18.786 Number Theory II **Spring 2018**
Problem Set #2 **Due: 3/14/2018**

Description

These problems are related to the material covered in Lectures 6-9. Your solutions are to be written up in latex and submitted as a pdf-file with a filename of the form `SurnamePset2.pdf` via e-mail to `drew@math.mit.edu` by **noon** on the date due. Collaboration is permitted/encouraged, but you must identify your collaborators, and any references consulted. If there are none, write "**Sources consulted: none**" at the top of your problem set. The first person to spot each non-trivial typo/error in any of the problem sets or lecture notes will receive 1-5 points of extra credit.

Instructions: In some of the problems below we reference terms and definitions from earlier problems, so be sure to read through the whole problem set first. You may use results proved in an earlier problem in any later problem (even if you don't choose to solve the earlier problem). Pick a combination of problems to solve that sum to 100 and write up your answers in latex, then complete the survey problem 5.

As in [1], we use $\mathbf{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ to denote the complex upper half plane.

Problem 1. Geodesic spaces (50 points)

Let (X, d) be a metric space. Recall that a *path* $\gamma\colon [a,b] \to X$ is a continuous function whose domain is a real interval, which we may also denote $\gamma\colon x \to y$, where $x = \gamma(a)$ and $y = \gamma(b)$. The *length* of a path is

$$l(\gamma) := \sup_{a=t_0<\cdots<t_n=b} \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)).$$

where the supremum is taken over all finite subdivisions of $[a, b]$. A metric space is a *length space* if for all $x, y \in X$ we have $d(x, y) = \inf_{\gamma\colon x\to y} l(\gamma)$.

(a) For a path $\gamma\colon [a,b] \to \mathbf{H}$ we define the *hyperbolic length*

$$\ell(\gamma) := \int_a^b \frac{|\gamma'(t)|}{\mathrm{Im}(\gamma(t))} dt.$$

Show that $d(x, y) := \inf_{\gamma\colon x\to y} \ell(\gamma)$ is a metric on $\mathbf{H}$ that makes $(\mathbf{H}, d)$ a length space with $l(\gamma) = \ell(\gamma)$. We call d the *hyperbolic metric* on $\mathbf{H}$.

Definition. A *geodesic path* $\gamma\colon [a,b] \to X$ is a distance preserving path; that is, we have $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [a, b]$. A *geodesic line* (resp. *geodesic ray*) is a distance preserving continuous map $\mathbb{R} \to X$ (resp. $\mathbb{R}_{\geq 0} \to X$).[1] By a geodesic path/line/ray in X we mean the image of a geodesic path/line/ray. A *geodesic space* is a metric space in which every pair of points can be connected by a geodesic path. A geodesic space is *straight* if every geodesic path lies in a geodesic line, and *uniquely geodesic* if every $x, y \in X$ are connected by a unique geodesic path. A subset of a geodesic space is *convex* if it contains all geodesic paths between points in the set.

[1] This definition differs from the shortest path definition used in class, and from the two definitions most commonly used in Riemannian geometry (it is slightly stronger).

(b) Show that $\mathbf{H}$ is a geodesic space that is a uniquely geodesic space under both the Euclidean metric $|z_1 - z_2|$ and the hyperbolic metric defined above, but that it is only straight for the hyperbolic metric, and automorphisms of $\mathbf{H}$ (as a Riemann surface) are isometries only for the hyperbolic metric.

(c) Show that the topologies on $\mathbf{H}$ induced by the Euclidean metric and the hyperbolic metric coincide.

(d) Show that Euclidean and hyperbolic open balls are convex under both the Euclidean and hyperbolic metric, and give examples of subsets of $\mathbf{H}$ that are convex under one metric but not the other (in both directions).

(e) Show that under the hyperbolic metric (i) for any distinct $z_1, z_2 \in \mathbf{H}$ the set

$$C_{z_1,z_2} := \{z \in \mathbf{H} : d(z, z_1) = d(z, z_2)\}$$

is a geodesic line $\gamma\colon \mathbb{R} \to \mathbf{H}$, (ii) $\mathbf{H} - C_{z_1,z_2}$ is the union of two disjoint convex sets that can meaningfully be called left and right half-planes (relative to γ), (iii) every geodesic line arises in this way. Does this apply to all Riemann surfaces that are straight uniquely geodesic spaces for a given metric?

Problem 2. Topological group actions (50 points)

Recall that locally compact groups are (by definition) Hausdorff and that whenever a topological group acts on a topological space the action is understood to be continuous, that is, the map $(g, x) \mapsto gx$ is a morphism of topological spaces $G \times X \to X$. A group acting on a metric space (X, d) is said to *act via isometries* if $x \mapsto gx$ is an isometry for every $g \in G$, in other words, $d(x, y) = d(gx, gy)$ for all $x, y \in X$ and $g \in G$.

A morphism of topological spaces is *proper* if it is a closed map and inverse images of compact sets are compact. A topological group G acts *properly* on a topological space X if the *action map* $(g, x) \mapsto (x, gx)$ defines a proper morphism $G \times X \to X \times X$.

(a) Let G be a locally compact group acting on a locally compact metric space (X, d) via isometries. Prove that the following are equivalent

(1) All $x, y \in X$ have open neighborhoods U, V with $\{g \in G : gU \cap V \neq \emptyset\}$ finite.

(2) Every $x \in X$ has an open neighborhood U with $\{g \in G : gU \cap U \neq \emptyset\}$ finite.

(3) For all compact $A, B \subseteq X$ the set $\{g \in G : gA \cap B \neq \emptyset\}$ is finite.

(4) For all compact $A \subseteq X$ the set $\{g \in G : gA \cap A \neq \emptyset\}$ is finite.

(5) G is discrete and acts properly on X.

(6) For every $x \in X$ the orbit Gx is discrete and the stabilizer G_x is finite.

(b) Suppose we drop the assumption in **(a)** that X is a metric space (so we cannot speak of isometries) and instead only assume X (and G) is locally compact Hausdorff. Which of the six properties listed in (a) are equivalent to each other?

(c) Let G be a Hausdorff group acting on a Hausdorff space X. Let $\pi\colon X \to G\backslash X$ be the quotient map. Show that the following are equivalent:

(1) $G\backslash X$ is Hausdorff.

(2) All $x, y \in X$ with $\pi(x) \neq \pi(y)$ have open neighborhoods U, V with $gU \cap V = \emptyset$ for all $g \in G$.

(3) The image of the action map $G \times X \to X \times X$ is closed.

(d) Let G be a Hausdorff group that acts properly on a Hausdorff space X. Show that the following hold:

(1) $G\backslash X$ is Hausdorff.

(2) Every orbit Gx is closed.

(3) Every stabilizer G_x is compact.

(4) For all $x \in X$ the map $G/G_x \to Gx$ defined by $g \mapsto gx$ is a homeomorphism.

(e) Let the discrete group $G = \mathbb{Z}$ act on $\mathbb{R}^2 - \{(0,0)\}$ via $n\,(x,y) := (2^n x, y/2^n)$, where $\mathbb{R}^2 - \{(0,0)\}$ is equipped with the Euclidean metric. Determine which of the 6 properties listed in **(a)** and the 4 properties listed in **(d)** are enjoyed by G and X. You should find that G and X satisfy some but not all of the properties of both **(a)** and **(d)**, implying that they do not satisfy the hypotheses of either **(a)** or **(d)**. Which hypotheses fail?

Problem 3. Fuchsian groups (50 points)

For any subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ we call $z \in \mathbb{CP}^1 := \mathbb{C} \cup \{\infty\}$ a *limit point* of Γ if there is a $w \in \mathbb{CP}^1$ and an infinite sequence of distinct $\gamma_n \in \Gamma$ such that $\lim \gamma_n w = z$, Let $\Lambda(\Gamma)$ to denote the set of of all limit points of Γ.

(a) Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{R})$ and let $\mathbf{H}$ be the upper half plane equipped with the hyperbolic metric defined in Problem 1. Show that the following are equivalent.

(1) $G = \Gamma$ and $X = \mathbf{H}$ satisfy property (1) listed in part **(a)** of problem 2.

(2) Γ is discrete.

(3) Any sequence in Γ converging to 1 is eventually constant.

(4) For every real $B > 0$ the subset of Γ with matrix entries in $[-B, B]$ is finite.

(5) $\Lambda(\Gamma)$ is a closed subset of $\mathbb{RP}^1 := \mathbb{R} \cup \{\infty\}$.

Such a group is called a *Fuchsian group*.

(b) Let $\Gamma = \langle \gamma \rangle$ be a cyclic subgroup of $\mathrm{SL}_2(\mathbb{R})$ with $\gamma \neq \pm 1$ and consider the three cases where γ is elliptic, parabolic, or hyperbolic. Compute $\Lambda(\Gamma)$ in each case and determine whether (or under what conditions) Γ is Fuchsian.

(c) Show that if a Fuchsian group Γ is abelian then $\Gamma/Z(\Gamma)$ is cyclic.

(d) Show that the $\mathrm{SL}_2(\mathbb{R})$-normalizer of a nonabelian Fuchsian group is Fuchsian.

(e) Let Γ be a Fuchsian group. Show that the following are equivalent.

(1) $\#\Lambda(\Gamma) \leq 2$

(2) There is a finite Γ-orbit in $\mathbf{H} \cup \mathbb{RP}^1$.

(3) $\Gamma/Z(\Gamma)$ is abelian or conjugate to $\langle D, S \rangle$, where D is diagonal and $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Such a group is called an *elementary Fuchsian group*.

Problem 4. Dirichlet domains (100 points)

Throughout this problem $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group. Recall that given any $z_0 \in \mathbf{H}$ that is not an elliptic point we can construct a Dirichlet domain

$$F_{z_0} := \{z \in \mathbf{H} : d(z, z_0) \le d(z, \gamma z_0) \text{ for all } \gamma \in \Gamma\},$$

where d is the hyperbolic distance metric on $\mathbf{H}$. We proved in class that F_{z_0} is a convex locally finite fundamental domain for Γ whose boundary ∂F is a union of geodesic segments $L_\gamma := F_{z_0} \cap \gamma F_{z_0}$ called *sides*, whose intersections are *vertices*; we include vertices in $\mathbb{R} \cup \{\infty\}$ that are the common endpoint of two geodesic rays. Recall that sides are required to have nonzero (possibly infinite) length. *Locally finite* means that every compact $A \subseteq \mathbf{H}$ intersects only finitely many Γ-translates of F_{z_0}.

(a) Let F be a Dirichlet domain for Γ and let $T := \{\gamma \in \Gamma : F \cap \gamma F \neq \emptyset\}$. Show that T generates Γ and includes all elliptic elements of Γ.

We now refine our notion of sides and vertices as follows. For sides L_γ with $\gamma^2 = \pm 1$ we have $\gamma L_\gamma = L_\gamma$ and γ thus fixes the midpoint of L_γ (and swaps its end points). In this case it will be convenient to view L_γ as two distinct sides that intersect at the midpoint of L_γ, which we now consider a vertex.

Each vertex of F lies in two sides of F, one of which we unambiguously distinguish as *the clockwise side of* v_1 (the one on the right if we look at v_1 from the interior of F).

Let S denote the set of sides of a Dirichlet domain F, and define

$$P := \{(\gamma, L, L') : L' = \gamma L\} \subseteq \Gamma \times (S \times S).$$

Let $G(P)$ be the projection of P to Γ modulo $Z(\Gamma)$ (keep only one of γ and $-\gamma$).

(b) Define a relation $\sim$ on S by letting $L \sim L'$ if and only if $(\gamma, L, L') \in P$ for some $\gamma \in \Gamma$. Show that this is an equivalence relation that partitions S into pairs of the form $\{(\gamma, L, L'), (\gamma^{-1}, L', L)\}$ with $L = F \cap \gamma^{-1}F$ and $L' = F \cap \gamma F$.

(c) Show that $G(P)$ generates $\Gamma/Z(\Gamma)$ and that if S is finite then $G(P)$ is finite.

(d) Let v_1 be a vertex of F. Show that there is an open neighborhood U of v_1 that contains no other vertices of F and intersects no sides of F that do not contain v_1, and a finite list of elements $1 = \delta_0, \delta_1, \ldots, \delta_n \in \Gamma$ such that the sets $\delta_i F$ are all distinct and $U \subseteq \cup_{i=0}^n \delta_i F$ with $v_1 \in \delta_i F$ for $0 \le i \le n$.

(e) Let v_1 be a vertex of F, let L_1 be the clockwise side of F, let $L_1' = \gamma_1 L_1$ be the unique side of F paired with L_1 via **(c)** (such a γ_1 exists, it need not be unique), and let $v_2 = \gamma_1 v_1$. Show that v_2 is a vertex of F, so we can similarly let L_2 be its clockwise side, let $L_2' = \gamma_2 L_2$ be the paired side, and define $v_3 = \gamma_2 v_2$. Continuing in this fashion, we get a sequence of vertices $v_1, v_2, \ldots$ and group elements $\gamma_1, \gamma_2, \ldots$ Show that after finitely many steps we return to $v_m = v_1$.

(f) With v_i and γ_i as in part **(e)**, let $\delta = (\gamma_m \cdots \gamma_1)^{-1}$. Show that the value of m in part **(e)** divides $n+1$, where n is as in part **(d)**, that $F \cap \Gamma v_1 = \{v_1, \ldots, v_{m-1}\}$ (with the v_i as in part **(e)**), and that δ fixes v_1 and has order $e := (n+1)/m$. We call $\delta^e = 1$ the *vertex cycle relation for* v_1. Show that if $v_1' = \gamma v_1$ is a vertex of F then the cycle relation for v_1' is $(\gamma^{-1}\delta\gamma)^e = \gamma^{-1}\delta^e\gamma = 1$. Let $R(P)$ be the set of vertex cycle relations for vertices in F.

(g) Show that $\Gamma/Z(\Gamma)$ is isomorphic to the free group on $G(P)$ with relations $R(P)$.

(h) For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, compute a Dirichlet domain using $z_0 = 2i$ and compute generators and relations for $\Gamma/\mathbb{Z}(\Gamma) = \mathrm{PSL}_2(\mathbb{Z})$ using the above method.

(i) Do the same for $\Gamma = \Gamma(2) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv 1_{\mathrm{SL}_2(\mathbb{Z})} \bmod 2\}$.

Problem 5. Survey

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material to you (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
2/26	Automorphisms of **H**				
2/28	Fuchsian groups				
3/5	Dirichlet domains				
3/7	Quotients by Fuchsian groups				

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

References

[1] T. Miyake, *Modular forms*, Springer, 2006.