31 Proof of local Artin reciprocity

Let K be a nonarchimedean field with separable closure $K^{\rm sep}$. In the previous lecture we defined the invariant map

$$\mathrm{inv}_K\colon H^2(\mathrm{Gal}(K^{\rm sep}/K), K^{\rm sep\times}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z},$$

which for every finite Galois extension L/K induces an isomorphism

$$\mathrm{inv}_{L/K}\colon H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow{\sim} \tfrac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$$

via composition with the inflation map $\mathrm{Inf}\colon H^2(\mathrm{Gal}(L/K), L^\times) \to H^2(\mathrm{Gal}(K^{\rm sep}/K), K^{\rm sep\times})$, and we showed that inv_K is canonical and natural in K; see Theorem 30.25

In this lecture we will use the invariant map to define the local Artin homomorphism and prove the local version of Artin reciprocity.

31.1 The fundamental class

The invariant map is that it gives us a canonical generator for $H^2(\mathrm{Gal}(L/K), L^\times)$ known as the *fundamental class.*

Definition 31.1. Let L/K be a finite extension of nonarchimedean local fields with Galois group $G := \mathrm{Gal}(L/K)$. The *fundamental class* $u_{L/K} \in H^2(G, L^\times)$ is the inverse image of $\frac{1}{[L:K]}$ under the invariant map $\mathrm{inv}_{L/K}\colon H^2(G, L^\times) \xrightarrow{\sim} \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$.

A key feature of the fundamental class is that is natural in both L and K, that is, it commutes with the restriction and corestriction maps (see Definition 29.16) that relate cohomology groups of Galois groups in towers of field extensions.

Lemma 31.2. *Let $K \subseteq E \subseteq L$ be a tower of finite Galois extensions of nonarchimedean local fields with restriction map* $\mathrm{Res}\colon H^2(\mathrm{Gal}(L/K), L^\times) \to H^2(\mathrm{Gal}(L/E), L^\times)$ *and corestriction map* $\mathrm{CoRes}\colon H^2(\mathrm{Gal}(L/E), L^\times) \to H^2(\mathrm{Gal}(L/K), L^\times)$. *Then* $u_{E/K} = [L:E]u_{L/K}$, $\mathrm{Res}(u_{L/K}) = u_{L/E}$, *and* $\mathrm{CoRes}(u_{L/E}) = [E:K]u_{L/K}$.

Proof. The first equality follows from $[L:E]\frac{1}{[L:K]} = \frac{1}{[E:K]}$. From Theorem 30.25 we have a commutative diagram

$$\begin{array}{ccccc}
H^2(\mathrm{Gal}(L/K), L^\times) & \xrightarrow{\mathrm{Inf}} & H^2(\mathrm{Gal}(K^{\rm sep}/K, K^{\rm sep\times}) & \xrightarrow{\mathrm{inv}_K} & \mathbb{Q}/\mathbb{Z} \\
\downarrow{\scriptstyle\mathrm{Res}} & & \downarrow{\scriptstyle\mathrm{Res}} & & \downarrow{\scriptstyle[E:K]} \\
H^2(\mathrm{Gal}(L/E), L^\times) & \xrightarrow{\mathrm{Inf}} & H^2(\mathrm{Gal}(K^{\rm sep}/E, K^{\rm sep\times}) & \xrightarrow{\mathrm{inv}_E} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

that implies $\mathrm{Res}(u_{L/K}) = u_{L/E}$, since $\frac{1}{[L:E]} = [E:K]\frac{1}{[L:K]}$. We then have

$$\mathrm{CoRes}(u_{L/E}) = \mathrm{CoRes}(\mathrm{Res}(u_{L/K})) = [E:K]u_{L/K},$$

by Proposition 29.21. ∎

We now observe that the G-module $L^\times$ satisfies the hypotheses of Tate's Theorem (Theorem 29.26): for all subgroups $H \le G$ we have $H^1(H, L^\times) = 0$ and $H^2(H, L^\times)$ is cyclic of order $\#H$ (to see this, apply Corollaries 30.11 and 30.24 to the extension L/L^H). If we take the fundamental class $u_{L/K}$ as our generator for $H^2(G, L^\times)$, Tate's theorem gives us canonical isomorphisms $\hat{H}^n(G, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^{n+2}(G, L^\times)$ for all $n \in \mathbb{Z}$.

In particular, for $n = -2$ we have

$$\hat{H}^{-2}(G, \mathbb{Z}) \simeq \hat{H}^0(G, L^\times) = (L^\times)^G/(N_G L^\times) = K^\times/\mathrm{N}(L^\times),$$

where N denotes the field norm $\mathrm{N}_{L/K}$, which coincides with the action of $N_G = \sum_{g\in G} g$.

Recall that $\hat{H}^{-2}(G, \mathbb{Z})$ is, by definition, the homology group $H_1(G, \mathbb{Z})$. The following theorem allows us to identify $H_1(G, \mathbb{Z})$ with the maximal abelian quotient of G, which we recall is the quotient of G by its commutator subgroup

$$[G, G] := \{ghg^{-1}h^{-1} : g, h \in G\}.$$

Theorem 31.3. *For any group G, we have a canonical isomorphism $G^{\mathrm{ab}} \xrightarrow{\sim} H_1(G, \mathbb{Z})$, where the G-action on $\mathbb{Z}$ is trivial and G^{ab} is the maximal abelian quotient of G.*

Proof. Recall that we have an exact sequence of G-modules

$$0 \longrightarrow I_G \xrightarrow{\iota} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where ε is the augmentation map $\sum_g n_g g \mapsto \sum_g n_g$ and $I_G := \ker \varepsilon$ is the augmentation ideal. The G-module $\mathbb{Z}[G] = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z} = \mathrm{Ind}^G(\mathbb{Z})$ is induced, hence $H_n(G, \mathbb{Z}[G]) = 0$ for all $n > 0$, by Lemma 23.26. In particular $H_1(G, \mathbb{Z}[G]) = 0$, so the long exact sequence in homology given by Theorem 23.21 ends with

$$0 \longrightarrow H_1(G, \mathbb{Z}) \xrightarrow{\delta_0} H_0(G, I_G) \xrightarrow{\iota_0} H_0(G, \mathbb{Z}[G]) \xrightarrow{\varepsilon_0} H_0(G, \mathbb{Z}) \longrightarrow 0$$

where δ_0 is the connecting homomorphism given by the snake lemma. Taking coinvariants, we have

$$H_0(G, I_G) = I_G/I_G^2, \qquad H_0(G, \mathbb{Z}[G]) = \mathbb{Z}[G]/I_G\mathbb{Z}[G] \simeq \mathbb{Z}, \qquad H_0(G, \mathbb{Z}) = \mathbb{Z}/I_G\mathbb{Z} \simeq \mathbb{Z},$$

so ε_0 is a surjective homomorphism $\mathbb{Z} \to \mathbb{Z}$, hence an isomorphism, which implies that ι_0 is the zero map and $\delta_0 \colon H_1(G, \mathbb{Z}) \to I_G/I_G^2$ is an isomorphism.

We now show that I_G/I_G^2 is (canonically) isomorphic to G^{ab}. Consider the map

$$\begin{aligned} \psi \colon G &\to I_G/I_G^2 \\ g &\mapsto (g-1) + I_G^2. \end{aligned}$$

It is a homomorphism, since $gh - 1 = (g-1)(h-1) + (g-1) + (h-1)$ for $g, h \in G$ and

$$\psi(gh) = (gh-1) + I_G^2 = (g-1) + (h-1) + I_G^2 = \psi(g) + \psi(h)$$

The group I_G/I_G^2 is commutative, so $[G, G] \subseteq \ker \psi$, and ψ is surjective, since I_G is generated by $\{g - 1 : g \in G\}$. It follows that ψ induces a homomorphism $\bar{\psi} \colon G^{\mathrm{ab}} \to I_G/I_G^2$.

Now let $\pi \colon G \twoheadrightarrow G/[G, G] = G^{\mathrm{ab}}$ be the quotient map and consider the map

$$\begin{aligned} \phi \colon I_G &\to G^{\mathrm{ab}} \\ (g-1) &\mapsto \pi(g). \end{aligned}$$

which is clearly a surjective group homomorphism (note that I_G is the free $\mathbb{Z}$-module generated by $\{g-1 : g \in G\}$). We have

$$\phi((g-1)(h-1)) = \phi((gh-1)-(g-1)-(h-1)) = \pi(gh)\pi(g)^{-1}\pi(h)^{-1} = 1,$$

since G^{ab} is abelian, so $I_G^2 \subseteq \ker\phi$ and ϕ induces a homomorphism $\bar{\phi}\colon I_G/I_G^2 \to G^{\mathrm{ab}}$. It is clear that $\bar{\psi}\circ\bar{\phi}$ and $\bar{\phi}\circ\bar{\psi}$ are both identity maps, so $\bar{\psi}$ and $\bar{\phi}$ are inverse isomorphisms and $I_G/I_G^2 \simeq G^{\mathrm{ab}}$ as claimed. This isomorphism is canonical, since we made no arbitrary choices when defining $\bar{\psi}$ and $\bar{\phi}$. □

Definition 31.4. Let L/K be a finite Galois extension of nonarchimedean local fields with $G := \mathrm{Gal}(L/K)$. The *local Artin map* $\theta_{L/K}\colon K^\times/\mathrm{N}(L^\times) \xrightarrow{\sim} G^{\mathrm{ab}}$ is the inverse of the isomorphisms

$$G^{\mathrm{ab}} \simeq H_1(G,\mathbb{Z}) = \hat{H}^{-2}(G,\mathbb{Z}) \xrightarrow{\Phi_{L/K}} \hat{H}^0(G,L^\times) = K^\times/\mathrm{N}(L^\times),$$

where $\Phi_{L/K} := \Phi_{u_{L/K}}$ is the isomorphism given by Tate's Theorem (Theorem 29.26) when we take the fundamental class $u_{L/K}$ as our generator for $H^2(G,L^\times)$. We may also view $\theta_{L/K}$ as a surjective homomorphism $K^\times \to G^{\mathrm{ab}}$ with kernel $\mathrm{N}(L^\times)$.

Lemma 31.5. *Let $K \subseteq E \subseteq L$ be a tower of finite Galois extensions of nonarchimedean local fields. The following diagrams commute:*

$$\begin{array}{ccc} E^\times & \xrightarrow{\theta_{L/E}} & \mathrm{Gal}(L/E)^{\mathrm{ab}} \\ \downarrow{\scriptstyle \mathrm{N}_{E/K}} & & \downarrow \\ K^\times & \xrightarrow{\theta_{L/K}} & \mathrm{Gal}(L/K)^{\mathrm{ab}} \end{array} \qquad\qquad \begin{array}{ccc} K^\times & \xrightarrow{\theta_{L/K}} & \mathrm{Gal}(L/K)^{\mathrm{ab}} \\ \| & & \downarrow{\scriptstyle \sigma\mapsto\sigma_{|E}} \\ K^\times & \xrightarrow{\theta_{E/K}} & \mathrm{Gal}(E/K)^{\mathrm{ab}} \end{array}$$

Proof. Let $G := \mathrm{Gal}(L/K)$ and $H := \mathrm{Gal}(L/E)$. The vertical maps in the first diagram are corestriction maps induced by the inclusion $H \subseteq G$. The left vertical map is $\mathrm{CoRes}\colon H^0(H,L^\times) \to H^0(G,L^\times)$, defined by $a \mapsto N_{G/H}a$ (see Definition 29.18), which is equivalent to the norm map $\mathrm{N}_{E/K}\colon E^\times \to K^\times$. The right vertical map is the natural corestriction map in homology $\mathrm{CoRes}\colon H_1(H,\mathbb{Z}) \to H_1(G,\mathbb{Z})$, which is equivalent to the natural map $H/[H,H] \to G/[G,G]$ that sends $h[H,H]$ to $h[G,G]$ (via the canonical isomorphism $\bullet^{\mathrm{ab}} \simeq H_1(\bullet,\mathbb{Z})$ given by Theorem 31.3). From Lemma 31.2 and the compatibility of $\Phi_{L/K}$ with corestriction given by Theorem 29.26 we have

$$\mathrm{CoRes}\circ\Phi_{u_{L/E}} = \mathrm{CoRes}\circ\Phi_{\mathrm{Res}(u_{L/K})} = \Phi_{u_{L/K}}\circ\mathrm{CoRes},$$

and the commutativity of the first diagram follows, since $\theta_{L/E} = \Phi_{L/E}^{-1}$ and $\theta_{L/K} = \Phi_{L/K}^{-1}$.

You will prove the commutativity of the second diagram on the Problem Set.[1] □

Definition 31.6. Let K be a nonarchimedean local field. The *local Artin map* θ_K is the continuous homomorphism $\theta_K\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ defined by the compatible system of homomorphisms $\{\theta_{L/K}\}$, where L ranges over finite abelian extensions of K; it is uniquely determined by the property $\theta_K(a)_{|L} = \theta_{L/K}(a)$ for all $a \in K^\times$ and finite abelian L/K.

[1] Alternatively, one can give a more direct proof by noting that the vertical maps in the second diagram are *residuacity* and *deflation* maps (respectively), and then appeal to [1, Thm. 1] (or prove just the special case $n = -2$ needed in this instance); we also refer the interested reader to the appendix of [2] which describes the residuacity/deflation maps in more detail.

Recall that the Galois group $\mathrm{Gal}(L/K)$ of a finite unramified extension of nonarchimedean local fields has a canonical generator $\mathrm{Frob}_{L/K}$ corresponding to the Frobenius automorphism of the residue field extension; we used this in our definition of the invariant map for unramified extensions. The (arithmetic) Frobenius element Frob_K is the corresponding element of $\mathrm{Gal}(K^{\mathrm{unr}}/K) = \varprojlim_L \mathrm{Gal}(L/K)$, where L ranges over finite unramified extensions of K; it is determined by the property $\mathrm{Frob}_{K|_L} = \mathrm{Frob}_{L/K}$ for all finite unramified extensions L/K.

Theorem 31.7. *Let K be a nonarchimedean local field and let $\theta_K\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ be the local Artin map. Then $\theta_K(\pi)_{|K^{\mathrm{unr}}} = \mathrm{Frob}_K$ for every uniformizer π of K.*

Proof. Let π be a uniformizer for K. It suffices to show that $\theta_{L/K}(\pi) = \mathrm{Frob}_{L/K}$ for every finite unramified extension L/K. So let L/K be such an extension, let $G := \mathrm{Gal}(L/K)$, and to simplify the notation, put $\sigma := \mathrm{Frob}_{L/K}$ and $n := [L\colon K]$. We have $G^{\mathrm{ab}} = G = \langle\sigma\rangle$, and it follows from the definition of $\theta_{L/K}$ that we need to show that $\pi\,\mathrm{N}(L^\times)$ is the image of σ under the composition of isomorphisms

$$G = G^{\mathrm{ab}} \xrightarrow{\psi_0} H_1(G,\mathbb{Z}) = \hat{H}^{-2}(G,\mathbb{Z}) \xrightarrow{\Phi_{L/K}} \hat{H}^0(G,L^\times) = K^\times/\mathrm{N}(L^\times), \tag{1}$$

where ψ_0 is the isomorphism given by Theorem 31.3 and $\Phi_{L/K} := \Phi_{u_{L/K}}$ is the isomorphism $\Phi_{L/K}\colon \hat{H}^{-2}(G,\mathbb{Z}) \to \hat{H}^0(G,L^\times)$ given by Theorem 29.26 (Tate's theorem) when we take the fundamental class $u_{L/K} = \mathrm{inv}_{L/K}^{-1}(\frac{1}{n})$ as our generator for $H^2(G,L^\times)$.

From the proof of Theorem 31.3 we see that ψ_0 is the composition of the inverse of the connecting homomorphism $\delta_0\colon H_1(G,\mathbb{Z}) \xrightarrow{\sim} H_0(G,I_G) = I_G/I_G^2$ induced by the exact sequence $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ with the isomorphism $\psi\colon G^{\mathrm{ab}} \xrightarrow{\sim} I_G/I_G^2$ defined by $\sigma \mapsto (\sigma-1) + I_G^2$, where I_G is the augmentation ideal.

From the proof of Theorem 29.26, we see that $\Phi_{L/K}$ is the composition of the connecting homomorphism $\hat{\delta}_0\colon \hat{H}^{-2}(G,\mathbb{Z}) \xrightarrow{\sim} \hat{H}^{-1}(G,I_G)$ and the connecting homomorphism $\hat{\delta}_{L/K}\colon \hat{H}^{-1}(G,I_G) \xrightarrow{\sim} \hat{H}^0(G,L^\times)$ induced by the short exact sequence of G-modules

$$0 \longrightarrow L^\times \xrightarrow{\iota} L^\times(\varphi) \xrightarrow{\phi} I_G \longrightarrow 0. \tag{2}$$

Here $\varphi\colon G^2 \to L^\times$ is a 2-cocycle representing the fundamental class $u_{L/K} \in H^2(G,L^\times)$, and

$$L^\times(\varphi) := L^\times \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}x_{\sigma^i},$$

where the x_{σ^i} are formal variables, and the G-action on $L^\times(\varphi)$ is defined by

$$\sigma^i x_{\sigma^j} := x_{\sigma^{i+j}} - x_{\sigma^i} + \varphi(\sigma^i,\sigma^j), \tag{3}$$

with $x_1 := \varphi(1,1) \in L^\times$. The map $\phi\colon L^\times(\varphi) \to I_G$ in the exact sequence (2) is defined by $a \mapsto 0$ for $a \in L^\times$ and $x_i \mapsto \sigma^i - 1$ for $1 \le i < n$.

We now observe that

$$\hat{H}^{-1}(G,I_G) = \hat{H}_0(G,I_G) = H_0(G,I_G) = I_G/I_G^2,$$

since $N_G I_G = 0$, so $\hat{\delta}_0 = \delta_0$ and therefore

$$\Phi_{L/K} \circ \psi_0 = (\hat{\delta}_{L/K} \circ \hat{\delta}_0) \circ (\delta_0^{-1} \circ \psi) = \hat{\delta}_{L/K} \circ \psi.$$

As noted above, we have $\psi(\sigma) = [\sigma - 1] \coloneqq (\sigma - 1) + I_G^2$, so we only need to show that $\hat\delta_{L/K}([\sigma - 1]) = [\pi]$, where we now view $\hat\delta_{L/K}$ as an isomorphism $I_G/I_G^2 \xrightarrow{\sim} K^\times/\mathrm{N}(L^\times)$.

Recall from Theorem 23.32 that $\hat\delta_{L/K}\colon \hat H_0(G, I_G) \to \hat H^0(G, L^\times)$ is defined by applying the snake lemma to the diagram

$$\begin{array}{ccccccc} & & L^\times_G & \longrightarrow & L^\times(\varphi)_G & \xrightarrow{\hat\phi} & (I_G)_G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \hat N_G} & & \downarrow{\scriptstyle \hat N_G} & & \downarrow{\scriptstyle \hat N_G} & & \\ 0 & \longrightarrow & (L^\times)^G & \xrightarrow{\hat\imath} & (L^\times(\varphi))^G & \longrightarrow & (I_G)^G & & \end{array}$$

and we want to compute the element of $K^\times = (L^\times)^G$ obtained by tracing $[\sigma - 1] \in (I_G)_G$ along the highlighted path. The pre-image of $[\sigma - 1]$ in $L^\times(\varphi)_G$ is, by the definition of ϕ, the class $[x_\sigma]$. We thus want to compute

$$\theta_{L/K}^{-1}(\sigma) = (\Phi_{L/K} \circ \psi_0)(\sigma) = (\hat\delta_{L/K} \circ \psi)(\sigma) = \hat N_G[x_\sigma] = N_G x_\sigma \in K^\times.$$

Applying the group action on $L^\times(\varphi)$ defined in (3), we have

$$N_G x_\sigma = \sum_{i=0}^{n-1} \sigma^i x_\sigma = \sum_{i=0}^{n-1} \bigl(x_{\sigma^{i+1}} - x_{\sigma^i} + \varphi(\sigma^i, \sigma)\bigr) = \sum_{i=0}^{n-1} \varphi(\sigma^i, \sigma).$$

To compute the sum on the right (a product in the multiplicative group $L^\times$) we need to pick a 2-cocycle $\varphi\colon G^2 \to L^\times$ that represents the fundamental class $u_{L/K} = \mathrm{inv}_{L/K}^{-1}(\frac{1}{n})$; we will get the same answer modulo $\mathrm{N}(L^\times)$, no matter which φ we pick.

Recall from Definition 30.20 that $\mathrm{inv}_{L/K}$ is defined by the composition

$$H^2(G, L^\times) \xrightarrow{v} H^2(G, \mathbb{Z}) \xrightarrow{\delta^{-1}} H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{f \mapsto f(\sigma)} \mathbb{Q}/\mathbb{Z},$$

where v is induced by the discrete valuation on $L^\times$. The connecting homomorphism

$$\delta\colon H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} H^2(G, \mathbb{Z}),$$

is induced by the short exact sequence of trivial G-modules $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. We have $H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$, since $\mathbb{Q}/\mathbb{Z}$ is a trivial G-module (so every crossed homomorphism is a homomorphism and every principal crossed homomorphism is trivial), and since $\mathrm{inv}_{L/K}(u_{L/K}) = \frac{1}{n}$, the image of $u_{L/K}$ in $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ must be the homomorphism $f\colon G \to \mathbb{Q}/\mathbb{Z}$ defined by $\sigma \mapsto \frac{1}{n}$.

To compute $\delta(f)$ we need to again trace through a snake lemma diagram, which in this case consists entirely of trivial G-modules.

$$\begin{array}{ccccccccc} & & \frac{C^1(G,\mathbb{Z})}{B^1(G,\mathbb{Z})} & \longrightarrow & \frac{C^1(G,\mathbb{Q})}{B^1(G,\mathbb{Q})} & \longrightarrow & \frac{C^1(G,\mathbb{Q}/\mathbb{Z})}{B^1(G,\mathbb{Q}/\mathbb{Z})} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d^1} & & \\ 0 & \longrightarrow & Z^2(G,\mathbb{Z}) & \longrightarrow & Z^2(G,\mathbb{Q}) & \longrightarrow & Z^2(G,\mathbb{Q}/\mathbb{Z}) & & \end{array}$$

where $C^n(G, \bullet)$, $B^n(G, \bullet)$, $Z^n(G, \bullet)$ denote n-cochains, n-coboundaries, n-cocycles. It follows that $\delta(f) \in H^2(G, \mathbb{Z})$ is represented by the coboundary of a 1-cocycle $\hat f\colon G \to \mathbb{Q}$ (modulo $B_1(G, \mathbb{Q})$) that agrees with f modulo $\mathbb{Z}$. So let $\hat f\colon G \to \mathbb{Q}$ be the homomorphism defined by $\sigma \mapsto \frac{1}{n}$ (as above, for trivial G-modules homomorphisms are the same thing as 1-cocycles). Applying the coboundary formula and remembering that the G-action is trivial, we find that $\delta(f)$ is represented by the 2-cocycle

$$d^1(\hat{f})(\sigma^i,\sigma^j) = \sigma^i\hat{f}(\sigma^j) - \hat{f}(\sigma^i\sigma^j) + \hat{f}(\sigma^i) = \tfrac{i+j}{n} - \tfrac{i+j \bmod n}{n} = \begin{cases} 0 & \text{if } i+j<n,\\ 1 & \text{if } i+j\ge n.\end{cases}$$

The map $v\colon H^2(G,L^\times)\to H^2(G,\mathbb{Z})$ simply sends the class represented by a 2-cocycle $G^2\to L^\times$ to the class of its composition with the discrete valuation v. It follows that

$$\varphi(\sigma^i,\sigma^j) \coloneqq \begin{cases} 1 & \text{if } i+j<n,\\ \pi & \text{if } i+j\ge n,\end{cases}$$

represents the fundamental class $u_{L/K}\in H^2(G,L^\times)$. We can now compute

$$N_G x_\sigma = \prod_{i=0}^{n-1}\varphi(\sigma^i,\sigma) = 1\cdot 1\cdots 1\cdot\pi = \pi$$

and the theorem follows (note that the abelian group $L^\times$ is multiplicative, so the expression for $N_G x_\sigma$ in the G-module $L^\times(\varphi)$ written above as a sum is really a product in $L^\times$). □

We have now proved the local Artin reciprocity theorem stated in Theorem 29.1: the existence of a continuous homomorphism with the required properties follows immediately from Definition 31.6 and Theorem 31.7, and we proved uniqueness in Proposition 27.12.

31.2 The norm limitation theorem

We conclude with a theorem that shows we cannot hope to extend the local Artin reciprocity theorem to nonabelian extensions, because in general, finite Galois extensions L/K of a local field K are not determined by norm groups $K^\times/\mathrm{N}(L^\times)$. Indeed, the norm group depends only on the maximal abelian subextension (even when L/K is not Galois).

Theorem 31.8 (Norm Limitation)**.** *Let L/K be a finite extension of nonarchimedean local fields and let E/K be the maximal abelian subextension. Then $\mathrm{N}(L^\times)=\mathrm{N}(E^\times)$, where N denotes the norm map down to K.*

Proof. By transitivity of the norm map, $\mathrm{N}_{L/K}=\mathrm{N}_{E/K}\circ\mathrm{N}_{L/E}$, so $\mathrm{N}(L^\times)\subseteq\mathrm{N}(E^\times)$, we just need to show that equality holds. If L/K is Galois then the isomorphisms given by the local Artin maps

$$\theta_{E/K}\colon K^\times/\mathrm{N}(E^\times)\xrightarrow{\sim}\mathrm{Gal}(E/K)^{\mathrm{ab}}=\mathrm{Gal}(E/K),$$
$$\theta_{L/K}\colon K^\times/\mathrm{N}(L^\times)\xrightarrow{\sim}\mathrm{Gal}(L/K)^{\mathrm{ab}}=\mathrm{Gal}(E/K),$$

imply $\mathrm{N}(L^\times)=\mathrm{N}(E^\times)$.

For the general case, let M/K be a finite Galois extension of K containing L, and put $G\coloneqq\mathrm{Gal}(M/K)$ and $H\coloneqq\mathrm{Gal}(M/L)$. Then $M^{[G,G]}$ is the maximal abelian subextension of M/K, and

$$E = M^{[G,G]}\cap L = M^{[G,G]}\cap M^H = M^{[G,G]H},$$

so $\mathrm{Gal}(M/E)=[G,G]H$. Noting that $[H,H]=H\cap[G,G]$, we have a diagram

$$
\begin{array}{ccccc}
L^\times & \xrightarrow{\theta_{M/L}} & H^{\mathrm{ab}} & = & H/[H,H] \\
\big\downarrow{\scriptstyle \mathrm{N}} & & \big\downarrow{\scriptstyle \iota} & & \big\downarrow \\
K^\times & \xrightarrow{\theta_{M/K}} & G^{\mathrm{ab}} & = & G/[G,G] \\
\big\| & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\
K^\times & \xrightarrow{\theta_{E/K}} & \mathrm{Gal}(E/K) & = & G/([G,G]H),
\end{array}
$$

where commutativity of the two squares on the left is given by Lemma 31.5 and the middle column is part of an exact sequence

$$1 \longrightarrow H^{\mathrm{ab}} \xrightarrow{\iota} G^{\mathrm{ab}} \xrightarrow{\pi} \mathrm{Gal}(E/K) \longrightarrow 1$$

Consider any $a \in \mathrm{N}(E^\times)$. Then $a \in \ker \theta_{E/K}$, so $\theta_{M/K}(a) \in \ker \pi = \mathrm{im}\, \iota$. The map $\theta_{M/L}$ is surjective, so there is a $b \in L^\times$ such that

$$\theta_{M/K}(a) = \iota(\theta_{M/L}(b)) = \theta_{M/K}(\mathrm{N}(b)),$$

and therefore $a/\mathrm{N}(b) \in \ker \theta_{M/K} = \mathrm{N}(M^\times)$. Now let $c \in M^\times$ satisfy $\mathrm{N}(c) = a/\mathrm{N}(b)$ and observe that $a = \mathrm{N}(b)\mathrm{N}(c) = \mathrm{N}(b\mathrm{N}_{M/L}(c)) \in \mathrm{N}(L^\times)$ as desired. $\square$

References

[1] K. Horie and M. Horie, *Deflation and residuation for class formation*, J. Algebra, 2001.

[2] T. P. Pollio and A. S. Rapinchuk, *The multinorm principle for linearly disjoint Galois extensions*, J. Number Theory **133** (2013), 802–821.