These notes summarize the material in §2.7–2.8 of [1] covered in lecture, along with some relevant background on permutation modules.

10.1 Group rings and permutation modules

Let R be a commutative ring. For any set X we use $R[X]$ to denote the free R-module generated by the elements of S; it consists of all finite formal sums $\sum_{x\in X} r_x x$, with addition scalar multiplication and addition defined in the obvious way: we let $r_1x + r_2x := (r_1 + r_2)x$ and $r_1(r_2x) = (r_1r_2)x$ and extend these definitions R-linearly to all of $R[X]$.

If G is a group, the R-module $R[G]$ is a ring with $r_1g_1 \cdot r_2g_2 := (r_1r_2)(g_1g_2)$ (to define the product of elements or $R[G]$ use the distributive law), called the group ring $R[G]$ of G over R; note that this ring is commutative if and only if G is. If X is a left/right G-set then the R-module $R[X]$ can be given the structure of a left/right $R[G]$-module by extending the G-action R-linearly. Note that this ensures that the G-action is compatible with the R-module structure of $R[X]$ in the sense that the action of each $g \in G$ defines an endomorphism of $R[X]$.

For subgroups $H \le G$ we use $[G/H]$ and $[H\backslash G]$ to denote the G-sets of left and right cosets of H equipped with the obvious left and right G-actions (g' sends gH to $g'gH$ and Hg to Hgg'). We then have permutation modules $R[G/H]$ and $R[H\backslash G]$ that are left and right $R[G]$-modules, respectively. If K is another subgroup of G we have an R-module of double cosets $R[H\backslash G/K]$ which admits left and right G-actions in which $g' \in G$ sends HgK to $Hg'gK$ and $Hgg'K$; extending these G-actions R-linearly makes $R[H\backslash G/K]$ an $R[G]$-bimodule.

For $H, K \le G$ and $g \in G$, the decomposition of the double coset $H\backslash g/K$ into right H-cosets is given by the orbit of $Hg \in [H\backslash G]$ under the right action of K. If this orbit is finite we can formally sum cosets to obtain an element of $R[H\backslash G]$. If every right H-coset in $[H\backslash G]$ has a finite K-orbit, we can use this to define an injective right $R[G]$-module homomorphism

$$\begin{aligned} R[H\backslash G/H] &\hookrightarrow R[H\backslash G] \\ HgK &\mapsto \sum_{Hg' \in \{Hgk : k \in K\}} Hg' \end{aligned}$$

Elements in the image of this homomorphism are R-linear combination of sums of K-orbits in $[H\backslash G]$, hence invariant under the action of K. Let $R[H\backslash G]^K$ denote the R-submodule of $R[H\backslash G]$ whose elements are fixed by K. Every such element must be an R-linear sum of K-orbits in $R[H\backslash G]$, corresponding to the image of an R-linear sum of double cosets in $[H\backslash G/K]$ that lies in the image of the map above. It follows that we then have an R-module isomorphism

$$R[H\backslash G/K] \simeq [H\backslash G]^K,$$

and if ever K-coset in $[G/K]$ has a finite H-orbit we similarly obtain an R-module isomorphism

$$R[H\backslash G/K] \simeq [G/K]^H.$$

We now that that everything above applies more generally to any semigroup $\Delta \le G$. The semigroup ring $R[\Delta]$ is defined in the same way as the group ring $R[G]$ (and it is indeed a ring, even though Δ need not be a group). For $H, K \subseteq \Delta$ we have the right $R[\Delta]$-module $R[H\backslash\Delta]$ and the $R[\Delta]$-bimodule $R[H\backslash\Delta/K]$, and if every K orbit in $[H\backslash\Delta]$ and every H-orbit in $[\Delta/K]$ is finite, we have R-module isomorphisms

$$R[H\backslash\Delta/K] \simeq [H\backslash\Delta]^K \simeq [\Delta/K]^H$$

10.2 Commensurability

We now investigate conditions under which the finite orbit assumptions above are satisfied.

Definition 10.1. Let G be a group. Two subgroups $H,K \le G$ are commensurable if their intersection has finite index in both H and K; we write $H \sim K$ to indicate this relationship.

For any $H,K,J \le G$ we have

$$[H : H\cap J] \le [H : H\cap K\cap J] = [H : H\cap K][H\cap K : H\cap K\cap J] \le [H : H\cap K][K : K\cap J],$$

which implies that commensurability is transitive; it is clearly reflexive and symmetric, hence an equivalence relation.

For $g \in G$, let H^g denote $g^{-1}Hg \le G$. For any $H,K \le G$ and $g \in G$ we have $H \sim K$ if and only if $H^g \sim K^g$. It follows that if H is commensurable with H^{g_1} and H^{g_2} then it is also commensurable with $H^{g_1g_2}$ and therefore the commensurator

$$\widetilde{H} := \{g \in G : H \sim H^g\}$$

is a subgroup of G that contains the normalizer of H in G.

If $H,K \le G$ are commensurable, then $\tilde{H} = \tilde{K}$ and for any $g \in \widetilde{H} = \widetilde{K}$ we have finite left and and right coset decompositions of the double coset HgK via

$$HgK = \coprod_{i=1}^{m} h_i gK = \coprod_{j=1}^{n} Hgk_j,$$

with $h_i \in H$ ranging over $H/(H\cap K^{g^{-1}})$ representatives, of which there are $m := [H : H\cap K^{g^{-1}}]$, and k_j ranging over $(K\cap H^g)\backslash K$ representatives, of which there are $n = [K : K\cap H^g]$.

10.3 Hecke algebras

Now let S be a set of commensurable subgroups $\Gamma \le G$, and let $\Delta \subseteq G$ be a semigroup containing every $\Gamma \in S$ and is contained in their common commensurator in G. We will eventually apply this with $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \le \mathrm{GL}_2^+(\mathbb{R}) = G$ and S the set of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, but rather letting Δ be the commensurator of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{GL}_2^+(\mathbb{R})$ (which is $\mathbb{R}^\times\mathrm{GL}_2^+(\mathbb{Q})$), we will take Δ to be a semigroup of integer matrices with positive determinant.

Let M be an R-module equipped with a right Δ-action that is compatible with the R-module structure of M, making M an $R[\Delta]$-module. Let $\Gamma,\Gamma' \in S$, and consider a double coset $\Gamma\alpha\Gamma'$. If $\Gamma\alpha\Gamma' = \coprod_i \Gamma\alpha_i$ is a right coset decomposition, we define an action of $\Gamma\alpha\Gamma'$ on M^Γ via

$$m|\Gamma\alpha\Gamma' := \sum_i m^{\alpha_i}.$$

This definition does not depend on the choice of the choice of the $\alpha_i \in \Gamma\alpha\Gamma'$, since if $\Gamma\alpha_i' = \Gamma\alpha_i$ for some then $\alpha_i' = \gamma_i\alpha_i$ for some $\gamma_i \in \Gamma$ and $m^{\alpha_i'} = m^{\gamma_i\alpha_i} = (m_i^\gamma)^{\alpha_i} = m^{\alpha_i}$ for all $m \in M^\Gamma$. We also note that $m|\Gamma\alpha\alpha\Gamma' \in M$ is Γ'-invariant, since for any $\gamma' \in \Gamma'$, if we let $\alpha_i' = \alpha_i\gamma'$ then

$$(m|\Gamma\alpha\Gamma')^{\gamma'} = \sum_i m^{\alpha_i\gamma'} = \sum_i m^{\alpha_i'} = m|\Gamma\alpha\Gamma',$$

because $\Gamma\alpha\Gamma' = \coprod_i \Gamma\alpha_i'$ is just another right coset decomposition. This defines an R-module homomorphism $M^\Gamma \to M^{\Gamma'}$. Extend this R-linearly to an action of $R[\Gamma\backslash\Delta/\Gamma']$ yields an R-module homomorphism $R[\Gamma\backslash\Delta/\Gamma'] \to \mathrm{Hom}_R(M^\Gamma, M^{\Gamma'})$.

Now Let $\Gamma_1\alpha\Gamma_2 = \coprod_i \Gamma_1\alpha_i$ and $\Gamma_2\beta\Gamma_3 = \coprod_j \Gamma_2\beta_j$, for some $\Gamma_1,\Gamma_2,\Gamma_3 \in S$ and $\alpha,\beta\in\Delta$, and define the product

$$\Gamma_1\alpha\Gamma_2\cdot\Gamma_2\beta\Gamma_3 = \sum_{\Gamma_1\gamma\Gamma_3} c_\gamma\Gamma_1\gamma\Gamma_3 \qquad\text{where}\qquad c_\gamma = \#\{(i,j) : \Gamma_1\alpha_i\beta_j = \Gamma_1\gamma\} \tag{1}$$

with the summation over distinct double cosets $\Gamma_1\gamma\Gamma_3$ (not over all $\gamma\in\Delta$). Note that c_γ can be nonzero only when $\Gamma_1\gamma\Gamma_3$ is equal to one of the finitely many double cosets $\Gamma_1\alpha_i\beta_j\Gamma_3$, so the product of $\Gamma_1\alpha\Gamma_2$ and $\Gamma_2\beta\Gamma_3$ is an element of $R[\Gamma_1\backslash\Delta/\Gamma_3]$, and this element does not depend on the choice of the $\alpha_i,\beta_j,\gamma\in\Delta$.

The decomposition $\Gamma_1\alpha\Gamma_2 = \coprod_i \Gamma_1\alpha_i$ defines an R-module isomorphism

$$R[\Gamma_1\backslash\Delta/\Gamma_2] \simeq R[\Gamma_1\backslash\Delta]^{\Gamma_2},$$

as in §10.1. We can thus view $R[\Gamma_1\backslash\Delta/\Gamma_2]$ as a Γ_2-invariant R-module on which $R[\Gamma_2\backslash\Delta/\Gamma_3]$ acts, and this action is given by the multiplication induced by (1), which extends to multiplication of elements of $R[\Gamma_1\backslash\Delta/\Gamma_2]$ by elements of $R[\Gamma_2\backslash\Delta/\Gamma_3]$ via

$$\left(\sum_{\Gamma_1\alpha\Gamma_2} a_\alpha\Gamma_1\alpha\Gamma_2\right)\left(\sum_{\Gamma_2\beta\Gamma_3} b_\beta\Gamma_2\beta\Gamma_3\right) = \sum_{\Gamma_1\alpha\Gamma_2,\Gamma_2\beta\Gamma_3} a_\alpha b_\beta(\Gamma_1\alpha\Gamma_2\cdot\Gamma_2\beta\Gamma_3)\in R[\Gamma_1\backslash\Delta/\Gamma_3].$$

It is easy to verify that for all $m\in M^{\Gamma_1}$, $A_1\in R[\Gamma_1\backslash\Delta/\Gamma_2]$, $A_2\in R[\Gamma_2\backslash\Delta/\Gamma_3]$ we have

$$(m|A_1)|A_2 = m|(A_1A_2)\in M^{\Gamma_3},$$

and that for all $A_1\in R[\Gamma_1\backslash\Delta/\Gamma_2]$, $A_2\in R[\Gamma_2\backslash\Delta/\Gamma_3]$, $A_3\in R[\Gamma_3\backslash\Delta/\Gamma_4]$ we have

$$(A_1A_2)A_3 = A_1(A_2A_3)\in R[\Gamma_1\backslash\Delta/\Gamma_4].$$

If we now consider the case where $\Gamma_1=\Gamma_2$, the multiplication defined above makes $R[\Gamma\backslash\Delta/\Gamma]$ into a ring, in fact an R-algebra, with identity $\Gamma = \Gamma 1_\Delta \Gamma$.

Definition 10.2. Let G be a group, let $\Gamma\le G$ be a subgroup, let $\Gamma\subseteq\Delta\subseteq\widetilde{\Gamma}$ be a semigroup, and let R be a ring. The Hecke algebra of Γ over R with respect to Δ is the R-algebra $R[\Gamma\backslash\Delta/\Gamma]$.

Recall that an anti-involution of a semigroup is a bijection $\iota\colon\Delta\to\Delta$ with $\iota(\alpha\beta)=\iota(\beta)\iota(\alpha)$ for all $\alpha,\beta\in\Delta$.

Theorem 10.3. *Let $\Gamma\le G$ be a subgroup, $\Gamma\le\Delta\le\widetilde{\Gamma}$ a semigroup, R a commutative ring, and $\iota\colon\Delta\to\Delta$ an anti-involution for which $\iota(\Gamma)=\Gamma$ and $\Gamma\iota(\alpha)\Gamma=\Gamma\alpha\Gamma$. The following hold:*

- *For all $\alpha\in\Delta$ the double coset $\Gamma\backslash\Gamma\alpha\Gamma$ has a common set of left and right Γ-coset representatives: there exist $\alpha_1,\ldots,\alpha_r\in\Gamma\alpha\Gamma$ such that $\Gamma\alpha\Gamma=\coprod_i^r\Gamma\alpha_i=\coprod_{i=1}^r\alpha_i\Gamma$.*
- *The Hecke algebra $R[\Gamma\backslash\Delta/\Gamma]$ is commutative.*

Proof. The existence of the involution ι implies that decompositions of $\Gamma\alpha\Gamma$ into left or right cosets all have the same cardinality. Indeed, if $\Gamma\alpha\Gamma=\coprod_{i=1}^r\Gamma\alpha_i$ then

$$\Gamma\alpha\Gamma=\iota(\Gamma\alpha\Gamma)=\coprod_{i=1}^r\iota(\Gamma\alpha_i)=\coprod_{i=1}^r\iota(\alpha_i)\iota(\Gamma)=\coprod_{i=1}^r\iota(\alpha_i)\Gamma.$$

Now suppose $\Gamma\alpha\Gamma = \coprod_{i=1}^r \Gamma\beta_i = \coprod_{i=1}^r \delta_i\Gamma$ then $\Gamma\beta_i \cap \delta_j\Gamma \neq \emptyset$ for $1 \le i, j \le r$, otherwise $\Gamma\beta_i \subseteq \bigcup_{k\neq j} \delta_k\Gamma$ for some i, j and $\Gamma\alpha\Gamma = \Gamma\beta_i\Gamma = \bigcup_{k\neq j} \delta_k\Gamma$, which is impossible. So we may pick $\alpha_i \in \Gamma\beta_i \cap \delta_i\Gamma$ so that $\Gamma\beta_i = \Gamma\alpha_i$ and $\delta_i\Gamma = \alpha_i\Gamma$ for $1 \le i \le r$. This proves the first claim.

For the second, given $\alpha, \beta \in \Delta$, the first claim lets us write $\Gamma\alpha\Gamma = \coprod_{i=1}^r \Gamma\alpha_i = \coprod_{i=1}^r \alpha_i\Gamma$ and $\Gamma\beta\Gamma = \coprod_{i=1}^r \Gamma\beta_i = \coprod_{i=1}^r \beta_i\Gamma$, and we then have

$$\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \sum_{\Gamma\gamma\Gamma} c_\gamma \Gamma\gamma\Gamma \qquad \text{and} \qquad \Gamma\beta\Gamma \cdot \Gamma\alpha\Gamma = \sum_{\Gamma\gamma\Gamma} c'_\gamma \Gamma\gamma\Gamma,$$

where

$$\begin{aligned} c_\gamma &= \#\{(i,j) : \Gamma\alpha_i\beta_j = \Gamma\gamma\} \\ &= \#\{(i,j) : \Gamma\alpha_i\beta_j\Gamma = \Gamma\gamma\Gamma\}/\#(\Gamma\backslash\Gamma\gamma\Gamma) \\ &= \#\{(i,j) : \Gamma\iota(\beta_j)\iota(\alpha_i)\Gamma = \Gamma\iota(\gamma)\Gamma\}/\#(\Gamma\backslash\Gamma\iota(\gamma)\Gamma) \\ &= \#\{(i,j) : \Gamma\iota(\beta_j)\iota(\alpha_i) = \Gamma\iota(\gamma)\} \\ &= c'_\gamma. \end{aligned}$$

It follows that $\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \Gamma\beta\Gamma \cdot \Gamma\alpha\Gamma$, and this implies that $R[\Gamma\backslash\Delta/\Gamma]$ is commutative. $\square$

10.4 Hecke operators on automorphic forms

We now consider a lattice $\Gamma \in \mathrm{SL}_2(\mathbb{R})$, viewed as a subgroup of $\mathrm{GL}_2^+(\mathbb{R})$, and fix a semigroup $\Gamma \subseteq \Delta \subseteq \widetilde{\Gamma}$. If we have a finite order character χ on Γ, we assume it extends to a character χ of Δ such that $\chi(\alpha\gamma\alpha^{-1}) = \chi(\gamma)$ whenever $\gamma, \alpha\gamma\alpha^{-1} \in \Gamma$, a condition that is obviously satisfied when χ is the trivial character.

Let S be a set of commensurable subgroups of Γ. For $\Gamma_1, \Gamma_2 \in S$ and $\alpha \in \Delta$ we have a decomposition $\Gamma_1\alpha\Gamma_2 = \coprod_{i=1}^r \Gamma_1\alpha_i$, and for any automorphic form $f \in A_k(\Gamma_1\chi)$ we define

$$\begin{aligned} (f|_k\Gamma_1\alpha\Gamma_2)(z) &:= \det(\alpha)^{k/2-1} \sum_{i=1}^r \overline{\chi}(\alpha_i)(f|_k\alpha_i)(z) \\ &= \det(\alpha)^{k-1} \sum_{i=1}^r \overline{\chi}(\alpha_i) j(\alpha_i, z)^{-k} f(\alpha_i z) \end{aligned}$$

This definition does not depend on the choice of the α_i; see [1, Theorem 2.8.1].

We now specialize to the case $R = \mathbb{Z}$.

Theorem 10.4. *Let $\Gamma_1, \Gamma_2, \Gamma_3$ be finite index subgroups of a lattice $\Gamma \le \mathrm{SL}_2(\mathbb{R})$, let Δ be a semigroup containing Γ that lies in its commensurator in $\mathrm{GL}_2^+(\mathbb{R})$, let k be an integer, and let χ be a character of finite order on Γ extending to Δ as above. The following hold:*

- *If f is an automorphic/modular/cusp form of weight k for Γ_1 with character χ then $f|\Gamma_1\alpha\Gamma_2$ is an automorphic/modular/cusp form of weight k for Γ_2 with character χ.*
- *For all finite index $\Gamma_1, \Gamma_2, \Gamma_3 \le \Gamma$ and $\alpha, \beta \in \Delta$ we have*

$$(f|_k\Gamma_1\alpha\Gamma_2)|_k\Gamma_2\beta\Gamma_3 = f|_k(\Gamma_1\alpha\Gamma_2 \cdot \Gamma_2\beta\Gamma_3).$$

- *The spaces $S_k(\Gamma, \chi) \subseteq M_k(\Gamma, \chi) \subseteq A_k(\Gamma, \chi)$ are right $\mathbb{Z}[\Gamma\backslash\Delta/\Gamma]$-submodules.*

Proof. This is Theorem 2.8.1 in [1]. $\square$

Recall the Petersson inner product $\langle f, g\rangle$, defined for $f \in S_k(\Gamma)$ and $g \in M_k(\Gamma)$.

Theorem 10.5. *Let $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ and define $\alpha' := \det(\alpha)\alpha^{-1}$. Let Γ_1, Γ_2 be commensurable to a lattice $\Gamma \in \mathrm{SL}_2(\mathbb{R})$ with commensurator $\widetilde{\Gamma}$ in $\mathrm{GL}_2^+(\mathbb{R})$, and let k be an integer. We then have:*

- $\langle f|_k\alpha, g\rangle = \langle f, g|_k\alpha'\rangle$ *for all* $f \in S_k(\Gamma_1)$ *and* $g \in M_k(\Gamma_2)$;
- $\langle f|_k\Gamma\alpha\Gamma, g\rangle = \langle f, g|_k\Gamma\alpha'\Gamma\rangle$ *for all* $f \in S_k(\Gamma)$ *and* $g \in M_k(\Gamma)$.

Proof. This is Theorem 2.8.2 in [1]. □

Corollary 10.6. *Let χ, ψ be distinct finite order characters of a lattice $\Gamma \in \mathrm{SL}_2(\mathbb{R})$. Then $\langle f, g\rangle = 0$ for all $f \in S_k(\Gamma, \chi)$ and $g \in M_k(\Gamma, \psi)$.*

Proof. Pick $\gamma \in \Gamma$ so that $\chi(\gamma) \neq \psi(\gamma)$. Then $\gamma' := \det(\gamma)\gamma^{-1} = \gamma^{-1}$ and

$$\chi(\gamma)\langle f, g\rangle = \langle f|_k\gamma, g\rangle = \langle f, g|_k\gamma^{-1}\rangle = \psi(\gamma)\langle f, g\rangle,$$

by Theorem 10.5, which is possible only if $\langle f, g\rangle = 0$. □

Corollary 10.7. *If $f \in \mathscr{E}_k(\Gamma)$ then $f|_k\Gamma\alpha\Gamma \in \mathscr{E}_k(\Gamma)$ for all $\alpha \in \widetilde{\Gamma}$*

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.