These notes summarize the material in §4.3 and §4.5 of [1] covered in lecture

11.1 Modular forms for congruence subgroups

Definition 11.1. For each positive integer N the principal congruence subgroup of level N is

$$\Gamma(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv 1 \bmod N\}.$$

A subgroup $\Gamma \leq \Gamma(1)$ that contains $\Gamma(N)$ for some $N \geq 1$ is a congruence subgroup. The least such $N \geq 1$ is the level of Γ. The set of congruence subgroups of level N includes the groups

$$\Gamma_0(N) := \{\gamma \in \Gamma(1) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod N\},$$
$$\Gamma_1(N) := \left\{\gamma \in \Gamma(1) : \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \bmod N\right\}.$$

Congruence subgroups are lattices in $\mathrm{SL}_2(\mathbb{R})$, so Fuchsian groups of the first kind.

If Γ is a congruence subgroup containing $\Gamma(N)$, then $M_k(\Gamma) \subseteq M_k(\Gamma(N))$ for any $k \in \mathbb{Z}$. Moreover, for $\delta_N := \left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ we have

$$\delta_N^{-1}\Gamma(N)\delta_N = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N^2, a \equiv d \equiv 1 \bmod N\right\} \supseteq \Gamma_1(N^2).$$

Thus for any $f \in M_k(\Gamma(N))$ we have

$$f(Nz) = N^{-k/2}(f|_k\delta_N)(z) \in M_k(\Gamma_1(N^2)),$$

and if $f(z) = \sum_{n\geq 0} a_n q^{n/N}$ is the q-expansion of f at ∞ (with $q(z) := e^{2\pi i nz}$), then

$$f(Nz) = \sum_{n\geq 0} a_n q^n,$$

and we can read off the Fourier coefficients of $f(Nz)$ from those of $f(z)$. It follows that provided we are willing to square the level when necessary, the study of modular forms for congruence subgroups reduces to the study of modular forms for $\Gamma_1(N)$.

Now let χ be a Dirichlet character of modulus N; this means that $\chi : \mathbb{Z} \to \mathbb{C}$ is a periodic multiplicative function that is the extension by zero of a group homomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$. The set of all such χ form a group $X(N)$ that can be identified with the character group of $(\mathbb{Z}/N\mathbb{Z})^\times$, which is abstractly isomorphic to its dual $(\mathbb{Z}/N\mathbb{Z})^\times$, since $(\mathbb{Z}/N\mathbb{Z})^\times$ is abelian.

We define a character $\chi : \Gamma_0(N) \to \mathbb{C}$ via

$$\chi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) := \chi(d)$$

Lemma 11.2. *For each positive integer N we have $\mathbb{C}$-vector space decompositions*

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi\in X(N)} M_k(\Gamma_0(N), \chi) \qquad \textit{and} \qquad S_k(\Gamma_1(N) = \bigoplus_{\chi\in X(N)} S_k(\Gamma_0(N), \chi).$$

Proof. We have $\Gamma_1(N) \trianglelefteq \Gamma_0(N)$ and an action of Γ_0 on $M_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N))$ via $f \mapsto f|_k\gamma$ in which elements of $\Gamma_1(N)$ act trivially. This induces a representation of $\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$. Now $(\mathbb{Z}/N\mathbb{Z})^\times$ is abelian, with character group $X(N)$, so every such representation is induced by a Dirichlet character $\chi \in X(N)$, and the decompositions above are simply decompositions into irreducible representations. □

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The lemma implies that we can restrict our study of modular forms for congruence subgroups to the spaces $M_k(\Gamma_0(N),\chi)$ and $S_k(\Gamma_0(N),\chi)$, and to simplify the notation we define

$$M_k(N,\chi) := M_k(\Gamma_0(N),\chi) \qquad \text{and} \qquad S_k(N,\chi) := S_k(\Gamma_0(N),\chi),$$

and just write $M_k(N)$ and $S_k(N)$ when the character χ is the trivial character of modulus N. For any multiple M of N, each $\chi \in X(N)$ induces a character $\chi \in X(M)$, and we have

$$M_k(N,\chi) \subseteq M_k(M,\chi) \qquad \text{and} \qquad S_k(N,\chi) \subseteq S_k(M,\chi).$$

Definition 11.3. A Dirichlet character χ has even parity if $\chi(-1) = 1$, and odd parity otherwise (in which case we necessarily have $\chi(-1) = -1$).

Lemma 11.4. *If k and χ do not have the same parity then $M_k(N,\chi) = \{0\}$.*

Proof. We have $-1 \in \Gamma_0(N)$ and $f = \chi(-1) f_k|-1 = \chi(-1)(-1)^k f$ cannot hold for $f \neq 0$ unless k and χ have the same parity. □

Definition 11.5. Fix $N \geq 1$. For each positive integer d coprime to N we define the diamond operator $\langle d \rangle$ on $M_k(\Gamma_1(N))$ via

$$\langle d \rangle f := f|_k \alpha,$$

for any $\alpha = \left(\begin{smallmatrix} a & b \\ c & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$ with $\delta \equiv d \bmod N$. The map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \to d$ on $\Gamma_0(N)$ has kernel $\Gamma_1(N)$, so the definition of $\langle d \rangle$ does not depend on the choice of α, since $f = f|_k \gamma$ for $\gamma \in \Gamma_1(N)$.

The spaces $M_k(N,\chi)$ and $S_k(N,\chi)$ are χ-eigenspaces of diamond operators:

$$M_k(N,\chi) = \{f \in M_k(\Gamma_1(N)) : \langle d \rangle f = \chi(d) f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^\times\},$$
$$S_k(N,\chi) = \{f \in S_k(\Gamma_1(N)) : \langle d \rangle f = \chi(d) f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^\times\}.$$

For each positive integer N we define

$$\omega(N) = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q}).$$

Lemma 11.6. *The map $f \mapsto f|_k \omega(N)$ induces isomorphisms*

$$M_k(N,\chi) \simeq M_k(N,\bar{\chi}) \qquad \textit{and} \qquad S_k(N,\chi) \simeq S_k(N,\bar{\chi}).$$

Proof. Let $f \in M_k(N,\chi)$ and put $g = f|_k \omega(N)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have

$$\omega(N)\gamma\omega_N^{-1} = \left(\begin{smallmatrix} d & -c \\ -bN & a \end{smallmatrix}\right),$$

thus $\Gamma_0(N)^{\omega(N)^{-1}} = \Gamma_0(N)$ and $g|_k\gamma = \chi(a) g = \bar{\chi}(d) = \bar{\chi}(\gamma) g$, since $ad = 1$. □

For $f \in M_k(N,\chi)$ with q-expansion $\sum_{n\geq 0} a_n q^n$, we define $\tilde{f} = \sum_{n \geq 0} \bar{a}_n q^n$.

Lemma 11.7. *For $f \in M_k(N,\chi)$ we have $\tilde{f}(z) = \overline{f(-\bar{z})} \in M_k(N,\bar{\chi})$, and for $f \in S_k(N,\chi)$ we have $\tilde{f} \in S_k(N,\bar{\chi})$.*

Proof. We have $q(z) = e^{2\pi i z}$, so $\overline{q(z)} = q(-\bar{z})$ and $\tilde{f}(z) = \overline{f(-\bar{z})}$ follows. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, if we put $\gamma' = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, then $\tilde{f}|_k\gamma = \widetilde{f|_k\gamma'} = \bar{\chi}(\gamma)\tilde{f}$. □

11.2 Hecke operators for congruence subgroups

Congruence subgroups have finite index in $\mathrm{SL}_2(\mathbb{Z})$ and are thus mutually commensurable (and commensurable with finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ that are not congruence subgroups).

Lemma 11.8. *Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then $\widetilde{\Gamma}$ is $\mathbb{R}^\times\mathrm{GL}_2^+(\mathbb{Q})$.*

Proof. It suffices to consider $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Given $\alpha \in \mathbb{R}^\times\mathrm{GL}_2^+(\mathbb{Q})$, we may choose $c \in \mathbb{R}^\times$ so that $\beta = c\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ is an integer matrix, with $\alpha^{-1}\Gamma\alpha = \beta^{-1}\Gamma\beta$. Let $m = \det(\beta) \in \mathbb{Z}_{\geq 1}$ so that $m\beta^{-1}$ has integer entries. For $\gamma \in \Gamma(m)$ the matrix $m\beta^{-1}\gamma\beta \in \mathrm{GL}_2^+(\mathbb{Q})$ has integer entries divisible by m, and it follows that $\beta^{-1}\gamma\beta \in \mathrm{GL}_2^+(\mathbb{Q})$ is an integer matrix with determinant 1, hence an element of Γ. Therefore $\beta^{-1}\Gamma(m)\beta = \alpha^{-1}\Gamma(m)\alpha \subseteq \Gamma$; it follows that $\alpha\Gamma\alpha^{-1} \cap \Gamma$ contains $\Gamma(m)$ and has finite index in Γ. Applying the same argument to $\alpha^{-1} \in \mathbb{R}^\times\mathrm{GL}_2^+(\mathbb{Q})$ and conjugating by α shows that $\alpha\Gamma\alpha^{-1} \cap \Gamma$ has finite index in $\alpha\Gamma\alpha^{-1}$. Thus Γ and $\alpha\Gamma\alpha^{-1}$ are commensurable, so $\alpha \in \widetilde{\Gamma}$, which proves that $\mathbb{R}^\times\mathrm{GL}_2^+(\mathbb{Q}) \subseteq \widetilde{\Gamma}$, since α was arbitrary.

Now suppose $\alpha \in \widetilde{\Gamma} \subseteq \mathrm{GL}_2^+(\mathbb{R})$. Then $\alpha\Gamma\alpha^{-1}$ is commensurable with Γ, as is $\beta\Gamma\beta^{-1}$, where β is the inverse transpose of α. It follows that $\Gamma, \alpha\Gamma\alpha^{-1}, \beta\Gamma\beta^{-1}$ all have the same set of cusps, namely $\mathbb{P}^1(\mathbb{Q})$, since taking a finite index subgroup of a Fuchsian group does not change the cusps, as proved in Lecture 4. So $\alpha^{-1}0, \alpha^{-1}\infty, \beta^{-1}0, \beta^{-1}\infty$ are all elements of $\mathbb{P}^1(\mathbb{Q})$, and this implies that every ratio of entries of $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ lies in $\mathbb{P}^1(\mathbb{Q})$, therefore $\alpha \in \mathbb{R}^\times\mathrm{GL}_2^+(\mathbb{Q})$. □

We now define the following semigroups $\Delta_0(N), \Delta_0^*(N) \subseteq \mathrm{GL}_2^+(\mathbb{Q})$:

$$\Delta_0(N) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) : c \equiv 0 \bmod N, a \perp N, ad - bc > 0\right\},$$
$$\Delta_0^*(N) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) : c \equiv 0 \bmod N, d \perp N, ad - bc > 0\right\}.$$

We also note that

$$\Delta_0(N) \cap \Delta_0^*(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) : c \equiv 0 \bmod N, ad - bc \perp N, ad - bc > 0\right\}.$$

We now want to consider the Hecke algebras

$$\mathbb{T}(N) := \mathbb{Z}[\Gamma_0(N)\backslash\Delta_0(N)/\Gamma_0(N)],$$
$$\mathbb{T}^*(N) := \mathbb{Z}[\Gamma_0(N)\backslash\Delta_0^*(N)/\Gamma_0(N)].$$

Lemma 11.9. *For each $\alpha \in \Delta_0(N)$ there exist unique positive integers $l|m$ with $l \perp N$ such that*

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)\left(\begin{smallmatrix} l & 0 \\ 0 & m \end{smallmatrix}\right)\Gamma_0(N),$$

and for each $\alpha \in \Delta_0^*(N)$ there exist unique positive integers $l|m$ with $l \perp N$ such that

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)\left(\begin{smallmatrix} m & 0 \\ 0 & l \end{smallmatrix}\right)\Gamma_0(N).$$

Proof. This is Lemma 4.5.2 in [1]. □

Theorem 11.10. *The Hecke algebras $\mathbb{T}(N), \mathbb{T}^*(N)$ are commutative, and for $\alpha \in \Delta_0(N) \cup \Delta_0^*(N)$ the double coset $\Gamma_0(N)\alpha\Gamma_0(N)$ admits a common set of representatives for its decomposition into left and right $\Gamma_0(N)$-cosets.*

Proof. By Theorem 10.3 proved in Lecture 10, it suffices to exhibit an anti-involution ι for which $\iota(\Gamma_0(N)) = \Gamma_0(N)$ and $\Gamma_0(N)\iota(\alpha)\Gamma_0(N) = \Gamma_0(N)\alpha\Gamma_0(N)$, for all $\alpha \in \Delta_0(N)$ and $\alpha \in \Delta_0^*(N)$. It follows from Lemma 11.9 that the map

$$\left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & c \\ bN & d \end{smallmatrix}\right)$$

is such an involution. □

For $\chi \in X(N)$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(N)$ we define $\chi(\alpha) = \overline{\chi}(a)$. Then χ is an extension of the induced character χ of $\Gamma_0(N)$ to $\Delta_0(N)$, and we claim that

$$\chi(\alpha\gamma\alpha^{-1}) = \chi(\gamma)$$

for all $\gamma \in \Gamma$ and $\alpha \in \Delta_0(N)$ for which $\alpha\gamma\alpha^{-1} \in \Gamma_0(N)$. By Lemma 11.9 we may assume $\alpha = \left(\begin{smallmatrix} l & 0 \\ 0 & m \end{smallmatrix}\right)$ with $l|m$ and $l \perp N$.

For $\gamma = \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma_0(N)$, if $\gamma' := \alpha\gamma\alpha^{-1} \in \Gamma_0(N)$ then $bl \equiv 0 \bmod m$ and $\gamma' = \left(\begin{smallmatrix} a & bl/m \\ cNm/l & d \end{smallmatrix}\right)$, in which case $\chi(\gamma') = \chi(\alpha\gamma\alpha^{-1}) = \chi(\gamma')$, as required. It follows from Theorem 10.4 of Lecture 10 that the Hecke algebra $\mathbb{T}(N)$ acts on $M_k(N,\chi)$ and $S_k(N,\chi)$.

For $\Delta_0^*(N)$ we extend χ to $\chi^*(\alpha) = \chi(d)$ and one similarly shows that $\mathbb{T}^*(N)$ acts on $M_k(N,\chi)$ and $S_k(N,\chi)$.

Remark 11.11. For double cosets $\Gamma_0(N)\alpha\Gamma_0(N) \in \mathbb{T}(N) \cap \mathbb{T}^*(N)$ the actions of $\Gamma_0(N)\alpha\Gamma_0(N)$ as an element of $\mathbb{T}(N)$ and $\mathbb{T}^*(N)$ need not coincide! However, the map $\alpha \mapsto \omega(N)^{-1}\alpha\omega(N)$ induces an isomorphism $\Delta_0(N) \simeq \Delta_0^*(N)$ and we have

$$\chi^*(\omega(N)^{-1}\alpha\omega(N)) = \overline{\chi}(\alpha)$$

for $\alpha \in \Delta_0(N)$.

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.