1 Geometry of the complex upper half plane

This is summary of the material from §1.1–1.4 of [1] presented in lecture, with proofs omitted.

1.1 Automorphisms of the upper half plane

The group $\mathrm{GL}_2(\mathbb{C})$ acts on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ via linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} \qquad (z \in \mathbb{P}^1(\mathbb{C}))$$

This defines a left group action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$. The maps $z \mapsto \alpha z$ are meromorphic.

We now define the upper half plane

$$\mathbf{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

For $z \in \mathbf{H}$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ we have

$$\mathrm{Im}(\alpha z) = \frac{\det(\alpha)\,\mathrm{Im}(\alpha z)}{|cz+d|^2}.$$

For $\det(\alpha) > 0$ we have $|cz+d| \neq 0$ and $\mathrm{Im}(\alpha z) > 0$, which implies that $\alpha \mapsto \alpha z$ is a holomorphic automorphism of $\mathbf{H}$ for all $\alpha \in \mathrm{GL}_2^+(\mathbb{R}) := \{\alpha \in \mathrm{GL}_2(\mathbb{R}) : \det(\alpha) > 0\}$.

Theorem 1.1. *The following hold:*

- *For every $z \in \mathbf{H}$ there exists $\alpha \in \mathrm{SL}_2(\mathbb{R})$ such that $\alpha i = z$.*
- $\mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^\times \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm 1\} \simeq \mathrm{Aut}(\mathbf{H})$
- *The $\mathrm{SL}_2(\mathbb{R})$-stabilizer of $i \in \mathbf{H}$ is $\mathrm{SL}_2(\mathbb{R})_i = \mathrm{SO}_2(\mathbb{R})$.*

Proof. See [1, Theorem 1.1.3]. □

Recall that a topological group G is a group object in the category of topological spaces (this means the maps $G \times G \to G$ and $G \times G$ defined by $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous). All the topological groups we shall consider in this course are Hausdorff, but we won't bake that into the definition (some authors do). We view $\mathrm{GL}_2(\mathbb{R})$ and all its subgroups and quotients as topological groups, where $\mathrm{GL}_2(\mathbb{R})$ has the subspace topology induced by $\mathrm{GL}_2(\mathbb{R}) \subseteq \mathrm{M}_2(\mathbb{R}) \simeq \mathbb{R}^4$.

When a topological group G acts on a topological space X the action map $G \times X \to X$ defined by $(g,x) \mapsto gx$ is required to be continuous, in addition to satisfying the usual properties of a (left) group action. For each $g \in G$ the map $x \mapsto gx$ is an automorphism of X (a homeomorphism $X \to X$). For $x \in X$ we have the G-stabilizer G_x and G-orbit Gx defined by

$$G_x := \{g \in G : gx = x\} \subseteq G \qquad \text{and} \qquad Gx := \{gx : g \in G\} \subseteq X$$

and we use $G\backslash X$ to denote the topological space defined by the G-orbits of X (with the quotient topology), and we use G/G_x to denote the topological space consisting of the right G_x-cosets of G (with the quotient topology).

Theorem 1.2. *Let G be a second countable locally compact Hausdorff topological group acting on a locally compact Hausdorff space X. Then for every $x \in X$ the map $gG_x \mapsto gx$ defines a homeomorphism $G/G_x \xrightarrow{\sim} X$.*

Proof. See [1, Theorem 1.2.1]. We will prove a more general theorem in the next lecture. $\square$

Corollary 1.3. *The map* $\alpha \mathrm{SO}_2(\mathbb{R}) \mapsto \alpha i$ *is a homeomorphism* $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \xrightarrow{\sim} \mathbf{H}$.

The corollary allows us to represent elements z of $\mathbf{H}$ as right cosets $\beta\mathrm{SO}_2(\mathbb{R})$, in which case the action of $\alpha\mathrm{SL}_2(\mathbb{R})$ on $\mathbf{H}$ is simply (left) matrix multiplication $\alpha z = \alpha\beta\mathrm{SO}_2(\mathbb{R})$ rather than a linear fractional transformation.

1.2 Classification of linear fractional transformations

For $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ we define $\mathrm{disc}(\alpha) := \mathrm{tr}(\alpha)^2 - 4\det(\alpha)$ to be the discriminant of its characteristic polynomial $t^2 - \mathrm{tr}(\alpha)t + \det(\alpha)$, whose sign determines the type of eigenvalues α has (distinct complex conjugates, a single real (double) eigenvalue, or two real eigenvalues).

Definition 1.4. We call a nonscalar $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$

elliptic if $\mathrm{disc}(\alpha) < 0$, parabolic if $\mathrm{disc}(\alpha) = 0$, hyperbolic if $\mathrm{disc}(\alpha) > 0$.

We denote the extended upper half-plane by $\mathbf{H}^* := \mathbf{H} \cup \mathbb{R} \cup \{\infty\} \subseteq \mathbb{P}^1(\mathbb{C})$.

Theorem 1.5. *For each nonscalar* $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ *we have*

- α *is elliptic if and only if its fixed points are* z *and* $\bar{z}$ *for some* $z \in \mathbf{H}$.
- α *is parabolic if and only if it has a unique fixed point* $x \in \mathbf{H}^* - \mathbf{H} = \mathbb{R} \cup \{\infty\}$.
- α *is hyperbolic if and only if it has distinct fixed points* $x_1, x_2 \in \mathbf{H}^* - \mathbf{H} = \mathbb{R} \cup \{\infty\}$.

In every case, the fixed points of α *on* $\mathbb{P}^1(\mathbb{C})$ *lie in the extended upper half-plane* $\mathbf{H}^*$*, and they lie in the upper half-plane if and only if* α *is elliptic.*

Proof. This follows immediately from an analysis of the eigenvalues in each case. $\square$

For $x, x' \in \mathbb{R} \cup \{\infty\}$ we define

$$\mathrm{GL}_2(R)_x^{(p)} := \{\alpha \in \mathrm{GL}_2^+(\mathbb{R})_x : \alpha \text{ is parabolic or scalar}\}, \quad \mathrm{GL}_2^+(\mathbb{R})_{x,x'} := \mathrm{GL}_2^+(\mathbb{R})_x \cap \mathrm{GL}_2^+(\mathbb{R})_{x'}$$

Lemma 1.6. *We have*

- $\mathrm{GL}_2^+(\mathbb{R})_i = \mathrm{SO}_2(\mathbb{R})$;
- $\mathrm{GL}_2^+(\mathbb{R})_\infty = \left\{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a, d \in \mathbb{R}^\times, b \in \mathbb{R}, ad > 0\right\}$;
- $\mathrm{GL}_2^+(\mathbb{R})_\infty^{(p)} = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) : a \in \mathbb{R}^\times, b \in \mathbb{R}\right\}$;
- $\mathrm{GL}_2^+(\mathbb{R})_{0,\infty} = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) : a, d \in \mathbb{R}^\times, ad > 0\right\}$.

Moreover, for $z \in \mathbf{H}$ *the stabilizer* $\mathrm{GL}_2^+(\mathbb{R})_z$ *is conjugate to* $\mathrm{GL}_2^+(\mathbb{R})_i$*, for* $x \in \mathbf{H}^* - \mathbf{H}$ *the stabilizer* $\mathrm{GL}_2^+(\mathbb{R})_x$ *is conjugate to* $\mathrm{GL}_2^+(\mathbb{R})_\infty$ *with* $\mathrm{GL}_2^+(\mathbb{R})_x^{(p)}$ *conjugate to* $\mathrm{GL}_2^+(\mathbb{R})_\infty^{(p)}$*, and for distinct* $x, x' \in \mathbf{H}^* - \mathbf{H}$ *the double stabilizer* $\mathrm{GL}_2(\mathbb{R})_{x,x'}^+$ *is conjugate to* $\mathrm{GL}_2^+(\mathbb{R})_{0,\infty}$*. In all cases, the conjugating element can be chosen to lie in* $\mathrm{SL}_2(\mathbb{R})$.

Proof. See [1, Lemma 1.3.2]. $\square$

1.3 The invariant metric and measure on the upper half plane

Let $z = x + iy \in \mathbf{H}$. The invariant metric ds and invariant measure on $\mathbf{H}$ are defined by

$$ds(z) = \frac{\sqrt{dx^2 + dy^2}}{y^2} \qquad \text{and} \qquad dv(z) = \frac{dxdy}{y^2}$$

We call the image $C_{z_0 \to z_1}$ of an injective function $\phi\colon [0,1] \to \mathbf{H}$ that is C^∞ at all but finitely many points a path from $z_0 = \phi(0)$ to $z_1 = \phi(1)$. If we put $\phi(t) = x(t) + iy(t)$, the length of $C_{z_0 \to z_1}$ is

$$\ell(C_{z_0 \to z_1}) := \int_0^1 ds(\phi(t)) = \int_0^1 \frac{\sqrt{(dx(t)/dt)^2 + (dy(t)/dt)^2}}{y(t)} dt,$$

which does not depend on the choice of ϕ,as long as it has image $C_{z_0 \to z_1}$.

A circle or line in $\mathbf{H}$ orthogonal to $\mathbb{R}$ is called a geodesic.

Lemma 1.7. *For any two distinct $z_0, z_1 \in \mathbf{H}$, there is a unique shortest path $C_{z_0 \to z_1}$ from z_0 to z_1, which lies on a geodesic. For any $z_0 \in \mathbf{H}$ and $r > 0$ the set $\{z \in \mathbf{H} \colon \ell(C_{z_0, z}) = r\}$ is a circle in $\mathbf{H}$ orthogonal to every geodesic that contains z_0.*

Proof. See Lemma 1.4.1 and Corollary 1.4.2 in [1] □

The isomorphism $\mathbf{H} \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ allows us to uniquely represent $\alpha \in \mathrm{SL}_2(\mathbb{R})$ in the form $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = h_z k_\theta$, where $z = x + iy = \alpha i \in \mathbf{H}$ and $\theta = -\arg(ci + d)$, with

$$h_z = y^{-1/2} \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}), \qquad k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in \mathrm{SO}_2(\mathbb{R}).$$

We now define a measure on $\mathrm{SL}_2(\mathbb{R})$ by

$$d\alpha = \frac{dxdyd\theta}{2\pi y^2}.$$

This measure is $\mathrm{SL}_2(\mathbb{R})$-invariant: one can check that $d\alpha = d\alpha\beta = d\beta\alpha$ for all $\alpha, \beta \in \mathrm{SL}_2(\mathbb{R})$; moreover, we have chosen the scaling so that $d\alpha$ is compatible with dv.

Theorem 1.8. $\mathrm{SL}_2(\mathbb{R})$ *is unimodular, and for any measurable $f : \mathbf{H} \to \mathbb{C}$ we have*

$$\int_{\mathbf{H}} f(z) dv(z) = \int_{\mathrm{SL}_2(\mathbb{R})} f(\alpha i) d\alpha$$

Proof. This is [1, Thm. 1.4.5]. □

References

[1] Toshitsune Miyake, *Modular forms*, Springer, 2006.