1 Geometry of the complex upper half plane

This is summary of the material from §1.1–1.4 of [1] presented in lecture, with proofs omitted.

1.1 Automorphisms of the upper half plane

The group $GL_2(\mathbb{C})$ acts on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ via linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d} \quad (z \in \mathbb{P}^1(\mathbb{C}))$$

This defines a left group action of $GL_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$. The maps $z \mapsto az$ are meromorphic.

We now define the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$$

For $z \in \mathbb{H}$ and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ we have

$$\text{Im}(\alpha z) = \frac{\det(\alpha) \text{Im}(az)}{|cz + d|^2}.$$ 

For $\det(\alpha) > 0$ we have $|cz + d| \neq 0$ and $\text{Im}(\alpha z) > 0$, which implies that $\alpha \mapsto \alpha z$ is a holomorphic automorphism of $\mathbb{H}$ for all $\alpha \in GL_2^+(\mathbb{R}) := \{\alpha \in GL_2(\mathbb{R}) : \det(\alpha) > 0\}$.

Theorem 1.1. The following hold:

- For every $z \in \mathbb{H}$ there exists $\alpha \in SL_2(\mathbb{R})$ such that $\alpha i = z$.
- $GL_2^+(\mathbb{R})/\mathbb{R}^+ \cong SL_2(\mathbb{R})/\{\pm 1\} \cong \text{Aut}(\mathbb{H})$
- The $SL_2(\mathbb{R})$-stabilizer of $i \in \mathbb{H}$ is $SL_2(\mathbb{R}) i = SO_2(\mathbb{R})$.

Proof. See [1, Theorem 1.1.3]. \qed

Recall that a topological group $G$ is a group object in the category of topological spaces (this means the maps $G \times G \rightarrow G$ and $G \times X$ defined by $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous). All the topological groups we shall consider in this course are Hausdorff, but we won’t bake that into the definition (some authors do). We view $GL_2(\mathbb{R})$ and all its subgroups and quotients as topological groups, where $GL_2(\mathbb{R})$ has the subspace topology induced by $GL_2(\mathbb{R}) \subseteq M_2(\mathbb{R}) \cong \mathbb{R}^4$.

When a topological group $G$ acts on a topological space $X$ the action map $G \times X \rightarrow X$ defined by $(g, x) \mapsto gx$ is required to be continuous, in addition to satisfying the usual properties of a (left) group action. For each $g \in G$ the map $x \mapsto gx$ is an automorphism of $X$ (a homeomorphism $X \rightarrow X$). For $x \in X$ we have the $G$-stabilizer $G_x$ and $G$-orbit $Gx$ defined by

$$G_x := \{g \in G : gx = x\} \subseteq G \quad \text{and} \quad Gx := \{gx : g \in G\} \subseteq X$$

and we use $G \setminus X$ to denote the topological space defined by the $G$-orbits of $X$ (with the quotient topology), and we use $G/G_x$ to denote the topological space consisting of the right $G_x$-cosets of $G$ (with the quotient topology).

Theorem 1.2. Let $G$ be a second countable locally compact Hausdorff topological group acting on a locally compact Hausdorff space $X$. Then for every $x \in X$ the map $gG_x \mapsto gx$ defines a homeomorphism $G/G_x \xrightarrow{\sim} X$. 

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Proof. See [1, Theorem 1.2.1]. We will prove a more general theorem in the next lecture. □

Corollary 1.3. The map $a \SO_2(\mathbb{R}) \mapsto a \iota$ is a homeomorphism $\SL_2(\mathbb{R}) / \SO_2(\mathbb{R}) \iso \mathbb{H}$.

The corollary allows us to represent elements $z$ of $\mathbb{H}$ as right cosets $\beta \SO_2(\mathbb{R})$, in which case the action of $a \SL_2(\mathbb{R})$ on $\mathbb{H}$ is simply (left) matrix multiplication $az = a\beta \SO_2(\mathbb{R})$ rather than a linear fractional transformation.

1.2 Classification of linear fractional transformations

For $a \in \GL_2^+(\mathbb{R})$ we define $\text{disc}(a) := \text{tr}(a)^2 - 4 \det(a)$ to be the discriminant of its characteristic polynomial $t^2 - \text{tr}(a)t + \det(a)$, whose sign determines the type of eigenvalues $a$ has (distinct complex conjugates, a single real (double) eigenvalue, or two real eigenvalues).

Definition 1.4. We call a nonscalar $a \in \GL_2^+(\mathbb{R})$

- elliptic if $\text{disc}(a) < 0$,
- parabolic if $\text{disc}(a) = 0$,
- hyperbolic if $\text{disc}(a) > 0$.

We denote the extended upper half-plane by $\mathbb{H}^* := \mathbb{H} \cup \mathbb{R} \cup \{\infty\} \subseteq \mathbb{P}^1(\mathbb{C})$.

Theorem 1.5. For each nonscalar $a \in \GL_2^+(\mathbb{R})$ we have

- $a$ is elliptic if and only if its fixed points are $z$ and $\bar{z}$ for some $z \in \mathbb{H}$.
- $a$ is parabolic if and only if it has a unique fixed point $x \in \mathbb{H}^* - \mathbb{H} = \mathbb{R} \cup \{\infty\}$.
- $a$ is hyperbolic if and only if it has distinct fixed points $x_1, x_2 \in \mathbb{H}^* - \mathbb{H} = \mathbb{R} \cup \{\infty\}$.

In every case, the fixed points of $a$ on $\mathbb{P}^1(\mathbb{C})$ lie in the extended upper half-plane $\mathbb{H}^*$, and they lie in the upper half-plane if and only if $a$ is elliptic.

Proof. This follows immediately from an analysis of the eigenvalues in each case. □

For $x, x' \in \mathbb{R} \cup \{\infty\}$ we define

$$\GL_2^+(\mathbb{R})_x := \{a \in \GL_2^+(\mathbb{R}) : a \text{ is parabolic or scalar}\}, \quad \GL_2^+(\mathbb{R})_{x,x'} := \GL_2^+(\mathbb{R})_x \cap \GL_2^+(\mathbb{R})_{x'}$$

Lemma 1.6. We have

- $\GL_2^+(\mathbb{R})_i = \SO_2(\mathbb{R})$;
- $\GL_2^+(\mathbb{R})_\infty = \{(s \, b \, 0 \, d) : a, d \in \mathbb{R}^+, b \in \mathbb{R}, ad > 0\}$;
- $\GL_2^+(\mathbb{R})_0^\circ = \{(s \, b \, 0 \, d) : a \in \mathbb{R}^+, b \in \mathbb{R}\}$;
- $\GL_2^+(\mathbb{R})_0^\circ = \{(s \, b \, 0 \, d) : a, d \in \mathbb{R}^+, ad > 0\}$.

Moreover, for $z \in \mathbb{H}$ the stabilizer $\GL_2^+(\mathbb{R})_z$ is conjugate to $\GL_2^+(\mathbb{R})_i$, for $x \in \mathbb{H}^* - \mathbb{H}$ the stabilizer $\GL_2^+(\mathbb{R})_x$ is conjugate to $\GL_2^+(\mathbb{R})_\infty$ with $\GL_2^+(\mathbb{R})_x^{(p)}$ conjugate to $\GL_2^+(\mathbb{R})_\infty^{(p)}$, and for distinct $x, x' \in \mathbb{H}^* - \mathbb{H}$ the double stabilizer $\GL_2(\mathbb{R})_{x,x'}$ is conjugate to $\GL_2^+(\mathbb{R})_{0,\infty}$. In all cases, the conjugating element can be chosen to lie in $\SL_2(\mathbb{R})$.

Proof. See [1, Lemma 1.3.2]. □
1.3 The invariant metric and measure on the upper half plane

Let \( z = x + iy \in \mathbb{H} \). The **invariant metric** \( ds \) and **invariant measure** on \( \mathbb{H} \) are defined by

\[
ds(z) = \frac{\sqrt{dx^2 + dy^2}}{y^2} \quad \text{and} \quad dv(z) = \frac{dx dy}{y^2}.
\]

We call the image \( C_{z_0 \to z_1} \) of an injective function \( \phi : [0,1] \to \mathbb{H} \) that is \( C^\infty \) at all but finitely many points a **path** from \( z_0 = \phi(0) \) to \( z_1 = \phi(1) \). If we put \( \phi(t) = x(t) + iy(t) \), the **length** of \( C_{z_0 \to z_1} \) is

\[
\ell(C_{z_0 \to z_1}) := \int_0^1 ds(\phi(t)) = \int_0^1 \frac{\sqrt{(dx(t)/dt)^2 + (dy(t)/dt)^2}}{y(t)} dt,
\]

which does not depend on the choice of \( \phi \), as long as it has image \( C_{z_0 \to z_1} \).

A circle or line in \( \mathbb{H} \) orthogonal to \( \mathbb{R} \) is called a **geodesic**.

**Lemma 1.7.** For any two distinct \( z_0, z_1 \in \mathbb{H} \), there is a unique shortest path \( C_{z_0 \to z_1} \) from \( z_0 \) to \( z_1 \), which lies on a geodesic. For any \( z_0 \in \mathbb{H} \) and \( r > 0 \) the set \( \{ z \in \mathbb{H} : \ell(C_{z_0,z}) = r \} \) is a circle in \( \mathbb{H} \) orthogonal to every geodesic that contains \( z_0 \).

**Proof.** See Lemma 1.4.1 and Corollary 1.4.2 in [1] \( \square \)

The isomorphism \( \mathbb{H} \simeq \text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R}) \) allows us to uniquely represent \( \alpha \in \text{SL}_2(\mathbb{R}) \) in the form \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = h z k_\theta \), where \( z = x + iy = ai \in \mathbb{H} \) and \( \theta = -\arg(ci + d) \), with

\[
h_z = y^{-1/2} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \text{SO}_2(\mathbb{R}).
\]

We now define a measure on \( \text{SL}_2(\mathbb{R}) \) by

\[
da = \frac{dx dy d\theta}{2\pi y^2}.
\]

This measure is \( \text{SL}_2(\mathbb{R}) \)-invariant: one can check that \( da = d\alpha \beta = \beta d\alpha \) for all \( \alpha, \beta \in \text{SL}_2(\mathbb{R}) \); moreover, we have chosen the scaling so that \( da \) is compatible with \( dv \).

**Theorem 1.8.** \( \text{SL}_2(\mathbb{R}) \) is unimodular, and for any measurable \( f : \mathbb{H} \to \mathbb{C} \) we have

\[
\int_{\mathbb{H}} f(z) dv(z) = \int_{\text{SL}_2(\mathbb{R})} f(ai) da
\]

**Proof.** This is [1, Thm. 1.4.5] \( \square \)

**References**