30 Galois cohomology and the invariant map for local fields

In order to complete our cohomological construction of the local Artin homomorphism we need to refine our notion of cohomology for profinite groups, for two reasons. First, we need to account for the topology, which we have already seen is critical to understanding Galois groups of infinite extensions; recall that the main theorem of Galois theory holds only when we restrict our attention to closed subgroups, see Theorem 26.22. Second, we want to be able to apply results that we have so far proved only for finite groups to profinite groups (by taking appropriate limits).

30.1 Cohomology of profinite groups

Definition 30.1. Let G be a topological group. A *topological G-module* A is an abelian topological group on which G acts continuously. In other words, the map $G \times A \to A$ defined by $(g, a) \mapsto ga$ is continuous. A *discrete G-module* A is a topological G-module whose topology is discrete. A morphism of topological G-modules is a morphism of topological abelian groups compatible with the G-action. For discrete G-modules this is the same thing as a morphism of G-modules.

By convention, if we refer to a G-module A as a discrete G-module then we are endowing it with the discrete topology (whether it already has a topology or not).

Example 30.2. Let G be a topological group. Every trivial G-module is a discrete G-module, and if G is discrete, every G-module is a discrete G-module. The $\mathbb{C}^\times$-module $\mathbb{C}$ (the action is multiplication) is not a discrete $\mathbb{C}^\times$-module, because $\{1\} \subseteq \mathbb{C}^\times$ is not open.

There are several inequivalent ways to define cohomology for topological G-modules, but in the case of interest to us, where G is a profinite group and A is a discrete G-module, the natural choice is *continuous cohomology*. We consider the cochain complex consisting of *continuous cochains*, functions $f\colon G^n \to A$ that are continuous with respect to the product topology on G^n and the topology on A (as usual, these need not be group homomorphisms), and we let $C^n(G, A)$ denote the the set of all continuous n-cochains, which is an abelian group under addition (because addition in A is continuous). If G has the discrete topology then continuous cochains are the same as cochains.

The coboundary maps $d^n\colon C^n(G, A) \to C^{n+1}(G, A)$ are defined as before (see Definition 23.3); coboundaries of continuous functions are necessarily continuous because the definition of d^n uses only addition and the G-action on A, both of which are continuous operations. Cohomology groups $H^n(G, A)$ is then defined as (continuous) cocycles modulo coboundaries as usual (see Definition 23.4). We note that $H^0(G, A) = A^G$ as with usual group cohomology, because constant functions are always continuous, but higher cohomology groups may differ.

If $\alpha\colon A \to B$ is a morphism of topological G-modules, the map $f \mapsto \alpha \circ f$ induces maps $\alpha^n\colon C^n(G, A) \to C^n(G, B)$ on continuous cochains (we are composing continuous maps. It follows that Lemma 23.6 still applies: the maps α^n commute with the coboundary maps, hence carry cocycles to cocycles and coboundaries to coboundaries, and thus induce homomorphisms $\alpha^n\colon H^n(G, A) \to H^n(G, B)$ in cohomology. We thus have functors $H^n(G, \bullet)$ from the category of topological G-modules to the category of abelian groups.

Remark 30.3. Many authors use the notation $H^n_c(G, A)$ or $H^n_{\text{cts}}(G, A)$ to emphasize that continuous cohomology is being used (as opposed to ordinary group cohomology or other types of cohomology for topological groups). As we will only be interested in the case where G is profinite and A is a discrete G-module, in which case it always makes sense to use continuous cohomology, so we will not make this distinction. To see why one might not want to use continuous cohomology in other settings, note that if G is connected and A is a discrete G-module, then every continuous cochain is a constant function and all cohomology groups are trivial. Note that profinite groups lie at the opposite extreme, since they are totally disconnected (see Theorem 26.17).

Lemma 30.4. *Let G be a compact group and let A be a G-module. The following are equivalent:*

(i) *A is a discrete G-module;*

(ii) *Every stabilizer $G_a := \{g \in G : ga = a\}$ is open;*

(iii) *$A = \bigcup_N A^N$, where N ranges over normal open subgroups of G.*

Proof. Let $\pi\colon G \times A \to A$ be the map $(g,a) \mapsto ga$. **(i) ⇒ (ii)**: for all $a \in A$ we have $\pi^{-1}(a) \cap (G \times \{a\})) = G_a \times \{a\}$ open in $G \times A$, so G_a open in G. **(ii) ⇒ (iii)**: if G_a is open it has finite index (since G is compact) and only finitely many conjugates; the intersection of these conjugates is a normal open subgroup N for which $a \in A^N$. **(iii) ⇒ (i)**: for each $a \in A$ we have $a \in A^N$ for some open N, and $Ng \times \{b\} \subseteq \pi^{-1}(a)$ is open for every b in the G-orbit of a; every $(g,b) \in \pi^{-1}(a)$ lies in such a set, so $\pi^{-1}(a)$ is a union of open sets. □

Remark 30.5. The equivalence of (i) and (ii) in Lemma 30.4 does not require compactness.

Lemma 30.6. *Let G be a topological group and let*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

be an exact sequence of discrete G-modules. Then for every $n \ge 0$ we have a corresponding exact sequence of continuous cochains

$$0 \longrightarrow C^n(G,A) \xrightarrow{\alpha^n} C^n(G,B) \xrightarrow{\beta^n} C^n(G,C) \longrightarrow 0.$$

Proof. The argument follows the proof of Lemma 23.7, but now we restrict to continuous cochains. We first note that for any morphism of discrete G-modules $\gamma\colon M \to N$ and $n \ge 0$ we have an induced map $\gamma^n\colon C^n(G,M) \to C^n(G,N)$ defined by $f \mapsto \gamma \circ f$; the map $\gamma \circ f$ is a continuous because both f and γ are (trivially so in the case of γ, since M is discrete). In particular, the maps α^n and β^n are well defined.

The injectivity of α^n follows immediately from the injectivity of α. For $f \in \ker \beta^n$ we have $\beta \circ f = 0$, so $\operatorname{im} f \subseteq \ker \beta = \operatorname{im} \alpha$, and using the bijection $\alpha^{-1}\colon \operatorname{im} \alpha \to A$ we can define $\alpha^{-1} \circ f \in C^n(G,A)$ whose image under α^n is f; thus $\ker \beta^n \subseteq \operatorname{im} \alpha^n$. Conversely, for any $f \in C^n(G,A)$ we have $\beta \circ \alpha \circ f = 0 \circ f = 0$ and therefore $\operatorname{im} \alpha^n \subseteq \ker \beta^n$.

It remains only to show that the map β^n is surjective. Let us fix $f \in C^n(G,C)$. The fibers of f partition G^n into a disjoint union of open sets U_c indexed by $c \in \operatorname{im} f \subseteq C$ (since f is continuous and C is discrete). For each $c \in \operatorname{im} f$ we now choose $b_c \in B$ such that $\beta(b_c) = c$ (this is possible because β is surjective). We now define $h \in C^n(G,b)$ by letting it map U_c to b_c; the fibers of h are then the same as those of f, so h is continuous, and we have $\beta \circ h = f$ by construction, so $\beta^n(h) = f$ and it follows that β^n is surjective. □

Corollary 30.7. *Every short exact sequence of discrete G-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

induces a long exact sequence of cohomology groups

$$0 \to H^0(G,A) \xrightarrow{\alpha^0} H^0(G,B) \xrightarrow{\beta^0} H^0(G,C) \xrightarrow{\delta^0} H^1(G,A) \longrightarrow \cdots$$

and commutative diagrams of short exact sequences of discrete G-modules induce corresponding commutative diagrams of long exact sequences of cohomology groups.

Remark 30.8. Lemma 30.6 and Corollary 30.7 do not hold for topological G-modules in general, so with continuous cohomology we really need to restrict our attention to discrete G-modules, otherwise we do not necessarily get δ-functors (families of functors that take short exact sequences to long exact sequences).

Recall that if N is normal subgroup of G then for any G-module A we have inflation maps $\operatorname{Inf}\colon H^n(G/N, A^N) \to H^n(G,A)$ induced by the quotient map $G \to G/N$ and the compatible inclusion $A^N \to A$; see Definition 29.22. More generally, if $H \subseteq K$ are normal subgroups of G, we can view K/H as a normal subgroup of G/H with corresponding quotient G/K, and we get an inflation map $\operatorname{Inf}\colon H^n(G/K, A^K) \to H^n(G/H, A^H)$. These maps are natural in the inclusion $H \subseteq K$. That is, if $H \subseteq K \subseteq L$ are normal subgroups of G then the following triangle commutes

$$\begin{array}{ccc} H^n(G/L, A^L) & \xrightarrow{\operatorname{Inf}} & H^n(G/K, A^K) \\ & \underset{\operatorname{Inf}}{\searrow} & \big\downarrow{\scriptstyle \operatorname{Inf}} \\ & & H^n(G/H, A^H). \end{array}$$

We can express any profinite group G as the inverse limit $G = \varprojlim_N G/N$ over the inverse system of quotients by open normal subgroups N, where the N are partially ordered by reverse inclusion and the connecting maps are projections. The inflation maps give us a corresponding *direct system* of cohomology groups $H^n(G/N, A^N)$.

Recall that a direct system is a collection of objects X_i indexed by a directed set I (a set with a partial order $\lesssim$ in which every pair $i,j \in I$ has an upper bound $i,j \lesssim k \in I$) with morphisms $f_{ij}\colon X_i \to X_j$ for all $i \lesssim j$ with $f_{ii} = \mathrm{id}$ and $f_{ik} = f_{jk} \circ f_{ij}$ for $i \lesssim j \lesssim k$. The *direct limit* X of a direct system is the disjoint union $\coprod_i X_i$ (coproduct) modulo the equivalence relation $\sim$ given by identifying $x_i \in X_i$ with $x_j \in X_j$ whenever $f_{ik}(x_i) = f_{jk}(x_j)$ for some $k \gtrsim i,j$. It comes equipped with natural maps $\phi_i\colon X_i \to X$ given by composing the natural inclusion $X_i \to \coprod_{i\in I} X_i$ with the map $x_i \mapsto [x_i]$ that sends elements to their equivalence classes. For a direct system of abelian groups the group operation is given by $[x_i] + [x_j] = [f_{ik}(x_i) + f_{jk}(x_j)]$, where $x_i \in X_i$ and $x_j \in X_j$ are equivalence class representatives and $k \gtrsim i,j$ is any upper bound.

In our setting, we have a directed set of open normal subgroups N partially ordered by reverse inclusion (so $N_1 \lesssim N_2$ if $N_1 \supseteq N_2$). Our profinite group G is the inverse limit $\varprojlim_N G/N$, and we now want to consider the direct limit $\varinjlim_N H^n(G/N, A^N)$.

Theorem 30.9. *Let G be a profinite group. For every discrete G-module A and $n \geq 0$ we have an isomorphism*

$$H^n(G,A) \simeq \varinjlim_N H^n(G/N, A^N),$$

where the direct limit is over open $N \trianglelefteq G$ ordered by inclusion and the maps are inflation maps. These isomorphisms are natural in the discrete G-module A.

Proof. Direct limits commute with kernels and cokernels, so it suffices to prove the analogous statement for cochains, that is, we want to show that

$$C^n(G,A) \overset{?}{\simeq} \varinjlim_N C^n(G/N, A^N),$$

where the maps $C^n(G/N_1, A^{N_1}) \to C^n(G/N_2, A^{N_2})$ for normal open subgroups $N_1 \supseteq N_2$ are given by inflating cocycles: given $f \in C^n(G/N_1, A^{N_1})$, its image in $C^n(G/N_2, A^{N_2})$ is defined by composing f with the quotient map $(G/N_2)^n \to (G/N_1)^n$.

There is an obvious homomorphism

$$\varphi\colon \varinjlim_N C^n(G/N, A^N) \to C^n(G,A).$$

Each element of $\varinjlim_N C^n(G/N, A^n)$ is represented by a cochain $f \in C^n(G/N, A^n)$ for some open $N \trianglelefteq G$. To get a cochain $G^n \to A$ we reduce inputs modulo N and apply f; this cochain is continuous because the quotient map is continuous (N is open) and f is continuous. This defines a homomorphism because addition of cochains is addition in A.

The homomorphism φ is injective, since its kernel is obviously trivial. We now show that it is surjective. Let $f\colon G^n \to A$ be a continuous n-cochain. Its image is compact and discrete, since G (and therefore G^n) is compact and A is discrete, and the image is therefore finite. The stabilizer of the image is thus an open subgroup of G and therefore contains an open normal subgroup N_1 of G (the intersection of its conjugates), and we have $\operatorname{im} f \subseteq A^{N_1}$. For each $a \in \operatorname{im} f$, its inverse image $f^{-1}(a)$ is open and therefore contains an n-fold product of open subgroups of G, each of which contains an open normal subgroup and the intersection of these is an open normal subgroup N_a for which $f(N_a^n) = a$. Let N be the intersection of N_1 with $\cap_{a\in \operatorname{im} f} N_a$; this is a finite intersection, since $\operatorname{im} f$ is finite, so N is a normal open subgroup of G, and N^n is contained in every fiber of f; it follows that f is the composition of the quotient map $G^n \to (G/N)^n$ and a continuous cochain $(G/N)^n \to A^N$ (note $\operatorname{im} f \subseteq A^{N_1} \subseteq A^N$ since $N \subseteq N_1$). Therefore f lies in the image of φ.

Naturality with respect to the discrete G-module A follows from from the fact that the inflation maps on cochains are natural: any morphism of discrete G-modules $A \to B$ yields a morphism of directed systems of cochains that commutes with the isomorphism φ constructed above (for both A and B). □

Corollary 30.10. *For every profinite group G and discrete G-module A the cohomology groups $H^n(G,A)$ are torsion for all $n > 0$.*

Proof. For every open $N \trianglelefteq G$ the group G/N is finite and $H^n(G/N, A^N)$ is torsion, by Proposition 29.11. A direct limit of torsion groups is obviously torsion. □

Corollary 30.11 (Hilbert Theorem 90)**.** *Let L/K be a (possibly infinite) Galois extension of fields. Then $H^1(\mathrm{Gal}(L/K), L^\times) = 0$.*

Proof. This follows immediately from Theorem 24.1 and Theorem 30.9. □

Theorem 30.12. *Let G be a profinite group and $A = \varinjlim_i A_i$ a direct limit of discrete G-modules M_i. Then A is a discrete G-module and $H^n(G,A) \simeq \varinjlim_i H^n(G,A_i)$ for $n \geq 0$.*

Proof. We first note that every $a \in A$ is represented by some $a_i \in A_i$ on which G acts with an open stabilizer; it follows from Lemma 30.4 that A is a discrete G-module. To prove the isomorphism in cohomology, as in the proof of Theorem 30.9, it suffices to prove it for continuous cochains. There is an obvious homomorphism

$$\varphi\colon \varinjlim_i C^n(G, A_i) \to C^n(G, A)$$

that sends $f_i\colon G^n \to A_i$ to $\phi_i \circ f_i$, where $\phi_i\colon A_i \to A$ is the natural map given by the direct limit $\varinjlim_i A_i = A$. The map φ is injective because its kernel is trivial: the zero maps $f_i\colon G^n \to A_i$ are all identified in the inverse limit $\varinjlim_i C^n(G, A_i)$ because the zero elements of A_i are identified in A.

To show that φ is surjective, we note that (as in the proof of Theorem 30.9), every continuous cochain $f\colon G^n \to A$ has finite image, since G is compact and A is discrete. If we let i be an upper bound in the directed indexing the A_i of the indices j of the A_j containing each $a_j \in \operatorname{im} f$, then $\operatorname{im} f \subseteq A_i$ (recall that the existence of upper bounds is part of the definition of a directed set). It follows that $f = \phi_i \circ f_i$ for some $f_i \in C^n(G, A_i)$ and φ is therefore surjective. □

As with G-modules, one can define a notion of compatibility for a continuous homomorphisms $\varphi\colon G \to G'$ of profinite groups and homomorphisms $\phi\colon A \to A'$ of abelian groups: we require $\varphi(g)\phi(a) = \phi(ga)$ for all $g \in G$ and $a \in A$. One can then use compatible pairs of morphisms to define restriction and inflation maps between cohomology groups of discrete G-modules as we did in Definitions 29.16 and Definition 29.22. Equivalently, one can define these as direct limits of restriction and inflation maps defined for finite quotients (the notions of G-module and discrete G-module coincide for profinite groups that are finite), via Theorem 30.9.

One key point is that, as in the fundamental theorem of Galois theory, we must restrict our attention to closed subgroups. Indeed, a subgroup of a profinite group is profinite if and only if it is closed. This follows immediately from the fact that a topological group is profinite if and only if it is totally disconnected and compact Hausdorff (Theorem 26.17). Every subset of a totally disconnected space is totally disconnected, but a subset of a compact Hausdorff space is compact Hausdorff if and only if it is closed.

We then obtain the following theorem.

Theorem 30.13 (Inflation-Restriction)**.** *Let H be a closed normal subgroup of a profinite group G, let A be a discrete G-module, and let $n \geq 1$. If $H^i(H, A) = 0$ for $1 \leq i < n$ then the sequence*

$$0 \longrightarrow H^n(G/H, A^H) \xrightarrow{\text{Inf}} H^n(G, A) \xrightarrow{\text{Res}} H^n(H, A)$$

is exact.

Proof. Copy the proof of Theorem 29.23 using continuous cocycles (or simply note that direct limits preserve exactness). □

Remark 30.14. At this point the reader might assume that everything we have done with G-modules easily extends to discrete G-modules. This is not quite true. Recall that when working with G-modules we frequently used the fact that we can view $\mathbb{Z}[G]$ and $\mathbb{Z}[G^n]$ as G-modules. We used this in our definition of the standard resolution of $\mathbb{Z}$ by G-modules, and in our definition of induced G-modules $\mathbb{Z}[G]\otimes_{\mathbb{Z}} A$ (which we used for dimension shifting to the left). But if G is an infinite profinite group, then $\mathbb{Z}[G]$ is *not* a discrete G-module, because every stabilizer is trivial and therefore not an open subgroup of G if G is infinite.

One can still define the notion of a coinduced discrete G-module: even though $\mathbb{Z}[G]$ is not a discrete G-module, the group of continuous $\mathbb{Z}$-module homomorphisms $\varphi\colon \mathbb{Z}[G]\to A$ with G-action defined as before by $g\varphi(z)=\varphi(zg)$ is a discrete G-module (the key point is that $g\varphi$ actually only depends on g modulo an open subgroup of G, because this is true of the continuous homomorphism φ). But in general the G-module $\mathbb{Z}[G]\otimes_{\mathbb{Z}} A$ does not have a natural structure as a discrete G-module.[1]

The fact that $\mathbb{Z}[G^n]$ is in general not a discrete G-module makes it difficult to directly define homology (and Tate cohomology) for discrete G-modules. But one can instead use coinflation to construct maps $H_n(G/V,A^V)\to H_n(G/U,A^U)$ for open normal subgroups $V\subseteq U$ of G and $n\ge 0$ and then define $H_n(G,A)$ for $n\ge 0$ as an inverse limit. We refer the interested reader to [2, §9].

30.2 The invariant map

Throughout this section K is a nonarchimedean local field. If L/K is any Galois extension (possibly infinite), then $\operatorname{Gal}(L/K)$ is a profinite group G, and the G-modules $L^\times$ and $\mathcal{O}_L^\times$ are both discrete G-modules, since any $\alpha\in L^\times$ generates a finite extension $K(\alpha)/K$ that is the fixed field of a finite index closed subgroup of G (by the fundamental theorem of Galois theory, see Theorem 26.22. A finite index closed subgroup is open, so every stabilizer G_α is open, which implies that $L^\times$ and $\mathcal{O}_L^\times$ are discrete G-modules, by Lemma 30.4.

Recall that every finite unramified extension of nonarchimedean local fields is cyclic (and in particular, Galois), with Galois group isomorphic to that of the residue field extension; this follows from Theorem 10.15 and the fact that finite extensions of finite fields are cyclic. In general, every unramified extension of local fields is a Galois extension generated by roots of unity, by Corollary 10.19.

To simplify the notation, we may write N for the norm map $\mathrm{N}_{L/K}\colon L^\times\to K^\times$ when the extension L/K is clear from context.

Proposition 30.15. *Let L/K be a finite unramified extension of nonarchimedean local fields with Galois group $G:=\operatorname{Gal}(L/K)$. Then $\hat H^n(G,\mathcal{O}_L^\times)=0$ for all $n\in\mathbb{Z}$. Moreover, $\hat H^n(H,\mathcal{O}_L^\times)=0$ for all $n\in\mathbb{Z}$ and all subgroups $H\le G$.*

Proof. Since L/K is unramified, the inertia group is trivial and G is isomorphic to the residue field extension, which is cyclic of order $m:=[L:K]$. It suffices to prove the proposition for $n=0,1$, since for cyclic groups G the Tate cohomology of every G-module is 2-periodic (Theorem 23.37). For any choice of uniformizer π for $\mathcal{O}_L$ we have an isomorphism

$$\begin{aligned} L^\times &\xrightarrow{\sim} \mathcal{O}_L^\times\times\pi^{\mathbb{Z}}\\ x&\mapsto (x/\pi^{v(x)},\pi^{v(x)}),\end{aligned}$$

where $v\colon L^\times\to\mathbb{Z}$ is the valuation of L. Since L/K is unramified, the valuation of L extends the valuation of K with index 1, by Theorem 8.20, so every uniformizer for $\mathcal{O}_K$ is also a uniformizer for $\mathcal{O}_L$, and we can assume $\pi\in\mathcal{O}_K$. In this case G acts trivially on $\pi^{\mathbb{Z}}$, allowing

[1] This is one reason why many authors use the term induced for what we have been calling coinduced.

us to view $L^\times$ as the direct sum of the G-module $\mathcal{O}_L^\times$ and the trivial G-module $\pi^{\mathbb{Z}} \simeq \mathbb{Z}$. We therefore have

$$\hat{H}^n(G, L^\times) \simeq \hat{H}^n(G, \mathcal{O}_L^\times) \oplus \hat{H}^n(G, \mathbb{Z}),$$

for all $n \in \mathbb{Z}$, where the G-action on $\mathbb{Z}$ is trivial, by Corollary 23.33. We have $\hat{H}^1(G, L^\times) = 0$, by Hilbert 90, so $\hat{H}^1(G, \mathcal{O}_L^\times) = 0$.

For $r \geq 1$ let $U_L^r = 1 + \mathfrak{q}^r$ and let $U_K^r = 1 + \mathfrak{p}^r$, where $\mathfrak{q} = (\pi_L)$ is the maximal ideal of $\mathcal{O}_L$ and $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_K$. Let $\ell := \mathcal{O}_L/\mathfrak{q}$ and $k := \mathcal{O}_K/\mathfrak{p}$ be the residue fields, so that $G \simeq \mathrm{Gal}(l/k)$. For $r \geq 1$ we have G-module isomorphisms

$$\mathcal{O}_L^\times / U_L^1 \xrightarrow{\sim} \ell^\times \qquad \text{and} \qquad U_L^r / U_L^{r+1} \xrightarrow{\sim} \ell$$

defined by $u \mapsto u \bmod \mathfrak{q}$ and $1 + a\pi^r \mapsto a \bmod \mathfrak{q}$. Now $H^1(G, \ell^\times) = 0$ by Hilbert 90, and the Herbrand quotient $\hat{H}^0((\ell^\times)/\hat{H}^1(\ell^\times) = 1$ since $\ell^\times$ is finite (by Lemma 23.43), so $\hat{H}^0(G, \ell^\times) = 0$, and $\hat{H}^0(G, \ell) = 0$, by 24.1, thus the norm and trace maps of the residue field extension ℓ/k are both surjective. If we now consider the commutative diagrams

$$\begin{array}{ccc} \mathcal{O}_L^\times & \xrightarrow{\bmod \mathfrak{q}} & \ell^\times \\ \downarrow{\scriptstyle \mathrm{N}} & & \downarrow{\scriptstyle \mathrm{N}} \\ \mathcal{O}_K^\times & \xrightarrow{\bmod \mathfrak{p}} & k^\times \end{array} \qquad\qquad \begin{array}{ccc} U_L^r & \xrightarrow{\bmod \mathfrak{q}} & \ell \\ \downarrow{\scriptstyle \mathrm{N}} & & \downarrow{\scriptstyle \mathrm{T}} \\ U_K^r & \xrightarrow{\bmod \mathfrak{p}} & k \end{array}$$

the surjectivity of $\mathrm{N}\colon \ell^\times \to k^\times$ and $\mathrm{T}\colon \ell \to k$ imply the surjectivity of $\mathrm{N}\colon \mathcal{O}_L^\times \to \mathcal{O}_K^\times$. Indeed, given $u \in \mathcal{O}_K^\times$ we can pick $v_1 \in \mathcal{O}_L^\times$ such that $\mathrm{N}(v_1 \bmod \mathfrak{q}) = u \bmod \mathfrak{p}$, which implies $u/\mathrm{N}(v_1) \in U_K^1$. We can then pick $w_2 \in U_L^1$ such that $\mathrm{N}(w_2) \equiv u/\mathrm{N}(v_1) \bmod U_K^2$, and if we put $v_2 = v_1 w_2$ then $u/\mathrm{N}(v_2) \in U_K^2$. Continuing in this fashion, we can construct v_r such $u/\mathrm{N}(v_r) \in U_K^r$ for all $r \geq 1$, and if we let $v \in \mathcal{O}_L^\times$ be the limit of the v_r then $u/\mathrm{N}(v) \in \cap_r U_K^r = \{1\}$ and $\mathrm{N}(v) = u$. It follows that $\hat{H}^0(G, \mathcal{O}_L^\times) = \mathcal{O}_K^\times/\mathrm{N}(\mathcal{O}_L^\times)$ is trivial. This proves the first conclusion of the proposition.

The second follows, since for any subgroup H of G we have $H = \mathrm{Gal}(L/L^H)$ and may apply the first conclusion to the finite unramified extension L/L^H. $\square$

The "moreover" part of Proposition 30.15 says that the unit group of a finite unramified extension of local fields is *cohomologically trivial*.

Definition 30.16. Let G be a finite group. A G-module A is *cohomologically trivial* if $\hat{H}^n(H, A) = 0$ for all $n \in \mathbb{Z}$ and subgroups $H \leq G$.

Remark 30.17. G-modules A for which $\hat{H}^n(G, A) = 0$ for all $n \in \mathbb{Z}$ are said to be *acyclic*. In general, acyclic G-modules need not be cohomologically trivial (see Problem Set 1).

Corollary 30.18. *For any finite unramified extension of nonarchimedean local fields L/K the norm map $\mathcal{O}_L^\times \to \mathcal{O}_K^\times$ is surjective.*

Proof. By Proposition 30.15, $\hat{H}^0(\mathrm{Gal}(L/K), \mathcal{O}_L^\times) = (\mathcal{O}_L^\times)^G / N_G \mathcal{O}_L^\times = \mathcal{O}_K^\times/\mathrm{N}(\mathcal{O}_L^\times) = 0$. $\square$

Corollary 30.19. *Let L/K be a not necessarily finite unramified extension of a nonarchimedean local field K with Galois group G. Then $H^n(G, \mathcal{O}_L^\times) = 0$ for all $n > 0$*

Proof. For every open normal subgroup N of G, the fixed field L^N is an unramified finite extension of K with $\mathrm{Gal}(L^N/K) \simeq G/N$, and $H^n(G/N, (\mathcal{O}_L^\times)^N) = 0$ for all $n > 0$, by Proposition 30.15. The corollary then follows from Theorem 30.9. $\square$

Now let L/K be an unramified extension of of a nonarchimedean local field (possibly infinite), let $G = \mathrm{Gal}(L/K)$, and let $v\colon L^\times \to \mathbb{Z}$ be the discrete valuation of L. Consider the following exact sequence of discrete G-modules

$$1 \longrightarrow \mathcal{O}_L^\times \longrightarrow L^\times \stackrel{v}{\longrightarrow} \mathbb{Z} \longrightarrow 0,$$

where $\mathbb{Z}$ is a trivial G-module (because $v(\sigma\alpha) = v(\alpha)$ for all $\alpha \in L^\times$, by Lemma 11.9). By Corollary 30.19, both $H^1(G, \mathcal{O}_L^\times)$ and $H^2(G, \mathcal{O}_L^\times)$ are trivial, so the corresponding long exact sequence in cohomology (by Lemma 30.7) contains an isomorphism

$$v\colon H^2(G, L^\times) \stackrel{\sim}{\longrightarrow} H^2(G, \mathbb{Z}).$$

If we now consider the exact sequence of trivial discrete G-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

we have $H^n(G, \mathbb{Q}) = 0$ for all $n > 0$ (by Corollary 29.12 and Lemma 30.9), and the long exact sequence in cohomology thus contains an isomorphism

$$\delta\colon H^1(G, \mathbb{Q}/\mathbb{Z}) \stackrel{\sim}{\longrightarrow} H^2(G, \mathbb{Z}),$$

where δ is a connecting homomorphism (coming from the snake lemma). We also observe that for the trivial discrete G-module $\mathbb{Q}/\mathbb{Z}$, the group $H^1(G, \mathbb{Q}/\mathbb{Z})$ is simply the group of continuous homomorphisms $f\colon G \to \mathbb{Q}/\mathbb{Z}$.

Recall that for any finite unramified extension L/K of nonarchimedean local fields, the Galois group $\mathrm{Gal}(L/K)$ has a canonical generator, the Frobenius element $\mathrm{Frob}_{L/K}$, defined as the unique element of $\mathrm{Gal}(L/K)$ that induces the Frobenius automorphism of the residue field extension ℓ/k (so $\mathrm{Frob}_{L/K}(x) \equiv rmx^{\#k} \bmod \mathfrak{q}$ where $\mathfrak{q}$ is the maximal ideal of $\mathcal{O}_L$). For infinite unramified extensions, $\mathrm{Frob}_{L/K}$ is the unique element of the profinite group $\mathrm{Gal}(L/K)$ whose restriction to finite E/K is $\mathrm{Frob}_{E/K}$.

Definition 30.20. Let L/K be an unramified extension of nonarchimedean local fields with Galois group $G = \mathrm{Gal}(L/K)$ and Frobenius element $\sigma \coloneqq \mathrm{Frob}_{L/K}$. The *invariant map* $\mathrm{inv}_{L/K}\colon H^2(G, L^\times) \to \mathbb{Q}/\mathbb{Z}$ is defined by the composition

$$H^2(G, L^\times) \stackrel{v}{\longrightarrow} H^2(G, \mathbb{Z}) \stackrel{\delta^{-1}}{\longrightarrow} H^1(G, \mathbb{Q}/\mathbb{Z}) \stackrel{f \mapsto f(\sigma)}{\longrightarrow} \mathbb{Q}/\mathbb{Z}.$$

The invariant map $\mathrm{inv}_{L/K}$ is functorial in L; that is, for any tower of unramified extensions of nonarchimedean local fields $K \subseteq L \subseteq M$ the following diagram commutes:

$$\begin{array}{ccc} H^2(\mathrm{Gal}(L/K), L^\times) & \xrightarrow{\mathrm{inv}_{L/K}} & \mathbb{Q}/\mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{Inf}} & & \| \\ H^2(\mathrm{Gal}(M/K), M^\times) & \xrightarrow{\mathrm{inv}_{M/K}} & \mathbb{Q}/\mathbb{Z} \end{array}$$

This follows from the functoriality of the inflation map and the maps used to define $\mathrm{inv}_{L/K}$.

We now fix a separable closure K^{sep} of our nonarchimedean local field K and let K^{unr} denote the maximal unramified extension of K in K^{sep}.

Theorem 30.21. *Let K be a nonarchimedean local field. There is a unique isomorphism*

$$\mathrm{inv}_K\colon H^2(\mathrm{Gal}(K^{\mathrm{unr}}/K), K^{\mathrm{unr}\times}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

such that for all finite Galois extensions L/K in K^{unr}, composition with the inflation map $\mathrm{Inf}\colon H^2(\mathrm{Gal}(L/K), L^\times) \to H^2(\mathrm{Gal}(K^{\mathrm{unr}}/K), K^{\mathrm{unr}\times})$ *induces the isomorphism*

$$\mathrm{inv}_{L/K}\colon H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow{\sim} \tfrac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}.$$

Proof. For every unramified extension L/K the invariant map $\mathrm{inv}_{L/K}$ is injective, since the maps v and δ^{-1} that appear in its definition are isomorphisms, and the map $f \to f(\sigma)$ is injective because every continuous homomorphism $f\colon G \to \mathbb{Q}/\mathbb{Z}$ is determined by $f(\sigma)$ (both when L/K is finite and when it is not).

If L/K is a finite unramified extension then $G := \mathrm{Gal}(L/K) = \langle\sigma\rangle$, where $\sigma := \mathrm{Frob}_{L/K}$ has order $[L\colon K]$. We have an $f \in H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ defined by $f(\sigma) = \frac{1}{[L:K]}$, so the image of $\mathrm{inv}_{L/K}$ contains the group $\frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$ of order $m := [L : K] = \#G$, and this containment must be an equality because $H^1(G, \mathbb{Q}/\mathbb{Z}) \simeq H^2(G, \mathbb{Z}) \simeq \hat{H}^0(G/\mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z}$. It follows that $\mathrm{inv}_{L/K}\colon H^2(\mathrm{Gal}(L/K), L^\times) \to \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$ is an isomorphism, and that $\mathrm{inv}_{K^{\mathrm{unr}}/K}$ is an isomorphism, since K has unramified extensions of degree n for every $n \in \mathbb{Z}$, by Corollary 10.19.

The map $\mathrm{inv}_K := \mathrm{inv}_{K^{\mathrm{unr}}/K}$ induces all the invariant maps $\mathrm{inv}_{L/K}$ for finite unramified extensions L/K, by the functoriality noted above, follows that inv_K has all the properties stated in the theorem. We now observe that for $G := \mathrm{Gal}(K^{\mathrm{unr}}/K)$ we have $H^2(G, K^{\mathrm{unr}\times}) \simeq \varinjlim_H H^2(G/H, K^{\mathrm{unr}\times})^H$, where H ranges over open normal subgroups of G, by Theorem 30.9. It follows that inv_K is uniquely determined by the maps $\mathrm{inv}_{L/K}$ it induces, where $L := (K^{\mathrm{unr}})^H$ ranges over all finite unramified extensions as H ranges over all open normal subgroups of G. □

Proposition 30.22. *Let L/K be a (not necessarily unramified) finite separable extension of nonarchimedean local fields of degree n. There is a canonical homomorphism ϕ such that the following diagram commutes*

$$\begin{array}{ccc} H^2(\mathrm{Gal}(K^{\mathrm{unr}}/K), K^{\mathrm{unr}\times}) & \xrightarrow{\phi} & H^2(\mathrm{Gal}(L^{\mathrm{unr}}/L), L^{\mathrm{unr}\times}) \\ \downarrow \mathrm{inv}_K & & \downarrow \mathrm{inv}_L \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow{[n]} & \mathbb{Q}/\mathbb{Z}. \end{array}$$

where $[n]$ denotes the multiplication by n map. When L/K is Galois, the kernel of ϕ can be canonically identified with a subgroup of $H^2(\mathrm{Gal}(L/K), L^\times)$ isomorphic to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Proof. We have $L^{\mathrm{unr}} = L \cdot K^{\mathrm{unr}}$, since unramified extensions of local fields are obtained by adjoining roots of unity (see Corollary 10.19). We first consider the case that L/K is a Galois extension. We have $H^1(\mathrm{Gal}(L^{\mathrm{unr}}/K), L^{\mathrm{unr}\times}) = 0$ by Corollary 30.11, so Theorem 30.13 gives us an exact sequence

$$0 \longrightarrow H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow{\mathrm{Inf}} H^2(\mathrm{Gal}(L^{\mathrm{unr}}/K), L^{\mathrm{unr}\times}) \xrightarrow{\mathrm{Res}} H^2(\mathrm{Gal}(L^{\mathrm{unr}}/L), L^{\mathrm{unr}\times}),$$

We similarly obtain an exact sequence

$$0 \longrightarrow H^2(\mathrm{Gal}(K^{\mathrm{unr}}/K), K^{\mathrm{unr}\times}) \xrightarrow{\mathrm{Inf}'} H^2(\mathrm{Gal}(L^{\mathrm{unr}}/K), L^{\mathrm{unr}\times}) \xrightarrow{\mathrm{Res}'} H^2(\mathrm{Gal}(L^{\mathrm{unr}}/K^{\mathrm{unr}}), L^{\mathrm{unr}\times}).$$

We now define $\phi := \text{Res} \circ \text{Inf}'$, which we note still makes sense even if L/K is not Galois (the maps Res and Inf′ are always defined, even if Inf is not). When L/K is Galois we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(\phi) & \longrightarrow & H^2(\text{Gal}(K^{\text{unr}}/K), K^{\text{unr}\times}) & \xrightarrow{\phi} & H^2(\text{Gal}(L^{\text{unr}}/L), L^{\text{unr}\times}) \\ & & \downarrow\varphi & & \downarrow\text{Inf}' & & \| \\ 0 & \longrightarrow & H^2(\text{Gal}(L/K), L^\times) & \xrightarrow{\text{Inf}} & H^2(\text{Gal}(L^{\text{unr}}/K), L^{\text{unr}\times}) & \xrightarrow{\text{Res}} & H^2(\text{Gal}(L^{\text{unr}}/L), L^{\text{unr}\times}) \end{array}$$

which canonically defines the injective map $\varphi\colon \ker(\phi) \to H^2(\text{Gal}(L/K), L^\times)$ given by the inclusion $\text{Inf}'(\ker\phi) \subseteq \ker(\text{Res}) = \text{im}(\text{Inf})$ when L/K is Galois

We now drop the assumption that L/K is Galois. The discrete valuation v_L of L extends the discrete valuation v_K of K with index equal to the ramification index e of the extension L/K, by Theorem 8.20. Let $\sigma_K := \text{Frob}_K$ and $\sigma_L := \text{Frob}_L$. Let f be the degree of the associated residue field extension, so that $n = [L:K] = ef$.

Writing out the definitions of inv_K and inv_L yields the following commutative diagram:

$$\begin{array}{ccccccc} H^2(\text{Gal}(K^{\text{unr}}/K), K^{\text{unr}\times}) & \xrightarrow{v_K} & H^2(\text{Gal}(K^{\text{unr}}/K), \mathbb{Z}) & \xrightarrow{\delta^{-1}} & H^1(\text{Gal}(K^{\text{unr}}/K), \mathbb{Q}/\mathbb{Z}) & \xrightarrow{g\mapsto g(\sigma_K)} & \mathbb{Q}/\mathbb{Z} \\ \downarrow\phi & & \downarrow[e]\circ\phi & & \downarrow[e]\circ\phi & & \downarrow[ef] \\ H^2(\text{Gal}(L^{\text{unr}}/L), L^{\text{unr}\times}) & \xrightarrow{v_L} & H^2(\text{Gal}(L^{\text{unr}}/L), \mathbb{Z}) & \xrightarrow{\delta^{-1}} & H^1(\text{Gal}(L^{\text{unr}}/L), \mathbb{Q}/\mathbb{Z}) & \xrightarrow{g\mapsto g(\sigma_L)} & \mathbb{Q}/\mathbb{Z}. \end{array}$$

The commutative square on the left is induced by the commutative square

$$\begin{array}{ccc} K^{\text{unr}\times} & \xrightarrow{v_K} & \mathbb{Z} \\ \downarrow & & \downarrow[e] \\ L^{\text{unr}\times} & \xrightarrow{v_L} & \mathbb{Z} \end{array}$$

where the vertical map on the left is inclusion; note that $\mathbb{Z} \xrightarrow{[e]} \mathbb{Z}$ is a morphism of trivial discrete G-modules that is also a morphism of $\mathbb{Z}$-modules and thus induces the same map in cohomology (multiplication by e). The commutative square in the middle is just a pair of isomorphisms, and the commutative square on the right follows from the fact that for $g \in H^1(\text{Gal}(K^{\text{unr}}/K), \mathbb{Q}/\mathbb{Z})$, we have $g(\sigma_L) = g(\sigma_K^f) = fg(\sigma_K)$, since g is a homomorphism because $\mathbb{Q}/\mathbb{Z}$ is a trivial $\text{Gal}(K^{\text{unr}}/K)$-module (note that the group operation in $\text{Gal}(K^{\text{unr}}/K)$ is multiplicative while the group operation in $\mathbb{Q}/\mathbb{Z}$ additive).

Having proved that commutativity of the diagram in the statement of the proposition containing isomorphisms inv_K and inv_L, it is clear that $\ker(\phi)$ is isomorphic to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, the kernel of $[n]\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. When L/K is Galois the injective map φ allows us to view $\ker(\phi)$ as a canonical subgroup of $H^2(\text{Gal}(L/K), L^\times)$ isomorphic to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$. □

We now want extend the invariant map inv_K we have defined to arbitrary separable extensions of K. For this we require what Neukirch calls the *class field axiom* [1].

Theorem 30.23 (Class field axiom)**.** *Let L/K be a cyclic extension of nonarchimedean local fields with Galois group $G := \text{Gal}(L/K)$ of order n. Then for all $k \in \mathbb{Z}$ we have*

$$\#\hat{H}^k(G, L^\times) = \begin{cases} n & \text{if } k \equiv 0 \bmod 2 \\ 1 & \text{if } k \equiv 1 \bmod 2. \end{cases}$$

Proof. We have $\hat{H}^k(G, L^\times) \simeq \hat{H}^{k+2}(G, L^\times)$ for all $k \in \mathbb{Z}$, by Theorem 23.37, since G is cyclic, and $\#\hat{H}^1(G, L^\times) = 1$ by Theorem 24.1, so it suffices to show $\#\hat{H}^0(G, L^\times) = n$.

We will construct a cohomologically trivial finite index G-submodule A of $\mathcal{O}_L^\times$, which will obtain from a finite index G-submodule M of $\mathcal{O}_L$. Let $G = \langle\sigma\rangle$, choose $\alpha \in L^\times$ so that $\{\sigma^i\alpha\}$ is a K-basis for L (such an α exists by the normal basis theorem), and pick $a \in K^\times$ so that $a\alpha \in \mathcal{O}_L$ (if $\alpha = \beta/\gamma$ with $\beta, \gamma \in \mathcal{O}_L$ we can take $a = \mathrm{N}_{L/K}(\gamma)$). We then have $z_i := a\sigma^i\alpha \in \mathcal{O}_L$ for $0 \le i < n$, and

$$M := \bigoplus_{i=1}^{n} z_i\mathcal{O}_K \subseteq \mathcal{O}_L$$

is a G-submodule of $\mathcal{O}_L$ isomorphic to $\mathcal{O}_K[G]$ (the isomorphism sends z_i to σ^i).

Let $\mathfrak{p} = (\pi)$ and $\mathfrak{q}$ be the maximal ideals of $\mathcal{O}_K$ and $\mathcal{O}_L$, respectively, and let $m := v_\mathfrak{q}(z_0)$. We have $v_\mathfrak{q}(z_i) = m$ for $0 \le i < n$ (since $v_\mathfrak{q}(\sigma^i\alpha) = v_\mathfrak{q}(\alpha)$), and $v_\mathfrak{q}(\pi^m) \ge v_\mathfrak{q}(z_i)$ for $0 \le i < n$. It follows that $\pi^m\mathcal{O}_L \subseteq M$, so

$$[\mathcal{O}_L : M] \le [\mathcal{O}_L : \pi^m\mathcal{O}_L] = mv_\mathfrak{q}(\pi)\#\mathbb{F}_\mathfrak{q}$$

is finite, and for all $i \ge 0$ we have

$$(1 + \pi^{m+i}M)(1 + \pi^{m+i}M) \subseteq (1 + \pi^{2m+2i})\mathcal{O}_L \subseteq 1 + \pi^{m+i}M.$$

For each $i \ge 0$ we thus have multiplicative G-modules

$$A_i := 1 + \pi^{m+i}M \subseteq \mathcal{O}_L^\times,$$

that each have finite index in $\mathcal{O}_L^\times$, and we have G-modules isomorphisms

$$A_i/A_{i+1} \simeq M/\pi M \simeq \mathbb{F}_\mathfrak{p}[G],$$

where the first is $[a_i] \mapsto [\pi^{-m-i}(a_i - 1)]$ and the second is defined by $[z_i] \mapsto \sigma^i \in \mathbb{F}_\mathfrak{p}[G]$. The G-module $\mathbb{F}_\mathfrak{p}[G]$ is cohomologically trivial (see Problem Set 1), so each A_i/A_{i+1} is cohomologically trivial, as is $A/A_i \simeq M/\pi^iM \simeq R[G]$, where $R = (\mathcal{O}_K/\mathfrak{p}^i)$ and $A := A_0$. By Lemma 30.27 below, $A \simeq \varprojlim_i A/A_i$ is cohomologically trivial, and in particular, the Herbrand quotient of A is trivial, that is

$$h(A) := \#\hat{H}^0(G, A)/\#\hat{H}_0(G, A) = 1,$$

which implies $h(\mathcal{O}_L^\times) = 1$, since A has finite index in $\mathcal{O}_L^\times$, by Corollary 23.48. We now observe that we have an exact sequence of G-modules

$$0 \longrightarrow \mathcal{O}_L^\times \to L^\times \xrightarrow{v_\mathfrak{q}} \mathbb{Z} \longrightarrow 0,$$

where $\mathbb{Z}$ is a trivial G-module with $h(\mathbb{Z}) = n$, by Corollary 23.46, and therefore

$$h(L^\times) = h(\mathcal{O}_L^\times)h(\mathbb{Z}) = n$$

by Corollary 23.41. This implies $\#\hat{H}^0(G, L^\times) = h(L^\times)\#\hat{H}_0(G, L^\times) = n \cdot 1$, since we already noted that $\hat{H}_0(G, L^\times) = \hat{H}^{-1}(G, L^\times) \simeq \hat{H}^1(G, L^\times)$ is trivial. □

Corollary 30.24. *Let L/K be a finite Galois extension of nonarchimedean local fields. Then $H^2(\mathrm{Gal}(L/K), L^\times)$ is cyclic of order $n = [L:K]$.*

Proof. We first note that the extension L/K is solvable: the maximal unramified subextension of L is a cyclic extension of K, by Theorem 10.15, the maximal tamely ramified subextension of M is a Galois extension of K that is a Kummer extension of the maximal unramified subextension, by Theorem 11.8, and L/M is either trivial or totally wildly ramified, in which case $\mathrm{Gal}(L/M)$ is a p-group, hence solvable.

We proceed by induction on n. If L/K is cyclic the corollary follows from Theorem 30.23. Otherwise $n > 1$ and we have a tower $K \subsetneq E \subsetneq L$ of Galois extensions with $\#H^2(\mathrm{Gal}(L/E), L^\times) = [L:E]$ and $\#H^2(\mathrm{Gal}(E/K), E^\times) = [E:K]$ by the inductive hypothesis. We have $H^1(\mathrm{Gal}(L/K), L^\times) = 0$ (by Hilbert 90), so Theorem 29.23 gives us an exact inflation-restriction sequence:

$$0 \longrightarrow H^2(\mathrm{Gal}(E/K), E^\times) \xrightarrow{\mathrm{Inf}} H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow{\mathrm{Res}} H^2(\mathrm{Gal}(L/E), L^\times),$$

and therefore

$$\begin{aligned}\#H^2(\mathrm{Gal}(L/K), L^\times) &\le \#H^2(\mathrm{Gal}(E/K), E^\times) \cdot \#H^2(\mathrm{Gal}(L/E), L^\times)\\ &\le [E:K]\cdot[L:E] = [L:K] = n.\end{aligned}$$

By Proposition 30.22, $H^2(\mathrm{Gal}(L/K), L^\times)$ contains a cyclic subgroup of order n, and by the inequality above, it must equal this subgroup. □

We now show that Theorem 30.21 and Proposition 30.22 still hold when K^{unr} is replaces by K^{sep}, inducing compatible invariant maps for all Galois extensions L/K.

Theorem 30.25. *Let K be a nonarchimedean local field. There is a unique isomorphism*

$$\mathrm{inv}_K \colon H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), K^{\mathrm{sep}\times}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

such that for all finite Galois extensions L/K in K^{sep}, composition with the inflation map $\mathrm{Inf}\colon H^2(\mathrm{Gal}(L/K), L^\times) \to H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), K^{\mathrm{sep}\times})$ *induces an isomorphism*

$$\mathrm{inv}_{L/K} \colon H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow{\sim} \tfrac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$$

that coincides with the invariant map $\mathrm{inv}_{L/K}$ *previously defined when L/K is unramified. Moreover, for every finite separable extension L/K the following diagram commutes*

$$\begin{array}{ccc} H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), K^{\mathrm{sep}\times}) & \xrightarrow{\mathrm{Res}} & H^2(\mathrm{Gal}(K^{\mathrm{sep}}/L), K^{\mathrm{sep}\times}) \\ \wr\downarrow \mathrm{inv}_K & & \wr\downarrow \mathrm{inv}_L \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow{[n]} & \mathbb{Q}/\mathbb{Z}. \end{array}$$

where $[n]$ denotes the multiplication by n map, and when L/K is Galois we have an isomorphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & H^2(\mathrm{Gal}(L/K), L^\times) & \xrightarrow{\mathrm{Inf}} & H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), K^{\mathrm{sep}\times}) & \xrightarrow{\mathrm{Res}} & H^2(\mathrm{Gal}(K^{\mathrm{sep}}/L), K^{\mathrm{sep}\times}) & \to & 0 \\ & & \wr\downarrow \mathrm{inv}_{L/K} & & \wr\downarrow \mathrm{inv}_K & & \wr\downarrow \mathrm{inv}_L & & \\ 0 & \to & \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \xrightarrow{[n]} & \mathbb{Q}/\mathbb{Z} & \to & 0. \end{array}$$

Proof. Corollary 30.24 implies that for all finite Galois extensions L/K the injective map $\varphi\colon \ker(\phi) \to H^2(\mathrm{Gal}(L/K), L^\times)$ constructed in the proof of Proposition 30.22 is actually an isomorphism, allowing us to canonically identify $H^2(\mathrm{Gal}(L/K), L^\times)$ with a subgroup of $H^2(\mathrm{Gal}(K^{\mathrm{unr}}/K), K^{\mathrm{unr}\times})$.

The inflation maps $\mathrm{Inf}\colon H^2(\mathrm{Gal}(L/K), L^\times) \to H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), K^{\mathrm{sep}\times})$ are all injective, by Corollary 30.11 and Theorem 30.13, and $H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), K^{\mathrm{sep}\times})$ is naturally isomorphic to the direct limit of the directed system formed by the inflation maps between finite Galois extensions of K. It follows that the inflation map

$$\mathrm{Inf}\colon H^2(\mathrm{Gal}(K^{\mathrm{unr}}/K), K^{\mathrm{unr}\times}) \to H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), K^{\mathrm{sep}\times})$$

is an isomorphism. This implies we can replace K^{unr} with K^{sep} and L^{unr} with $L^{\mathrm{sep}} = K^{\mathrm{sep}}$ in both Theorem 30.21 and Proposition 30.22, and the homomorphism $\phi = \mathrm{Res} \circ \mathrm{Inf}'$ defined in the proof of Proposition 30.22 reduces to Res because Inf' is the identity map.

The commutativity and exactness of the second diagram follows; note that the restriction map must be surjective because the multiplication map $[n]\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is. ☐

30.3 Cohomological triviality of inverse limits

In this section we prove that an inverse limit of cohomologically trivial G-modules is cohomologically trivial; we used this fact in the proof of the class field axiom (Theorem 30.23). We first note the following theorem, which you will have an opportunity to prove on the problem set.

Theorem 30.26. *Let G be a finite group and let A be a G-module. Then A is cohomologically trivial if and only if for every prime p dividing $\#G$ there is a p-Sylow subgroup $H \le G$ and an integer n such that $\hat{H}^n(H, A) = \hat{H}^{n+1}(H, A) = 0$.*

Proof. See Problem Set 1. ☐

Lemma 30.27. *Let G be a finite group. Suppose we have an inverse system of cohomologically trivial G-modules*

$$\cdots \longrightarrow A_{i+1} \xrightarrow{\alpha_i} A_i \longrightarrow \cdots \longrightarrow A_1 \xrightarrow{\alpha_0} A_0,$$

then the inverse limit $A := \varprojlim_i A_i$ is cohomologically trivial.

Proof. We first note that an inverse limit of G-modules is indeed a G-module: if $(a_i) \in \prod_i A_i$ represents an element of A (so the map $A_{i+1} \to A_i$ sends a_{i+1} to a_i), then G acts componentwise: $g(a_i) = (ga_i)$, and the natural maps $\phi_i\colon A \to A_i$ given by the inverse limit are morphisms of G-modules since $g\phi_i((a_i)) = ga_i = \phi_i((ga_i)) = \phi_i(g(a_i))$.

By Theorem 30.26, it suffices to prove that $\hat{H}^n(H, A) = \hat{H}^{n+1}(H, A) = 0$ for some $n \in \mathbb{Z}$, for every subgroup H of G (in fact we only need to consider p-Sylow subgroups H, but this makes no difference to us). So let H be a subgroup of G. Then $\hat{H}^n(H, A_i) = 0$ for all $n \in \mathbb{Z}$ and $i \ge 0$, since the A_i are cohomologically trivial.

For each $n > 0$ and subgroup $H \le G$ our hypothesis implies equality of coboundaries and cocycles: $B^n(H, A_i) = Z^n(H, A_i)$ for all $i \ge 0$. By Lemma 23.6, the G-morphisms α_i induce a compatible system of morphisms of cochains $\alpha_i^n\colon C^n(H, A_{i+1}) \to C^n(H, A_i)$ defined by $f \mapsto \alpha_i \circ f$ that commute with the boundary operators. We claim that

$$C^n(H, A) \simeq \varprojlim_i C^n(H, A_i).$$

Indeed, we have inverse isomorphisms $f \mapsto (\phi_i \circ f)$ and $(f_i) \mapsto (\vec{g} \mapsto (f_i(\vec{g}))$ (note that $\alpha_i(f_{i+1}(\vec{g})) = f_i(\vec{g})$ for $i \geq 0$, so the second map is well defined). These isomorphisms commute with the boundary operators because the morphisms ϕ_i and α_i do. Therefore $B^n(H, A) = Z^n(H, A)$, since $B^n(H, A_i) = Z^n(H, A_i)$ for all $i \geq 0$. The lemma follows. □

Remark 30.28. In general we do not have $\hat{H}^n(G, \varprojlim_i A_i) = \varprojlim_i \hat{H}^n(G, A_i)$; the key issue is that the inverse limit functor is only left exact. But if one assumes that the connecting morphisms in the inverse system and the induced morphisms in Tate cohomology are all surjective, then the Tate cohomology functors will commutes with inverse limits.

References

[1] J. Neukirch, *Class field theory*, Springer, 1986

[2] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, Springer, 2008.