dirichlet character $\chi$ has **even parity** if $\chi(-1) = 1$, and **odd parity** otherwise (in which case we necessarily have $\chi(-1) = -1$).

**Lemma 11.4.** If $k$ and $\chi$ do not have the same parity then $M_k(N, \chi) = \{0\}$.

*Proof.* We have $-1 \in \Gamma_0(N)$ and $f = \chi(-1)f_k|_{-1} = \chi(-1)(-1)^k f$ cannot hold for $f \neq 0$ unless $k$ and $\chi$ have the same parity. $\square$

**Definition 11.5.** Fix $N \geq 1$. For each positive integer $d$ coprime to $N$ we define the **diamond operator** $(d)$ on $M_k(\Gamma_1(N))$ via

$$\langle d \rangle f := f|_{d_k}\alpha,$$

for any $\alpha = \left( \begin{array}{ll} a & b \\ c & d \end{array} \right) \in \Gamma_1(N)$ with $d \equiv d \mod N$. The map $\left( \begin{array}{ll} a & b \\ c & d \end{array} \right) \rightarrow d$ on $\Gamma_1(N)$ has kernel $\Gamma_1(N)$, so the definition of $\langle d \rangle$ does not depend on the choice of $\alpha$, since $f = f|_{\gamma d}$ for $\gamma \in \Gamma_1(N)$.

The spaces $M_k(N, \chi)$ and $S_k(N, \chi)$ are $\chi$-eigenspaces of diamond operators:

$$M_k(N, \chi) = \{ f \in M_k(\Gamma_1(N)) : \langle d \rangle f = \chi(d)f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^N \},$$

$$S_k(N, \chi) = \{ f \in S_k(\Gamma_1(N)) : \langle d \rangle f = \chi(d)f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^N \}.$$

For each positive integer $N$ we define

$$\omega(N) = \left( \begin{array}{ll} 0 & -1 \\ N & 0 \end{array} \right) \in \text{GL}_2^+(\mathbb{Q}).$$

**Lemma 11.6.** The map $f \mapsto f|_{k}\omega(N)$ induces isomorphisms

$$M_k(N, \chi) \simeq M_k(N, \tilde{\chi}) \quad \text{and} \quad S_k(N, \chi) \simeq S_k(N, \tilde{\chi}).$$

*Proof.* Let $f \in M_k(N, \chi)$ and put $g = f|_{k}\omega(N)$. For $\gamma = \left( \begin{array}{ll} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)$ we have

$$\omega(N)^{\gamma} \omega(N)^{-1} = \left( \begin{array}{ll} d & -c \\ -bN & a \end{array} \right),$$

thus $\Gamma_0(N)^{\omega(N)^{-1}} = \Gamma_0(N)$ and $g|_{k}\gamma \chi = \chi(a)g = \tilde{\chi}(d) = \tilde{\chi}(\gamma)g$, since $ad = 1$. $\square$

For $f \in M_k(N, \chi)$ with $q$-expansion $\sum_{n \geq 0} a_n q^n$, we define $\tilde{f} = \sum_{n \geq 0} \tilde{a}_n q^n$.

**Lemma 11.7.** For $f \in M_k(N, \chi)$ we have $\tilde{f}(z) = \overline{f(-\overline{z})} \in M_k(N, \tilde{\chi})$, and for $f \in S_k(N, \chi)$ we have $\tilde{f} \in S_k(N, \tilde{\chi})$.

*Proof.* We have $q(z) = e^{2\pi i z}$, so $q(-z) = q(-\overline{z})$ and $\tilde{f}(z) = \overline{\tilde{f}(-\overline{z})}$ follows. For $\gamma = \left( \begin{array}{ll} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)$, if we put $\gamma' = \left( \begin{array}{ll} a' & b' \\ c' & d' \end{array} \right) \in \Gamma_0(N)$, then $\tilde{f}|_{k}\gamma' = \overline{\tilde{f}|_{k}\gamma} = \tilde{\chi}(\gamma)\tilde{f}$. $\square$
11.2 Hecke operators for congruence subgroups

Congruence subgroups have finite index in $\text{SL}_2(\mathbb{Z})$ and are thus mutually commensurable (and commensurable with finite index subgroups of $\text{SL}_2(\mathbb{Z})$ that are not congruence subgroups).

**Lemma 11.8.** Let $\Gamma$ be a finite index subgroup of $\text{SL}_2(\mathbb{Z})$. Then $\tilde{\Gamma}$ is $\mathbb{R}^* \text{GL}_2^+(\mathbb{Q})$.

**Proof.** It suffices to consider $\Gamma = \text{SL}_2(\mathbb{Z})$. Given $\alpha \in \mathbb{R}^* \text{GL}_2^+(\mathbb{Q})$, we may choose $c \in \mathbb{R}^*$ so that $\beta = c\alpha \in \text{GL}_2^+(\mathbb{Q})$ is an integer matrix, with $\alpha^{-1}\Gamma\alpha = \beta^{-1}\Gamma\beta$. Let $m = \det(\beta) \in \mathbb{Z}_{\geq 1}$ so that $m\beta^{-1}$ has integer entries. For $\gamma \in \Gamma(m)$ the matrix $m\beta^{-1}\gamma\beta \in \text{GL}_2^+(\mathbb{Q})$ has integer entries divisible by $m$, and it follows that $\beta^{-1}\gamma\beta \in \text{GL}_2^+(\mathbb{Q})$ is an integer matrix with determinant 1, hence an element of $\Gamma$. Therefore $\beta^{-1}\Gamma(m)\beta = \alpha^{-1}\Gamma(m)\alpha \subseteq \Gamma$; it follows that $\alpha\Gamma\alpha^{-1} \cap \Gamma$ contains $\Gamma(m)$ and has finite index in $\Gamma$. Applying the same argument to $\alpha^{-1} \in \mathbb{R}^* \text{GL}_2^+(\mathbb{Q})$ and conjugating by $\alpha$ shows that $\alpha\Gamma\alpha^{-1} \cap \Gamma$ has finite index in $\alpha\Gamma\alpha^{-1}$. Thus $\Gamma$ and $\alpha\Gamma\alpha^{-1}$ are commensurable, so $\alpha \in \tilde{\Gamma}$, which proves that $\mathbb{R}^* \text{GL}_2^+(\mathbb{Q}) \subseteq \tilde{\Gamma}$, since $\alpha$ was arbitrary.

Now suppose $\alpha \in \tilde{\Gamma} \subseteq \text{GL}_2^+(\mathbb{R})$. Then $\alpha\Gamma\alpha^{-1}$ is commensurable with $\Gamma$, as is $\beta\Gamma\beta^{-1}$, where $\beta$ is the inverse transpose of $\alpha$. It follows that $\Gamma, \alpha\Gamma\alpha^{-1}, \beta\Gamma\beta^{-1}$ all have the same set of cusps, namely $\mathbb{P}^1(\mathbb{Q})$, since taking a finite index subgroup of a Fuchsian group does not change the cusps, as proved in Lecture 4. So $\alpha^{-1}_0, \alpha^{-1}_0, \beta^{-1}_0, \beta^{-1}_0$ are all elements of $\mathbb{P}^1(\mathbb{Q})$, and this implies that every ratio of entries of $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ lies in $\mathbb{P}^1(\mathbb{Q})$, therefore $\alpha \in \mathbb{R}^* \text{GL}_2^+(\mathbb{Q})$. \qed

We now define the following semigroups $\Delta_0(N), \Delta_0^*(N) \subseteq \text{GL}_2^+(\mathbb{Q})$:

$$\Delta_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \mod N, \ a \perp N, \ ad - bc > 0 \right\},$$

$$\Delta_0^*(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \mod N, \ d \perp N, \ ad - bc > 0 \right\}.$$  

We also note that

$$\Delta_0(N) \cap \Delta_0^*(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \mod N, \ ad - bc \perp N, \ ad - bc > 0 \right\}.$$  

We now want to consider the Hecke algebras

$$\mathbb{T}(N) := \mathbb{Z}[\Gamma_0(N) \backslash \Delta_0(N)/\Gamma_0(N)],$$

$$\mathbb{T}^*(N) := \mathbb{Z}[\Gamma_0(N) \backslash \Delta_0^*(N)/\Gamma_0(N)].$$

**Lemma 11.9.** For each $\alpha \in \Delta_0(N)$ there exist unique positive integers $l|m$ with $l \perp N$ such that

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N) \left( \begin{array}{cc} l & 0 \\ 0 & m \end{array} \right) \Gamma_0(N),$$

and for each $\alpha \in \Delta_0^*(N)$ there exist unique positive integers $l|m$ with $l \perp N$ such that

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N) \left( \begin{array}{cc} m & 0 \\ 0 & l \end{array} \right) \Gamma_0(N).$$

**Proof.** This is Lemma 4.5.2 in [1]. \qed

**Theorem 11.10.** The Hecke algebras $\mathbb{T}(N), \mathbb{T}^*(N)$ are commutative, and for $\alpha \in \Delta_0(N) \cup \Delta_0^*(N)$ the double coset $\Gamma_0(N)\alpha\Gamma_0(N)$ admits a common set of representatives for its decomposition into left and right $\Gamma_0(N)$-cosets.

**Proof.** By Theorem 10.3 proved in Lecture 10, it suffices to exhibit an anti-involution $\iota$ for which $\iota(\Gamma_0(N)) = \Gamma_0(N)$ and $\Gamma_0(N)(\alpha)\Gamma_0(N) = \Gamma_0(N)\alpha\Gamma_0(N)$, for all $\alpha \in \Delta_0(N)$ and $\alpha \in \Delta_0^*(N)$. It follows from Lemma 11.9 that the map

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \mapsto \begin{pmatrix} a & bN \\ c & d \end{pmatrix}$$

is such an involution. \qed
For $\chi \in X(N)$ and $a = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0(N)$ we define $\chi(a) = \overline{\chi}(a)$. Then $\chi$ is an extension of the induced character $\chi$ of $\Gamma_0(N)$ to $\Delta_0(N)$, and we claim that

$$\chi(a \gamma a^{-1}) = \chi(\gamma)$$

for all $\gamma \in \Gamma$ and $a \in \Delta_0(N)$ for which $a \gamma a^{-1} \in \Gamma_0(N)$. By Lemma 11.9 we may assume $a = \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}$ with $l|m$ and $l \perp N$.

For $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$, if $\gamma' := a \gamma a^{-1} \in \Gamma_0(N)$ then $bl \equiv 0 \mod m$ and $\gamma' = \begin{pmatrix} a & bl/m \\ cN/m & d \end{pmatrix}$, in which case $\chi(\gamma') = \chi(a \gamma a^{-1}) = \chi(\gamma')$, as required. It follows from Theorem 10.4 of Lecture 10 that the Hecke algebra $T(N)$ acts on $M_k(N, \chi)$ and $S_k(N, \chi)$.

For $\Delta_\ast(N)$ we extend $\chi$ to $\chi^*(\alpha) = \chi(d)$ and one similarly shows that $T^\ast(N)$ acts on $\chi^*(\alpha)$. We have $\chi^*(\omega(N)^{-1}a\omega(N)) = \overline{\chi}(a)$ for $a \in \Delta_0(N)$.

**Remark 11.11.** For double cosets $\Gamma_0(N)a\Gamma_0(N) \in \mathbb{T}(N) \cap \mathbb{T}^\ast(N)$ the actions of $\Gamma_0(N)a\Gamma_0(N)$ as an element of $\mathbb{T}(N)$ and $\mathbb{T}^\ast(N)$ need not coincide! However, the map $\alpha \mapsto \omega(N)^{-1}a\omega(N)$ induces an isomorphism $\Delta_0(N) \cong \Delta_\ast(N)$ and we have

$$\chi^*(\omega(N)^{-1}a\omega(N)) = \overline{\chi}(a)$$

for $a \in \Delta_0(N)$.

**References**