These notes summarize the material in §2.7–2.8 of [1] covered in lecture, along with some relevant background on permutation modules.

10.1 Group rings and permutation modules

Let $R$ be a commutative ring. For any set $X$ we use $R[X]$ to denote the free $R$-module generated by the elements of $X$; it consists of all finite formal sums $\sum_{x \in X} r_x x$, with addition scalar multiplication and addition defined in the obvious way: we let $r_1 x + r_2 x := (r_1 + r_2) x$ and $r_1 (r_2 x) = (r_1 r_2) x$ and extend these definitions $R$-linearly to all of $R[X]$.

If $G$ is a group, the $R$-module $R[G]$ is a ring with $r_1 g_1 \cdot r_2 g_2 := (r_1 r_2) (g_1 g_2)$ (to define the product of elements or $R[G]$ use the distributive law), called the group ring $R[G]$ of $G$ over $R$; note that this ring is commutative if and only if $G$ is. If $X$ is a left/right $G$-set then the $R$-module $R[X]$ can be given the structure of a left/right $R[G]$-module by extending the $G$-action $R$-linearly. Note that this ensures that the $G$-action is compatible with the $R$-module structure of $R[X]$ in the sense that the action of each $g \in G$ defines an endomorphism of $R[X]$.

For subgroups $H \leq G$ we use $[G/H]$ and $[H\backslash G]$ to denote the $G$-sets of left and right cosets of $H$ equipped with the obvious left and right $G$-actions ($g' \in G$ sends $gH$ to $g'gH$ and $Hg$ to $Hgg'G$).

We then have permutation modules $R[G/H]$ and $R[H\backslash G]$ that are left and right $R[G]$-modules, respectively. If $K$ is another subgroup of $G$ we have an $R$-module of double cosets $R[H\backslash G/K]$ which admits left and right $G$-actions in which $g' \in G$ sends $HgK$ to $Hg'gK$ and $Hgg'K$; extending these $G$-actions $R$-linearly makes $R[H\backslash G/K]$ an $R[G]$-bimodule.

For $H,K \leq G$ and $g \in G$, the decomposition of the double coset $H \backslash g \mid K$ into right $H$-cosets is given by the orbit of $Hg \in [H\backslash G]$ under the right action of $K$. If this orbit is finite we can formally sum cosets to obtain an element of $R[H\backslash G]$.

If every right $H$-coset in $[H\backslash G]$ has a finite $K$-orbit, we can use this to define an injective right $R[G]$-module homomorphism

$$R[H\backslash G/H] \hookrightarrow R[H\backslash G]$$

$$HgK \mapsto \sum_{Hg'K : g' \in G} Hg'K$$

Elements in the image of this homomorphism are $R$-linear combination of sums of $K$-orbits in $[H\backslash G]$, hence invariant under the action of $K$. Let $R[H\backslash G]^K$ denote the $R$-submodule of $R[H\backslash G]$ whose elements are fixed by $K$. Every such element must be an $R$-linear sum of $K$-orbits in $R[H\backslash G]$, corresponding to the image of an $R$-linear sum of double cosets in $[H\backslash G/K]$ that lies in the image of the map above. It follows that we then have an $R$-module isomorphism

$$R[H\backslash G/K] \cong [H\backslash G]^K,$$

and if every $K$-coset in $[G/K]$ has a finite $H$-orbit we similarly obtain an $R$-module isomorphism

$$R[H\backslash G/K] \cong [G/K]^H.$$

We now that that everything above applies more generally to any semigroup $\Delta \leq G$. The semigroup ring $R[\Delta]$ is defined in the same way as the group ring $R[G]$ (and it is indeed a ring, even though $\Delta$ need not be a group). For $H,K \subseteq \Delta$ we have the right $R[\Delta]$-module $R[H\backslash \Delta]$ and the $R[\Delta]$-bimodule $R[H\backslash \Delta/K]$, and if every $K$ orbit in $[H\backslash \Delta]$ and every $H$-orbit in $[\Delta/K]$ is finite, we have $R$-module isomorphisms

$$R[H\backslash \Delta/K] \cong [H\backslash \Delta]^K \cong [\Delta/K]^H.$$
10.2 Commensurability

We now investigate conditions under which the finite orbit assumptions above are satisfied.

**Definition 10.1.** Let $G$ be a group. Two subgroups $H, K \leq G$ are **commensurable** if their intersection has finite index in both $H$ and $K$; we write $H \sim K$ to indicate this relationship.

For any $H, K, J \leq G$ we have

$$[H : H \cap J] \leq [H : H \cap K \cap J] = [H : H \cap K][H \cap K : H \cap J] \leq [H : H \cap K][K : K \cap J],$$

which implies that commensurability is transitive; it is clearly reflexive and symmetric, hence an equivalence relation.

For $g \in G$, let $H^g$ denote $g^{-1}Hg \leq G$. For any $H, K \leq G$ and $g \in G$ we have $H \sim K$ if and only if $H^g \sim K^g$. It follows that if $H$ is commensurable with $H^{g_1}$ and $H^{g_2}$ then it is also commensurable with $H^{g_1 g_2}$ and therefore the commensurator

$$\tilde{H} := \{ g \in G : H \sim H^g \}$$

is a subgroup of $G$ that contains the normalizer of $H$ in $G$.

If $H, K \leq G$ are commensurable, then $\tilde{H} = \tilde{K}$ and for any $g \in \tilde{H} = \tilde{K}$ we have finite left and right coset decompositions of the double coset $HgK$ via

$$HgK = \bigsqcup_{i=1}^{m} h_i gK = \bigsqcup_{j=1}^{n} Hgk_j,$$

with $h_i \in H$ ranging over $H/(H \cap K^{g^{-1}})$ representatives, of which there are $m := [H : H \cap K^{g^{-1}}]$, and $k_j$ ranging over $(K \cap H^g)\backslash K$ representatives, of which there are $n = [K : K \cap H^g]$.

10.3 Hecke algebras

Now let $S$ be a set of commensurable subgroups $\Gamma \leq G$, and let $\Delta \subseteq G$ be a semigroup containing every $\Gamma \in S$ and is contained in their common commensurator in $G$. We will eventually apply this with $\Gamma = SL_2(\mathbb{Z}) \leq GL_2^+(\mathbb{R}) = G$ and $S$ the set of congruence subgroups of $SL_2(\mathbb{Z})$, but rather letting $\Delta$ be the commensurator of $SL_2(\mathbb{Z})$ in $GL_2^+(\mathbb{R})$ (which is $\mathbb{R}^+GL_2^+(\mathbb{Q})$), we will take $\Delta$ to be a semigroup of integer matrices with positive determinant.

Let $M$ be an $R$-module equipped with a right $\Delta$-action that is compatible with the $R$-module structure of $M$, making $M$ an $R[\Delta]$-module. Let $\Gamma, \Gamma' \in S$, and consider a double coset $\Gamma \alpha \Gamma'$. If $\Gamma \alpha \Gamma' = \bigsqcup_i \Gamma \alpha_i$ is a right coset decomposition, we define an action of $\Gamma \alpha \Gamma'$ on $M^\Gamma$ via

$$m|\Gamma \alpha \Gamma' := \sum_i m^{\alpha_i}.$$

This definition does not depend on the choice of the choice of the $\alpha_i \in \Gamma \alpha \Gamma'$, since if $\Gamma \alpha_i' = \Gamma \alpha_i$ for some then $\alpha_i' = \gamma_i \alpha_i$ for some $\gamma_i \in \Gamma$ and $m^{\alpha_i} = m^{\gamma_i \alpha_i} = (m^{\gamma_i})^{\alpha_i} = m^{\alpha_i}$ for all $m \in M^\Gamma$. We also note that $m|\Gamma \alpha \Gamma' \in M$ is $\Gamma'$-invariant, since for any $\gamma' \in \Gamma'$, if we let $\alpha_i' = \alpha_i \gamma'$ then

$$(m|\Gamma \alpha \Gamma')^{\gamma'} = \sum_i m^{\alpha_i \gamma'} = \sum_i m^{\alpha_i} = m|\Gamma \alpha \Gamma',$$

because $\Gamma \alpha \Gamma' = \bigsqcup_i \Gamma \alpha_i'$ is just another right coset decomposition. This defines an $R$-module homomorphism $M^\Gamma \to M^{\Gamma'}$. Extend this $R$-linearly to an action of $R[\Gamma \backslash \Delta / \Gamma']$ yields an $R$-module homomorphism $R[\Gamma \backslash \Delta / \Gamma'] \to \text{Hom}_R(M^\Gamma, M^{\Gamma'})$.  

18.786 Spring 2024, Lecture #10, Page 2
Now let \( \Gamma_1 \alpha \Gamma_2 = \bigsqcup_i \Gamma_1 \alpha_i \) and \( \Gamma_2 \beta \Gamma_3 = \bigsqcup_j \Gamma_2 \beta_j \), for some \( \Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{S} \) and \( \alpha, \beta \in \Delta \), and define the product

\[
\Gamma_1 \alpha \Gamma_2 \cdot \Gamma_2 \beta \Gamma_3 = \sum_{\gamma} c_{\gamma} \Gamma_1 \gamma \Gamma_3 \quad \text{where} \quad c_{\gamma} = \# \{ (i, j) : \Gamma_1 \alpha_i \Gamma_2 \beta_j = \gamma \Gamma_3 \}
\]

with the summation over distinct double cosets \( \Gamma_1 \gamma \Gamma_3 \) (not over all \( \gamma \in \Delta \)). Note that \( c_{\gamma} \) can be nonzero only when \( \Gamma_1 \gamma \Gamma_3 \) is equal to one of the finitely many double cosets \( \Gamma_1 \alpha_1 \beta_1 \Gamma_3 \), so the product of \( \Gamma_1 \alpha \Gamma_2 \) and \( \Gamma_2 \beta \Gamma_3 \) is an element of \( R[\Gamma_1 \setminus \Delta / \Gamma_3] \), and this element does not depend on the choice of the \( \alpha_i, \beta_j, \gamma \in \Delta \).

The decomposition \( \Gamma_1 \alpha \Gamma_2 = \bigsqcup_i \Gamma_1 \alpha_i \) defines an \( R \)-module isomorphism

\[
R[\Gamma_1 \setminus \Delta / \Gamma_3] \cong R[\Gamma_1 \setminus \Delta]^{\Gamma_2},
\]

as in §10.1. We can thus view \( R[\Gamma_1 \setminus \Delta / \Gamma_3] \) as a \( \Gamma_2 \)-invariant \( R \)-module on which \( R[\Gamma_2 \setminus \Delta / \Gamma_3] \) acts, and this action is given by the multiplication induced by (1), which extends to multiplication of elements of \( R[\Gamma_1 \setminus \Delta / \Gamma_2] \) by elements of \( R[\Gamma_2 \setminus \Delta / \Gamma_3] \) via

\[
\left( \sum_{\alpha} a_{\alpha} \Gamma_1 \alpha \Gamma_2 \right) \left( \sum_{\beta} b_{\beta} \Gamma_2 \beta \Gamma_3 \right) = \sum_{\Gamma_1 \alpha \Gamma_2, \Gamma_2 \beta \Gamma_3} a_{\alpha} b_{\beta} (\Gamma_1 \alpha \Gamma_2 \cdot \Gamma_2 \beta \Gamma_3) \in R[\Gamma_1 \setminus \Delta / \Gamma_3].
\]

It is easy to verify that for all \( m \in M^{\Gamma_1}, A_1 \in R[\Gamma_1 \setminus \Delta / \Gamma_2], A_2 \in R[\Gamma_2 \setminus \Delta / \Gamma_3] \) we have

\[
(m|A_1)|A_2 = m|(A_1 A_2) \in M^{\Gamma_3},
\]

and that for all \( A_1 \in R[\Gamma_1 \setminus \Delta / \Gamma_2], A_2 \in R[\Gamma_2 \setminus \Delta / \Gamma_3], A_3 \in R[\Gamma_3 \setminus \Delta / \Gamma_4] \) we have

\[
(A_1 A_2) A_3 = A_1 (A_2 A_3) \in R[\Gamma_1 \setminus \Delta / \Gamma_4].
\]

If we now consider the case where \( \Gamma_1 = \Gamma_2 \), the multiplication defined above makes \( R[\Gamma \setminus \Delta / \Gamma] \) into a ring, in fact an \( R \)-algebra, with identity \( \Gamma = \Gamma_1 \cap \Gamma \).

**Definition 10.2.** Let \( G \) be a group, let \( \Gamma \leq G \) be a subgroup, let \( \Delta \subseteq \Gamma \subseteq \tilde{\Gamma} \) be a semigroup, and let \( R \) be a ring. The \textbf{Hecke algebra} of \( \Gamma \) over \( R \) with respect to \( \Delta \) is the \( R \)-algebra \( R[\Gamma \setminus \Delta / \Gamma] \).

Recall that an \textbf{anti-involution} of a semigroup is a bijection \( \iota : \Delta \rightarrow \Delta \) with \( \iota(\alpha \beta) = \iota(\beta) \iota(\alpha) \) for all \( \alpha, \beta \in \Delta \).

**Theorem 10.3.** Let \( \Gamma \leq G \) be a subgroup, \( \Delta \subseteq \Gamma \subseteq \tilde{\Gamma} \) a semigroup, \( R \) a commutative ring, and \( \iota : \Delta \rightarrow \Delta \) an anti-involution for which \( \iota(\Gamma) = \Gamma \) and \( \Gamma \iota(\alpha) = \iota(\alpha) \Gamma \). The following hold:

- For all \( \alpha \in \Delta \) the double coset \( \Gamma \setminus \Gamma \alpha \Gamma \) has a common set of left and right \( \Gamma \)-coset representatives: there exist \( \alpha_1, \ldots, \alpha_r \in \Gamma \alpha \Gamma \) such that \( \Gamma \alpha \Gamma = \bigsqcup_i \Gamma \alpha_i = \bigsqcup_i \alpha_i \Gamma \).

- The Hecke algebra \( R[\Gamma \setminus \Delta / \Gamma] \) is commutative.

**Proof.** The existence of the involution \( \iota \) implies that decompositions of \( \Gamma \alpha \Gamma \) into left or right cosets all have the same cardinality. Indeed, if \( \Gamma \alpha \Gamma = \bigsqcup_i \Gamma \alpha_i \) then

\[
\Gamma \alpha \Gamma = \iota(\Gamma \alpha \Gamma) = \bigsqcup_i \iota(\Gamma \alpha_i) = \bigsqcup_i \iota(\alpha_i) \iota(\Gamma) = \bigsqcup_i \iota(\alpha_i) \Gamma.
\]
Now suppose $\Gamma \alpha \Gamma = \bigsqcup_{i=1}^{r} \Gamma \beta_i = \bigsqcup_{i=1}^{r} \delta_i \Gamma$ then $\Gamma \beta_i \cap \delta_j \Gamma \neq \emptyset$ for $1 \leq i, j \leq r$, otherwise $\Gamma \beta_i \subseteq \bigcup_{k \neq j} \delta_k \Gamma$ for some $i, j$ and $\Gamma \alpha \Gamma = \Gamma \beta_i \Gamma = \bigcup_{k \neq j} \delta_k \Gamma$, which is impossible. So we may pick $\alpha_i \in \Gamma \beta_i \cap \delta_i \Gamma$ so that $\Gamma \beta_i = \Gamma \alpha_i$ and $\delta_i \Gamma = \alpha_i \Gamma$ for $1 \leq i \leq r$. This proves the first claim.

For the second, given $\alpha, \beta \in \Delta$, the first claim lets us write $\Gamma \alpha \Gamma = \bigsqcup_{i=1}^{r} \Gamma \alpha_i = \bigsqcup_{i=1}^{r} \alpha_i \Gamma$ and $\Gamma \beta \Gamma = \bigsqcup_{i=1}^{r} \Gamma \beta_i = \bigsqcup_{i=1}^{r} \beta_i \Gamma$, and we then have

$$\Gamma \alpha \Gamma \cdot \Gamma \beta \Gamma = \sum_{\gamma \Gamma} c_{\gamma} \Gamma \gamma \Gamma$$

and

$$\Gamma \beta \Gamma \cdot \Gamma \alpha \Gamma = \sum_{\gamma \Gamma} c'_{\gamma} \Gamma \gamma \Gamma,$$

where

$$c_{\gamma} = \#\{(i, j) : \Gamma \alpha_i \beta_j = \Gamma \gamma\}$$

$$= \#\{(i, j) : \Gamma \alpha_i \beta_j \Gamma = \Gamma \gamma \Gamma\} / \#(\Gamma \gamma \Gamma)$$

$$= \#\{(i, j) : \Gamma \alpha_i \beta_j \Gamma = \Gamma \gamma \Gamma\} / \#(\Gamma \gamma \Gamma)$$

$$= \#\{(i, j) : \Gamma \alpha_i \beta_j \Gamma = \Gamma \gamma \Gamma\}$$

$$= c'_{\gamma}.$$

It follows that $\Gamma \alpha \Gamma \cdot \Gamma \beta \Gamma = \Gamma \beta \Gamma \cdot \Gamma \alpha \Gamma$, and this implies that $R[\Gamma \Delta / \Gamma]$ is commutative.

\[\square\]

### 10.4 Hecke operators on automorphic forms

We now consider a lattice $\Gamma \subseteq \text{SL}_2(\mathbb{R})$, viewed as a subgroup of $\text{GL}_2^+(\mathbb{R})$, and fix a semigroup $\Gamma \subseteq \Delta \subseteq \overline{\Delta}$. If we have a finite order character $\chi$ on $\Gamma$, we assume it extends to a character $\chi$ of $\Delta$ such that $\chi(\Gamma \alpha \Gamma^{-1}) = \chi(\gamma)$ whenever $\gamma, \alpha \gamma \alpha^{-1} \in \Gamma$, a condition that is obviously satisfied when $\chi$ is the trivial character.

Let $\Delta$ be a set of commensurable subgroups of $\Gamma$. For $\Gamma_1, \Gamma_2 \in \Delta$ and $\alpha \in \Delta$ we have a decomposition $\Gamma_1 \alpha \Gamma_2 = \bigsqcup_{i=1}^{r} \Gamma_1 \alpha_i$, and for any automorphic form $f \in A_k(\Gamma_1 \chi)$ we define

$$\langle f | k \Gamma_1 \alpha \Gamma_2 \rangle(z) := \det(k \gamma \Gamma)^{k/2-1} \sum_{i=1}^{r} \chi(\alpha_i)(f | k \alpha_i)(z)$$

$$= \det(k \gamma \Gamma)^{k-1} \sum_{i=1}^{r} \chi(\alpha_i)(j(\alpha_i, z))^{-k} f(\alpha_i z).$$

This definition does not depend on the choice of the $\alpha_i$; see [1, Theorem 2.8.1].

We now specialize to the case $\Gamma = \mathbb{Z}$.

**Theorem 10.4.** Let $\Gamma_1, \Gamma_2, \Gamma_3$ be finite index subgroups of a lattice $\Gamma \subseteq \text{SL}_2(\mathbb{R})$, let $\Delta$ be a semigroup containing $\Gamma$ that lies in its commensurator in $\text{GL}_2^+(\mathbb{R})$, let $k$ be an integer, and let $\chi$ be a character of finite order on $\Gamma$ extending to $\Delta$ as above. The following hold:

- If $f$ is an automorphic/modular/cusp form of weight $k$ for $\Gamma_1$ with character $\chi$ then $f | k \Gamma_1 \alpha \Gamma_2$ is an automorphic/modular/cusp form of weight $k$ for $\Gamma_2$ with character $\chi$.

- For all finite index $\Gamma_1, \Gamma_2, \Gamma_3 \leq \Gamma$ and $\alpha, \beta \in \Delta$ we have

$$\langle f | k \Gamma_1 \alpha \Gamma_2 \rangle | k \Gamma_2 \beta \Gamma_3 = f | k \Gamma_1 \alpha \Gamma_2 \cdot \Gamma_2 \beta \Gamma_3 \rangle.$$

- The spaces $S_k(\Gamma, \chi) \subseteq M_k(\Gamma, \chi) \subseteq A_k(\Gamma, \chi)$ are right $\mathbb{Z}[\Gamma \Delta / \Gamma]$-submodules.

**Proof.** This is Theorem 2.8.1 in [1].

\[\square\]
Recall the Petersson inner product \( \langle f, g \rangle \), defined for \( f \in S_k(\Gamma) \) and \( g \in M_k(\Gamma) \).

**Theorem 10.5.** Let \( \alpha \in \text{GL}^+_2(\mathbb{R}) \) and define \( \alpha' := \det(\alpha)\alpha^{-1} \). Let \( \Gamma_1, \Gamma_2 \) be commensurable to a lattice \( \Gamma \in \text{SL}_2(\mathbb{R}) \) with commensurator \( \Gamma' \) in \( \text{GL}^+_2(\mathbb{R}) \), and let \( k \) be an integer. We then have:

- \( \langle f |_k \alpha, g \rangle = \langle f, g |_{k'} \rangle \) for all \( f \in S_k(\Gamma_1) \) and \( g \in M_k(\Gamma_2) \);
- \( \langle f |_{k\Gamma\alpha}, g \rangle = \langle f, g |_{k\Gamma'\alpha'} \rangle \) for all \( f \in S_k(\Gamma) \) and \( g \in M_k(\Gamma) \).

**Proof.** This is Theorem 2.8.2 in [1]. \qed

**Corollary 10.6.** Let \( \chi, \psi \) be distinct finite order characters of a lattice \( \Gamma \in \text{SL}_2(\mathbb{R}) \). Then \( \langle f, g \rangle = 0 \) for all \( f \in S_k(\Gamma, \chi) \) and \( g \in M_k(\Gamma, \psi) \).

**Proof.** Pick \( \gamma \in \Gamma \) so that \( \chi(\gamma) \neq \psi(\gamma) \). Then \( \gamma' := \det(\gamma)\gamma^{-1} = \gamma^{-1} \) and

\[
\chi(\gamma)(f, g) = \langle f |_{k\gamma}, g \rangle = \langle f, g |_{k\gamma} \rangle = \psi(\gamma)\langle f, g \rangle,
\]

by Theorem 10.5, which is possible only if \( \langle f, g \rangle = 0 \). \qed

**Corollary 10.7.** If \( f \in \mathcal{E}_k(\Gamma) \) then \( f |_{k\Gamma\alpha} \in \mathcal{E}_k(\Gamma) \) for all \( \alpha \in \Gamma' \).

**References**