18.786 Number Theory II **Spring 2018**

Problem Set #3 **Due: 4/4/2018**

Description

These problems are related to the material covered in Lectures 6-13. Your solutions are to be written up in latex and submitted as a pdf-file with a filename of the form `SurnamePset3.pdf` via e-mail to `drew@math.mit.edu` by **noon** on the date due. Collaboration is permitted/encouraged, but you must identify your collaborators, and any references consulted. If there are none, write "**Sources consulted: none**" at the top of your problem set. The first person to spot each non-trivial typo/error in any of the problem sets or lecture notes will receive 1-5 points of extra credit.[1]

Instructions: In many of the problems below we reference terms and definitions from earlier problems, so be sure to read through the problems in order. You may use results proved in an earlier problem in any later problem (even if you do not choose to solve the earlier problem). Pick any combination of problems to solve that sum to 200 and write up your answers in latex (be sure to include Problem 7, which is a survey).

Note on notation: $\mathbf{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, $\mathbf{D} := \{z \in \mathbb{C} : |z| < 1\}$, $Z(\Gamma) := \Gamma \cap \{\pm 1\}$.

Problem 1. Classification of Fuchsian groups (66 points)

As in Problem 3 of Problem Set 2, we call $z \in \mathbb{CP}^1 := \mathbb{C} \cup \{\infty\}$ a *limit point* of a group $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ if there is a $w \in \mathbb{CP}^1$ and an infinite sequence of distinct $\gamma_n \in \Gamma$ such that $\lim \gamma_n w = z$, and we use $\Lambda(\Gamma)$ to denote the set of of all limit points of Γ.

Recall that *Fuchsian groups* are subgroups Γ of $\mathrm{SL}_2(\mathbb{R})$ that satisfy any of the equivalent properties listed in part (a) of Problem 3 on Problem Set 2; in particular, such a Γ is discrete and its limit set $\Lambda(\Gamma)$ is a closed subset of $\mathbb{RP}^1 := \mathbb{R} \cup \{\infty\}$. *Elliptic points* are elements of $\mathbf{H}$ fixed by an elliptic $\gamma \in \Gamma$, *cusps* are elements of $\mathbb{RP}^1$ fixed by a parabolic $\gamma \in \Gamma$, and *hyperbolic points* are elements of $\mathbb{RP}^1$ fixed by a hyperbolic $\gamma \in \Gamma$.

Recall that in a topological space X a point $x \in X$ is a *limit point* of a set $S \subseteq X$ if every punctured neighborhood of x contains a point in S. A set $S \subseteq X$ is *perfect* if it is equal to its set of limit points and *nowhere dense* if its interior is empty, equivalently, every punctured neighborhood of $x \in S$ contains a point not in S.

(a) Show that every $x \in \mathbb{RP}^1$ fixed by an element of $\Gamma - Z(\Gamma)$ lies in $\Lambda(\Gamma)$; in other words, $\Lambda(\Gamma)$ contains all cusps and hyperbolic points of Γ.

(b) Show that Γ is countable and $\mathbf{H}$ contains uncountably many non-elliptic points.

(c) Let Γ be a Fuchsian group. Show that for any non-elliptic point $w \in \mathbf{H}$ the set $\Lambda(\Gamma)$ is equal to the set of limit points of the Γ-orbit of w, and in particular is Γ-invariant.

(d) Show that if $x \in \Lambda(\Gamma)$ is fixed by a hyperbolic $\gamma \in \Gamma$ and $y \in \Lambda(\Gamma)$ is not fixed by γ then either $\lim_{n\to\infty} \gamma^n y = x$ or $\lim_{n\to\infty} \gamma^{-n} y = x$, so x is a limit point of $\Lambda(\Gamma)$.

(e) Show that if $x \in \Lambda(\Gamma)$ is not a hyperbolic point and $\gamma \in \Gamma$ is hyperbolic then x is a limit of hyperbolic points (hint: choose w on the geodesic fixed by γ and use **(c)**).

[1] This is a long and completely new problem set; there are surely many points to be gained.

(f) Show that if $\#\Lambda(\Gamma) > 2$ then Γ contains a hyperbolic element γ and $\Lambda(\Gamma)$ is perfect, and therefore uncountable.

(g) Show that if $\Lambda(\Gamma) \neq \mathbb{RP}^1$ is perfect then $\Lambda(\Gamma)$ is nowhere dense. Use this to show that $\Lambda(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{RP}^1$.

It follows from what you have proved above that there are three kinds of Fuchsian groups:

- *elementary*: $\#\Lambda(\Gamma) \leq 2$.
- *the first kind*: $\Lambda(\Gamma) = \mathbb{RP}^1$.
- *the second kind*: $\Lambda(\Gamma)$ is perfect and nowhere dense.

Problem 2. Finitely generated Fuchsian groups (66 points)

Let Γ be a Fuchsian group, $w \in \mathbf{H}$ a non-elliptic point, and F_w the corresponding Dirichlet domain. Recall that the boundary of F_w is a union of geodesic segments of the form $L_\gamma := F_w \cap \gamma F_w$ called *sides*; we require geodesic segments to have nonzero (possibly infinite) length (so if L_γ is a point it is not a side).

If F_w has only finitely many sides then Γ is called *geometrically finite* (this does not depend on the choice of w). Problem 4 of Problem Set 2 proves that if Γ is geometrically finite then it is finitely generated. The goal of this problem is to prove the converse.

As in Problem Set 2, we split sides of the form L_γ with $\gamma^2 = \pm 1$ into two sides that intersect at the midpoint of L_γ and group sides into pairs that are Γ-translates (and not Γ-equivalent to any other sides). Let $G(\Gamma)$ be the subset of Γ consisting of elements γ that appear in a pair $(L, \gamma L)$. As proved in Problem Set 2, the set $G(\Gamma)$ generates Γ.

Now suppose that Γ is finitely generated. Out goal is to show F_w has finitely many sides.

(a) Show that if Γ is finitely generated then any generating set for Γ contains a finite subset that generates Γ. Let $\gamma_1, \ldots \gamma_n$ be a minimal subset of $G(\Gamma)$ generating Γ and let $L_1, \ldots, L_n$ be corresponding sides so that $(L_i, \gamma_i L_i)$ is a side pair for $1 \leq i \leq n$.

(b) Show that there exists a hyperbolic ball B_w about w whose interior contains arcs of positive length on all of the sides $L_1, \ldots, L_n, \gamma_1 L_1, \ldots \gamma_n L_n$, whose boundary contains no vertices of F_w and is not tangent to any side of F_w. Then show that only finitely many sides of F_w intersect B_w and that ΓB_w is connected.

(c) Let $s_1, \ldots, s_m$ be the arcs that make up $\partial B_w \cap F_w$. Pick an arc s_i and let e be one of its endpoints. Show that for some $\gamma \in G(\Gamma)$ the point γe is an endpoint of some s_j, so that $s_i \cup \gamma^{-1} s_j$ is a connected path. Show that by continuing in this fashion in both directions one can construct a connected path C_i composed of Γ-translates of the s_j that separates $\mathbf{H}$ into two disjoint sets, and that $\omega_i C_i = C_i$ for some $\omega_i \in \Gamma - Z(\Gamma)$. We thus obtain C_i and ω_i for each $1 \leq i \leq m$.

(d) Let $E_1, \ldots, E_m$ be the components of $F_w - B_w$ adjacent to $s_1, \ldots, s_m$. Show that every side of F_w disjoint from B_w intersects one of the E_i, so that it suffices to show that each E_i intersects only finitely many sides of F_w. Then show that if no element of $\Lambda(\Gamma)$ is a limit point of E_i then E_i can intersect only finitely many sides of F_w.

(e) Show that the path C_i constructed in part **(c)** separates the interior of E_i from Γw, so that any limit point of E_i that lies in $\Lambda(\Gamma)$ must be an endpoint of C_i, and show that C_i has 0, 1, or 2 endpoints, depending on whether ω_i is elliptic, parabolic, or hyperbolic, respectively.

(f) Show that if ω_i is hyperbolic then E_i has no limit points in $\Lambda(\Gamma)$.

(g) Show that if ω_i is parabolic and Γ is not elementary then E_i lies inside C_i, but even if E_i does have the end point of C_i as a limit point, E_i still intersects F_w in only finitely many sides.

(h) Finally, show that if Γ is elementary then F_w has finitely many sides.

Problem 3. Fuchsian groups of the first kind (66 points)

For a Fuchsian group Γ we use X_Γ to denote the Riemann surface $\Gamma\backslash\mathbf{H}^*_\Gamma$, where $\mathbf{H}^*_\Gamma$ is the topological space obtained by adjoining the cusps of Γ to $\mathbf{H}$ (and giving them open neighborhoods), and X_Γ is equipped with the complex structure defined in [3, §1.8].

Recall that under the measure induced by the hyperbolic metric every fundamental domain for Γ has the same area, and we say that Γ is *cofinite* if this volume is finite. Siegel's theorem [3, §1.9.1] implies that Γ is cofinite if and only if X_Γ is compact.

In Problem 1 (and many texts), a Fuchsian group Γ of the *first kind* is one that satisfies $\Lambda(\Gamma) = \mathbb{RP}^1$. Miyake [3, p. 28] uses this term to indicate that X_Γ is compact. The goal of this problem is to understand how these two notions are related.

(a) Show that if Γ is cofinite then it is geometrically finite, hence finitely generated.

(b) Show that if Γ is cofinite then $\Lambda(\Gamma) = \mathbb{RP}^1$.

(c) Show that elementary Fuchsian groups are geometrically finite but not cofinite.

(d) Show that for geometrically finite Fuchsian groups, if $\Lambda(\Gamma) = \mathbb{RP}^1$ then Γ is cofinite.

(e) Assume Γ is not elementary, and let $N(\Gamma)$ be the (hyperbolic) convex hull of $\Lambda(\Gamma)$ in $\mathbf{H}$, and let $D(\Gamma)$ be a Dirichlet domain. Show that $N(\Gamma) \cap D(\Gamma)$ has finite area if and only if Γ is geometrically finite, and $N(\Gamma) = \mathbf{H}$ if and only if $\Lambda(\Gamma) = \mathbb{RP}^1$.

(f) Prove that for finitely generated Fuchsian groups Γ, the Riemann surface X_Γ is compact if and only if $\Lambda(\Gamma) = \mathbb{RP}^1$.

Remark. In Poincaré's original exposition, Fuchsian groups were assumed to be finitely generated. Some authors still make this assumption (possibly implicitly), in which case the two definitions of *first kind* that we have been considering are equivalent. However, there are infinitely generated Fuchsian groups that are not cofinite for which $\Lambda(\Gamma) = \mathbb{RP}^1$; the next problem provides an explicit construction of such groups.

Problem 4. Schottky groups (66 points)

Recall that the upper half plane $\mathbf{H}$ is isomorphic (both as a Riemann surface and as a hyperbolic geodesic space) to the unit disk $\mathbf{D}$ via the map $\rho\colon \mathbf{H} \to \mathbf{D}$ defined by $z \mapsto (z - i)/(z + i)$. In this problem it will be more convenient to work with the unit disk, where geodesics are either diameters or semicircles orthogonal to the boundary and *horocycles* are circles tangent to the boundary. Horocycles bound neighborhoods of points on the boundary of $\mathbf{D}$ (in the topology of the extended disk $\overline{\mathbf{D}} := \{z \in \mathbb{C} : |z| = 1\}$, corresponding to the topology of the extended upper half plane $\mathbf{H} \cup \mathbb{R} \cup \{\infty\}$) and are orthogonal to all geodesics ending at the tangent point; horocycles tangent to $1 \in \overline{\mathbf{D}}$ correspond to horizontal lines in $\mathbf{H}$ (which bound neighborhoods of ∞).

Recall that the automorphism group of $\mathbf{H}$ and $\mathbf{D}$ (as Riemann surfaces) is the group of orientation preserving isometries $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. In this problem we shall view elements $\gamma \in \mathrm{SL}_2(\mathbb{R})$ as representatives of their class $\pm\gamma$ in $\mathrm{PSL}_2(\mathbb{R})$. The action of $\gamma \in \mathrm{SL}_2(R)$ on $w \in \mathbf{D}$ is defined by

$$\gamma w := \rho(\gamma \rho^{-1}(w))$$

where γ acts on $\rho^{-1}(w) \in \mathbf{H}$ in the usual way via linear fractional transformation.

For each $\gamma \in \mathrm{SL}_2(\mathbb{R})$ we define

$$D(\gamma) := \{w \in \mathbf{D} : d(w, \gamma 0) \leq d(w, 0)\},$$

where d denotes hyperbolic distance. For $\gamma \in \mathrm{SO}_2(\mathbb{R})$ we have $D(\gamma) = \mathbf{D}$, since in this case γ fixes 0, but otherwise $D(\gamma)$ is a closed half plane that does not contains 0 and its boundary is a geodesic semicircle. We use $D^0(\gamma)$ to denote the interior of $D(\gamma)$.

(a) Show that $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ is hyperbolic (resp. parabolic) if and only if $D(\gamma)$ and $D(\gamma^{-1})$ are disjoint (resp. tangent), and that $\gamma D(\gamma^{-1}) = \mathbf{D} - D^0(\gamma)$ for all non-trivial non-elliptic γ.

Definition. A *generalized Schottky group* $S(\gamma_1, \ldots, \gamma_r)$ (with respect to 0) is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by non-elliptic non-trivial $\gamma_1, \ldots, \gamma_n$ such that for all $i \neq j$ we have

$$(D(\gamma_i) \cup D(\gamma_i^{-1})) \cap (D(\gamma_j) \cup D(\gamma_j^{-1})) = \emptyset. \tag{1}$$

If the sets $D(\gamma_i) \cup D(\gamma_i^{-1})$ are also pairwise non-tangent, then $S(\gamma_1, \ldots, \gamma_r)$ is a *Schottky group* (note that $D(\gamma_i)$ and $D(\gamma_i^{-1})$ are still allowed to be tangent to each other). The *alphabet* of $S(\gamma_1, \ldots, \gamma_r)$ is the set of $2r$ *letters* $\{\gamma_i^{\pm 1} : 1 \leq i \leq n\}$. A *reduced word* of $S(\gamma_1, \ldots, \gamma_r)$ is a finite sequence of letters $a_1, \ldots, a_n$ with $a_i \neq a_{i+1}^{-1}$ for $1 \leq i < n$. For each reduced word $a_1, \ldots, a_n$ we define

$$D(a_1, \ldots, a_n) := a_1 \ldots a_{n-1} D(a_n),$$

which we note is a closed geodesic half plane in $\mathbf{D}$.

(b) Show that for all distinct reduced words $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ of a generalized Schottky group $S(\gamma_1, \gamma_2)$ we have:

(i) $a_1 \cdots a_n(\mathbf{D} - D^0(a_n^{-1})) \subsetneq D(a_1)$;
(ii) $D(a_1, \ldots, a_n) \subsetneq D(a_1, \ldots, a_{n-1})$ for $n \geq 2$;
(iii) $D^0(a_1, \ldots, a_n) \cap D^0(a'_1, \ldots, a'_n) = \emptyset$.

(c) Show that a generalized Schottky group $S(\gamma_1, \gamma_2)$ is a geometrically finite Fuchsian group with no elliptic elements with Dirichlet domain

$$(\mathbf{D} - D^0(g_1)) \cap (\mathbf{D} - D^0(g_1^{-1})) \cap (\mathbf{D} - D^0(g_2)) \cap (\mathbf{D} - D^0(g_2^{-1})),$$

and that $S(\gamma_1, \gamma_2)$ is isomorphic to the free group on two letters.

(d) Show that a Schottky group $S(\gamma_1, \gamma_2)$ is a geometrically finite Fuchsian group of the second kind (this need not hold for generalized Schottky groups, see Problem 6).

(e) Let $\gamma_1, \gamma_2, \ldots \in \mathrm{PSL}_2(\mathbb{R})$ be non-trivial non-elliptic elements satisfying (1) for $i \neq j$. Show that $\Gamma = \langle \gamma_1, \gamma_2, \ldots \rangle$ is a Fuchsian group that is not geometrically finite and therefore (by Problem 2) not finitely generated, and is in fact free on $\gamma_1, \gamma_2, \ldots$.

The groups Γ arising in **(d)** provide a potential source of Fuchsian groups of the first kind (meaning $\Lambda(\Gamma) = \mathbb{RP}^1$ as in Problem 1) for which X_Γ is not compact. Our goal is to construct such a Γ, following the recipe given by Basmajian in [1]. The tricky part is ensuring that we actually get a Fuchsian group of the first kind (Γ as a limit of generalized Schottky groups, all of which are Fuchsian groups of the second kind).

A Fuchsian group Γ is called a *pair of pants* if the quotient space $\Gamma\backslash\mathbf{D}$ is topologically a sphere with three holes. Let us call a Schottky group $S(\gamma_1, \gamma_2)$ *crossed* if when we order the sets $D(\gamma_i^{\pm 1})$ along the boundary of $\mathbf{D}$, we find that $D(\gamma_1)$ and $D(\gamma_1^{-1})$ are not adjacent (in which case the same applies to $D(\gamma_2)$ and $D(\gamma_2^{-1})$).

(f) Show that a Schottky group $S(\gamma_1, \gamma_2)$ that is not crossed is a pair of pants, in which case it may have 0, 1, 2 holes that are cusps (fixed points of a parabolic element), which we call *cusp-holes*. Assuming that $S(\gamma_1, \gamma_2)$ is a pair of pants, show that

(i) $S(\gamma_1, \gamma_2)$ has no cusp-holes if and only if γ_1, γ_2 are hyperbolic (as is $\gamma_1\gamma_2$);

(ii) If $S(\gamma_1, \gamma_2)$ has 1 cusp-hole we can choose γ_1 hyperbolic and γ_2 parabolic, with $\gamma_1\gamma_2$ hyperbolic;

(iii) If $S(\gamma_1, \gamma_2)$ has 2 cusp-holes we can choose γ_1, γ_2 parabolic, with $\gamma_1\gamma_2$ hyperbolic.

For a Schottky group $S(\gamma_1, \gamma_2)$ that is a pair of pants, the quotient space $\mathbf{D}/S(\gamma_1, \gamma_2)$ has three boundaries (one for each hole), and we want to choose γ_1, γ_2 so that $\gamma_1, \gamma_2, \gamma_1\gamma_2$ can be matched with the holes via their fixed points. For hyperbolic elements γ, we want its *axis* $A(\gamma)$, the geodesic connecting its fixed points, to bound a hole, and for parabolic elements γ we want its cusp to be the hole. When γ_1, γ_2 are chosen as in **(e)** so that $\gamma_1, \gamma_2, \gamma_1\gamma_2$ correspond to the holes in this way, we call them *standard generators* for the pair of pants $S(\gamma_1, \gamma_2)$. Notice that in every case $\gamma_1\gamma_2$ is hyperbolic, and γ_1 is parabolic if and only if $S(\gamma_1, \gamma_2)$ has 2 cusp-holes. Henceforth we assume that all pairs of pants are specified as Schottky groups $S(\gamma_1, \gamma_2)$ using standard generators.

To construct what Basmajian calls a *tight flute*, we start with a pair of pants $S(\alpha_1, \beta_1)$ with two cusp-holes, to which we glue a second pair of pants $S(\alpha_2, \beta_2)$ with one cusp-hole along the images of $A(\alpha_1\beta_1)$ and $A(\alpha_2)$; under suitable constraints this will yield a Schottky group $S(\alpha_1, \alpha_2, \beta_2)$ with $\beta_1 = \alpha^{-1}\alpha_2$ that contains $S(\alpha_1, \beta_1) = S(\alpha_1, \alpha_2)$ and $S(\alpha_2, \beta_2)$. We then glue a third pair of pants $S(\alpha_3, \beta_3)$ along the images of $A(\alpha_2\beta_2)$ and $A(\alpha_3)$ to obtain $S(\alpha_1, \alpha_2, \alpha_3, \beta_3)$, and continue in this fashion: in step $n \geq 2$ we glue a pair of paints $S(\alpha_n, \beta_n)$ with one cusp along the images of $A(\alpha_{n-1}\beta_{n-1})$ and $A(\alpha_n)$ to obtain $S(\alpha_1, \ldots, \alpha_n, \beta_n)$. In the limit we obtain a tight flute $\Gamma = \langle \alpha_1, \alpha_2, \alpha_3 \ldots \rangle$, as depicted in Figure 1 of [1] that satisfies the hypothesis of **(d)** above.

In [1, §5] Basmajian gives an explicit criterion for the existence of standard generators for each of the pairs of pants $S(\alpha, \beta)$ required by the construction above, and criteria that determine when the tight flute Γ will be a Fuchsian group of the first kind. For a hyperbolic element γ we define the *translation length* $T(\gamma)$ as the hyperbolic distance

$d(z, \gamma z)$ for any $z \in \mathbf{D}$ on the axis $A(\gamma)$. We then define the *collar width* $c(\gamma)$ of a non-trivial non-elliptic $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ as

$$c(\gamma) = \begin{cases} \log\coth T(\gamma)/4 & \text{if } \gamma \text{ is hyperbolic,} \\ \log 2 & \text{otherwise,} \end{cases}$$

where $\coth(x) := \frac{e^x + e^{-x}}{e^x - e^{-x}}$ is the hyperbolic cotangent. Let us define the *axis* $A(\gamma)$ of a parabolic element γ with cusp x as the unique *horocycle* about x bounding a region with hyperbolic measure 1 in $\mathbf{H}/\langle\gamma\rangle$.

In the construction of the tight flute Γ described above, we want the axes $A(\alpha_n)$ to be *nested* so that they approach a point on the boundary in the limit; see the bottom diagram in Figure 2 of [1], in which $L_1 = A(\alpha_1)$ corresponds to a horocycle depicted as a horizontal line in the upper half plane (in the unit disk it would be a circle tangent to the boundary lying entirely outside of L_1) and the $L_n = A(\alpha_n)$ for $n > 1$ are nested geodesic semicircles. We call such a sequence of $L_n = A(\alpha_n)$ a *nested sequence of geodesics* (even though $L_1 = A(\alpha_1)$ is a horocycle, not a geodesic).

The axes $A(\alpha_n)$ determine a sequence of distances $d_n := d(L_n, L_{n+1})$, where distance is measured along an orthogonal geodesic. The following theorem shows that the sequence of distances completely determines all the collar widths.

Theorem (Basmajian [1])**.** *Let $\gamma_1, \gamma_2 \in \mathrm{PSL}_2(\mathbb{R})$ be non-trivial non-elliptic elements with γ_1 hyperbolic and $d := d(A(\gamma_1), A(\gamma_2))$ the distance between their axes (measured along an orthogonal geodesic). Then γ_1, γ_2 are standard generators for a pair of pants if and only if $c(\gamma_1) + c(\gamma_2) \le d$, with equality if and only if the pair of pants has a cusp-hole.*

Finally we note the following the following consequence of Theorems 1-4 (and the proof of Theorem 1) in [1].

Theorem (Basmajian)**.** *Let $d > \log 2$ be a real number and let $\alpha_1, \alpha_2, \ldots$ be elements of $\mathrm{PSL}_2(\mathbb{R})$ with α_1 parabolic, α_n hyperbolic for $n > 1$, such that the $A(\alpha_n)$ are a nested sequence of geodesics with $d(A(\alpha_n), A(\alpha_{n+1})) = d$ for all $n \ge 1$. Then $\Gamma := \langle \alpha_1, \ldots, \alpha_n \rangle$ is a Fuchsian group of the first kind that is not finitely generated. Moreover, for any such $d > \log 2$ there exist such α_n.*

(g) Construct the first three elements of a sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ satisfying the hypothesis of the theorem. You should specify the α_i as 2×2 matrices whose entries are real numbers specified to some reasonable precision, along with their fixed points, and demonstrate that they satisfy the required constraints on their collar widths and distances between axes.

Problem 5. Automorphic forms in an adelic setting (66 points)

Let $\mathbb{A}_{\mathbb{Q}}$ denote the adele ring of $\mathbb{Q}$ (Definition 25.7), which we recall contains $\mathbb{Q}$ (embedded diagonally as the subgroup of principal adeles), let $\mathbb{A}_{\mathbb{Q}}^{\infty} := \{a \in \mathbb{A}_{\mathbb{Q}} : \|a\|_\infty = 0\}$ denote the subgroup of *finite adeles*. We embed each completion of $\mathbb{Q}$ in $\mathbb{A}_{\mathbb{Q}}$ in the obvious way $(x \mapsto (0, \ldots, 0, x, 0, \ldots 0))$, and similarly for $\mathbb{Z}_p \subseteq \mathbb{Q}_p$.

(a) Show that we have isomorphisms of topological groups

$$\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) \simeq \mathrm{GL}_2(\mathbb{A}^{\infty}) \times \mathrm{GL}_2(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\mathrm{GL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}),$$

and thus may view any subgroup of $\mathrm{GL}_2(\mathbb{R})$ or $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ as a subgroup of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$.

Let $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \simeq \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ denote the profinite completion of $\mathbb{Z}$, which we view as a subring of $\mathbb{A}_{\mathbb{Q}}^{\infty}$ (via the embeddings of $\mathbb{Z}_p \subseteq \mathbb{Q}_p$). For any positive integer N we define

$$K_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : c \equiv 0 \bmod N\} \subseteq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}),$$
$$\Gamma_0(N) := K_0(N) \cap \mathrm{GL}_2^+(\mathbb{Z}) \subseteq \mathrm{SL}_2(\mathbb{Z})$$

(b) Show that

$$\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2^+(\mathbb{R})K_0(N),$$

thus we can write any $a \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ as $g = \gamma\sigma\kappa$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $\sigma \in \mathrm{GL}_2^+(\mathbb{R})$, and $\kappa \in K_0(N)$, not necessarily uniquely. (hint: use strong approximation).

(c) Show that there is a natural isomorphism of topological coset/double-coset spaces

$$\Gamma_0(N)\backslash\mathrm{GL}_2^+(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K_0(N).$$

(d) Let $K_\infty^+ = \mathrm{SO}_2(\mathbb{R})$ and $Z_\infty^+ := Z(\mathrm{GL}_2^+(\mathbb{R}))$ (scalar matrices). Show that we have a natural isomorphism of topological spaces

$$\Gamma_0(N)\backslash\mathbf{H} \simeq \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K_0(N)K_\infty^+Z_\infty^+$$

and that $\Gamma_0(N)$ is a finitely generated Fuchsian group of the first kind, implying that $X_0(N) := X_{\Gamma_0(N)}$ is a compact Riemann surface (see Problem 3).

Let $k \in \mathbb{Z}_{>0}$ be even. For any cusp form $f \in S_k(\Gamma_0(N))$, we define the function $\varphi_f \colon \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$ by writing $g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ in the form $g = \gamma\sigma\kappa$ as in **(b)** and putting

$$\varphi_f(g) := (f_{|_k}\sigma)(i),$$

where $(f_{|_k}\sigma)(z) := \det(\sigma)^{k/2} j(\sigma,z)^{-k} f(\sigma z)$ is the slash operator and $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) := cz+d$.

(e) Show that φ_f is well defined, in other words, for all $\gamma, \gamma' \in \mathrm{GL}_2(\mathbb{Q})$, $\sigma, \sigma' \in \mathrm{GL}_2^+(\mathbb{R})$, $\kappa, \kappa' \in K_0(N)$ such that $\gamma\sigma\kappa = \gamma'\sigma'\kappa'$, we have $(f_{|_k}\sigma)(i) = (f_{|_k}\sigma')(i)$.

Define $\xi \colon K_\infty^+ \to \mathbb{C}$ by $\xi(r_\theta) := e^{ik\theta}$ for $r_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in K_\infty^+ = \mathrm{SO}_2(\mathbb{R})$.

(f) Let $K := K_0(N)K_\infty^+$. Show that $\varphi := \varphi_f$ satisfies the following:

(i) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ and $g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ (automorphy).

(ii) $\varphi(g r_\theta \kappa) = \xi(r_\theta)\varphi(g)$ for all $g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, $r_\theta \in K_\infty^+$, $\kappa \in K_0(N)$, so the $\mathbb{C}$-vector space spanned by $\{\varphi(gh) : h \in K\}$ has finite dimension (K-finite).

(iii) $\varphi(\lambda g) = \varphi(g)$ for all scalar $\lambda \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ and $g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ (central action).

(iv) For every $g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ the map $\mathbf{H} \to \mathbb{C}$ given by

$$z \mapsto \varphi(g\sigma) j(\sigma,i)^k \det(\sigma)^{-k/2},$$

where $\sigma \in \mathrm{GL}_2^+(\mathbb{R})$ satisfies $\sigma i = z$, is well-defined and holomorphic (Z-finite).

(v) There exist $c, r \in \mathbb{R}_{>0}$ such that for all $g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ the function ϕ_g on $\mathrm{GL}_2^+(\mathbb{R})$ defined by $\sigma \mapsto \varphi(g\sigma)$ satisfies the bound

$$|\phi_g(\sigma)| \le c\|\sigma\|^r,$$

with $\|\sigma\| := \mathrm{tr}(\sigma^{\mathrm{T}}\sigma)^{1/2}$ (moderate growth), and in fact ϕ_g is bounded on $\mathrm{GL}_2^+(\mathbb{R})$ (cuspidal).

(g) Show that any smooth function $\varphi\colon \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ satisfying conditions (i)-(v) above arises as φ_f for some $f \in S_k(\Gamma_0(N))$. Such functions are *cuspidal automorphic forms on* $\mathbb{A}_\mathbb{Q}$ *for* $K_0(N)$.

Remark. Our definition of K-finiteness depends not only on K, but also on the choice of the function ξ (a *fundamental idempotent*). Our definition of Z-finiteness is specific to the GL_2 setting, where holomorphicity is equivalent to being an eigenfunction for the Casimir element Δ, which together with 1 generates the center $Z(\mathfrak{g})$ of the universal enveloping algebra for the complexified Lie algebra $\mathfrak{g} := \mathfrak{gl}_2 \otimes_\mathbb{R} \mathbb{C}$; in general one specifies a finite index ideal of $Z(\mathfrak{g})$ that annihilates φ. An automorphic form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is *cuspidal* if for all $g \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ we have

$$\int_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} \varphi(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g)\, dx = 0.$$

See [2, Cor. 12.4] for a proof that this is equivalent to the (cuspidal) condition in (v).

Remark. The $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ automorphic forms for $K_0(N)$ described in this problem admit many generalizations, including: (1) replace $\mathbb{Q}$ with a number field F and $\widehat{\mathbb{Z}}$ with $\widehat{\mathcal{O}}_F$, (2) replace GL_2 with a connected linear algebraic group G and $\Gamma_0(N)$ with an *arithmetic* subgroup of $G(F)$ (and adjust $K_0(N)$ and $\mathrm{GL}_2^+(\mathbb{R})$ accordingly), (3) allow a character χ (and adjust the fundamental idempotent and the central action condition accordingly).

Problem 6. The modular group (66 points)

In this problem we consider the modular group $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ and some of its subgroups. We use $\Gamma(N)$ to denote the subgroup of $\Gamma(1)$ congruent to the identity modulo a positive integer N. For any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ we use $\overline{\Gamma}$ to denote $\Gamma/Z(\Gamma) \subseteq \mathrm{PSL}_2(\mathbb{Z})$.

(a) Show that $\Gamma(1)$ is a Fuchsian group of the first kind in two ways: (i) show directly that $\Lambda(\Gamma(1)) = \mathbb{RP}^1$ using results of Problem 1, (ii) show that $X(1) := X_{\Gamma(1)}$ has hyperbolic volume $\pi/3$ and is therefore compact.

(b) Show that every elliptic element of $\Gamma(1)$ is conjugate to a power of $R := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ or $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and every parabolic element is conjugate to a power of $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then show that $\overline{\Gamma}(1)$ is isomorphic to the free group on $\pm R$ and $\pm S$ (and thus has the presentation $\langle r, s \,|\, r^3 = s^2 = 1\rangle$), and show that its commutator subgroup $\overline{\Gamma}(1)' := \langle \alpha\beta\alpha^{-1}\beta^{-1} : \alpha, \beta \in \overline{\Gamma}(1)\rangle$ has index 6 and is isomorphic to the free group on two letters (with no relations).

(c) Show that $\Gamma(2)$ is a generalized Schottky group $S(T^2, U^2)$, where $U := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and $\Gamma(2)/\mathbf{H}$ is a pair of pants with 3 holes corresponding to the cusps of T^2, U^2, T^2U^2 (c.f. part **(e)** of problem 4). Conclude that $\overline{\Gamma}(2)$ is isomorphic to the free group on two letters.

(d) Show that $\Gamma(N)$ is a normal subgroup of Γ. Conclude that $\overline{\Gamma}(2)$ and $\overline{\Gamma}(1)'$ are isomorphic normal index 6 subgroups of $\overline{\Gamma}(1)$. Are they equal? If not, describe their intersection and determine the least integer N (if any) for which $\overline{\Gamma}(1)' \subseteq \overline{\Gamma}(N)$.

For a subgroup Γ of $\Gamma(1)$ let $e_n(\Gamma)$ count the Γ-inequivalent elliptic points of order n, and let $e_\infty(\Gamma)$ count the Γ-inequivalent cusps.

(e) Prove that for any finite index subgroup Γ of $\Gamma(1)$ the following formula holds

$$g(\Gamma) := g(X_\Gamma) = 1 + \frac{[\overline{\Gamma}(1) : \overline{\Gamma}]}{12} - \frac{e_2(\Gamma)}{4} - \frac{e_3(\Gamma)}{3} - \frac{e_\infty(\Gamma)}{2}.$$

Your proof should rely only on material covered in the first two chapters of [3].

(f) Compute an explicit (and reasonably sharp) lower bound on $g(\Gamma(N))$ as a function of N and determine its asymptotic growth. Use this to explicitly determine the sets of integers $\{N : g(\Gamma(N)) = g\}$ for $g \le 5$ (you can compute use Sage to compute $g(\Gamma(N))$ for any particular N, but these sets will be quite small in any case and easy to determine by hand in any case; the key point is to prove your sets are complete).

(g) Let $\Gamma := \Gamma(1)'$ be the commutator subgroup of $\Gamma(1)$. Compute $g(\Gamma)$ using your formula, and use Theorems 2.5.2, 2.5.4 of [3] to compute $\dim S_k(\Gamma)$ for $1 \le k \le 6$.

Definition. A *congruence subgroup* Γ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ that contains $\Gamma(N)$ for some positive integer N. The least such N is the *level* of Γ.

(h) Prove that for all $N \ge 1$, the group $\Gamma(1)/\Gamma(N) \simeq \mathrm{SL}_2(\mathbb{Z}/\mathrm{N}\mathbb{Z})$ contains a subgroup of index 7 only when $7|N$, in which case it projects on to an index 7 subgroup of $\mathrm{SL}(2,\mathbb{Z}/7\mathbb{Z})$. Conclude that every congruence subgroup of index 7 has level 7.

(i) Consider the subgroup $\Gamma := \left\langle \left(\begin{smallmatrix} 1 & 5 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right) \right\rangle$ of $\mathrm{SL}(2,\mathbb{Z})$ found in [4], which evidently has a cusp of width 5. Prove that if Γ contains $\Gamma(N)$ then the width of every cusp of Γ divides N. Then show that Γ has index 7 in $\mathrm{SL}(2,\mathbb{Z})$ and is therefore not a congruence subgroup, and compute $g(\Gamma)$ using your formula.

Problem 7. Survey (2 points)

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			
Problem 5			
Problem 6			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material to you (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
3/12	Automorphic forms				
3/14	Meromorphic differentials				
3/19	Genus and Dimension formulas				
3/21	Poincaré and Eisenstein Series				

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

References

[1] A. Basmajian, *Hyperbolic structures for surfaces of finite type*, Trans. Amer. Math. Soc. **336** (1993), 421–444.

[2] A. Knightly and C. Li, *Traces of Hecke operators*, American Mathematical Society, 2006.

[3] T. Miyake, *Modular forms*, Springer, 2006.

[4] A.J. Scholl, *On the Hecke algebra of a noncongruence subgroup*, Bull. London Math. Soc. **29** (1997), 395–399.