29 Tate's theorem

Over the next several lectures our goal is to complete the proof of class field theory that we started in 18.785, beginning with local class field theory. In Lecture 27 we presented the local version of Artin reciprocity which we restate below. As usual, K^{ab} denotes the maximal abelian extension of K inside a fixed separable closure of K.

Theorem 29.1 (Local Artin Reciprocity)**.** *Let K be a local field. There is a unique continuous homomorphism*

$$\theta_K\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

with the property that for each finite extension L/K in K^{ab}, the homomorphism

$$\theta_{L/K}\colon K^\times \to \mathrm{Gal}(L/K)$$

given by composing θ_K with the natural map $\mathrm{Gal}(K^{\mathrm{ab}}/K) \twoheadrightarrow \mathrm{Gal}(L/K)$ satisfies:

- *if K is nonarchimedean and L/K is unramified then $\theta_{L/K}(\pi) = \mathrm{Frob}_{L/K}$ for every uniformizer π of $\mathcal{O}_K$;*
- *$\theta_{L/K}$ is surjective with kernel $\mathrm{N}_{L/K}(L^\times)$, inducing $K^\times/\mathrm{N}_{L/K}(L^\times) \simeq \mathrm{Gal}(L/K)$.*

It is easy to see that Theorem 29.1 holds when K is archimedean: either $L = K$ and the theorem is trivially true or $L = \mathbb{C}$ and $K = \mathbb{R}$, in which case $\theta_{L/K}$ maps $\mathrm{N}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) = \mathbb{R}_{>0}$ to the trivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and maps $\mathbb{R}_{<0}$ to the non-trivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

To treat the nonarchimedean case we will use a cohomological result of Tate to explicitly construct the isomorphism $K^\times/\mathrm{N}_{L/K}(L^\times) \simeq \mathrm{Gal}(L/K)$ induced by $\theta_{L/K}$ as an isomorphism of cohomology groups. This isomorphism will be functorial in L, allowing us to extend it to a continuous homomorphism $\theta_K\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$, since we can view $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ as the profinite completion of the groups $\mathrm{Gal}(L/K)$ as L ranges over finite abelian extensions of K, via Theorem 26.21.

We have already seen that $K^\times/\mathrm{N}_{L/K}(L^\times)$ arises naturally as the Tate cohomology group $\hat{H}^0(\mathrm{Gal}(L/K), L^\times)$; see Theorem 24.1. Our main task is to show that we may also view $\mathrm{Gal}(L/K)$ as a Tate cohomology group that is naturally isomorphic to $\hat{H}^0(\mathrm{Gal}(L/K), L^\times)$. If L/K is unramified, the two groups are clearly isomorphic, since they are both cyclic groups of order $[L:K]$, but we want to make this isomorphism explicit and functorial.

In this lecture we will focus on proving Tate's theorem, which is the key cohomological result that we need. In the next lecture we will use it to prove Theorem 29.1. We assume the reader is familiar with the material on Tate cohomology in Section 23.4.

29.1 Shapiro's Lemma

In order to prove Tate's theorem we will need some standard results from group cohomology that allow us to relate the cohomoglogy of subgroups $H \le G$ to the cohomology of G. Recall that a G-module is an abelian group equipped with a G-action, equivalently, a module over the ring $\mathbb{Z}[G]$ (unless otherwise specified, we put the G-action on the left so that A is a left $\mathbb{Z}[G]$-module).

If A is a G-module then A is also an H-module, for any subgroup H of G; this is known as *restriction of scalars*, and the corresponding H-module may be denoted $\mathrm{Res}^G_H(A)$, although we will typically just write A when it is clear that we are viewing A as an H-module. Conversely, given an H-module A we can use induction or co-induction to construct a corresponding G-module.

Definition 29.2. Let H be a subgroup of G. For any H-module A we have G-modules

$$\operatorname{Ind}_H^G(A) \coloneqq \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A \qquad \text{and} \qquad \operatorname{CoInd}_H^G(A) \coloneqq \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], A),$$

where the G-action on $\operatorname{Ind}_H^G(A)$ is given by $g(\alpha \otimes a) \coloneqq g\alpha \otimes a$ and the G-action on $\operatorname{CoInd}_H^G(A)$ is given by $g\varphi = (\alpha \mapsto \varphi(\alpha g))$.[1]

Equivalently, $\operatorname{Ind}_H^G(A)$ is the $\mathbb{Z}[G]$-module obtained from the $\mathbb{Z}[H]$-module A by extension of scalars via the ring homomorphism $\mathbb{Z}[H] \to \mathbb{Z}[G]$ induced by the inclusion $H \hookrightarrow G$, and $\operatorname{CoInd}_H^G(A)$ is similarly obtained by co-extension of scalars.[2]

We have already seen this construction in the case that H is the trivial group, see Definitions 23.17 and 23.25, and we proved that whenever G is finite we have a canonical isomorphism $\operatorname{Ind}^G(A) \simeq \operatorname{CoInd}^G(A)$; see Lemma 23.27. This holds more generally.

Lemma 29.3. *Let G be a group with a finite index subgroup H, and let A be an H-module. We have a canonical isomorphism of G-modules*

$$\begin{aligned} \operatorname{CoInd}_H^G(A) &\xrightarrow{\sim} \operatorname{Ind}_H^G(A) \\ \varphi &\mapsto \sum_{Hg \in H\backslash G} g^{-1} \otimes \varphi(g) \\ (g^{-1} \mapsto a) &\mapsfrom g \otimes a \end{aligned}$$

where the sum is over a set of right coset representatives Hg of H in G.

Proof. We first note that for $h \in H$, $g \in G$ and any $\varphi \in \operatorname{CoInd}_H^G(A)$ we have

$$(hg)^{-1} \otimes \varphi(hg) = g^{-1}h^{-1} \otimes h\varphi(g) = g \otimes \varphi(g),$$

which shows that the isomorphism is well defined, independent of the choice of right coset representatives. The rest of the proof is identical to the proof of Lemma 23.27. $\square$

Remark 29.4. Recall that an *induced* G-module is one of the form $\operatorname{Ind}^G(A)$ (or $\operatorname{CoInd}^G(A)$, if G is finite), where A is an abelian group. If H is a non-trivial subgroup of G and A is an H-module the G-module $\operatorname{Ind}_H^G(A)$ is typically not an induced G-module in this sense.

Lemma 29.5 (Shapiro's Lemma)**.** *Let H be a subgroup of G and let A be an H-module. For all $n \geq 0$ we have canonical isomorphisms*

$$H_n(G, \operatorname{Ind}_H^G(A)) \simeq H_n(H, A) \qquad \textit{and} \qquad H^n(G, \operatorname{CoInd}_H^G(A)) \simeq H^n(H, A).$$

If H has finite index in G then we also have canonical isomorphisms

$$H^n(G, \operatorname{Ind}_H^G(A)) \simeq H^n(H, A) \qquad \textit{and} \qquad H_n(G, \operatorname{CoInd}_H^G(A)) \simeq H_n(H, A).$$

Proof. By the associativity of tensor products, for all $n \geq 0$ we have natural isomorphisms

$$\mathbb{Z}[G^n] \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A) \simeq (\mathbb{Z}[G^n] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]) \otimes_{\mathbb{Z}[H]} A \simeq \mathbb{Z}[G^n] \otimes_{\mathbb{Z}[H]} A, \tag{1}$$

[1]Note that $g(h\varphi) = g(\alpha \mapsto \varphi(\alpha h)) = (\alpha \mapsto \varphi(\alpha gh)) = (gh)\varphi$ as required. It is the *right* action of $\mathbb{Z}[G[$ on itself that induces a *left* action of $\mathbb{Z}[G]$ on $\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], A)$.

[2]Extension and co-extension of scalars are the left and right adjoint functors of restriction of scalars.

of abelian groups, where we are viewing $\mathbb{Z}[G^n]$ as both a left and right $\mathbb{Z}[G]$-module, and as a right $\mathbb{Z}[H]$-module. These isomorphisms maps $\alpha\otimes(\beta\otimes a)$ to $\alpha\beta\otimes a$ and thus commute with the left G-action: $g(\alpha\otimes(\beta\otimes a)) = g\alpha\otimes(\beta\otimes a)\mapsto g\alpha\beta\otimes a = g(\alpha\beta\otimes a)$.

The $\mathbb{Z}[H]$-module $\mathbb{Z}[G^n]$ is free, hence projective, so the standard resolution of $\mathbb{Z}$ by G-modules is also a projective resolution of $\mathbb{Z}$ by H-modules. It follows that

$$H_n(G,\operatorname{Ind}_H^G(A)) = \operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},\operatorname{Ind}_H^G(A)) \simeq \operatorname{Tor}_n^{\mathbb{Z}[H]}(\mathbb{Z},A) = H_n(H,A)$$

for all $n\ge 0$, since (1) implies that computing $\operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},\operatorname{Ind}_H^G(A))$ using the standard resolution of $\mathbb{Z}$ by G-modules is the same thing as computing $\operatorname{Tor}_n^{\mathbb{Z}[H]}(\mathbb{Z},A)$ using the standard resolution of $\mathbb{Z}$ by G-modules viewed as a projective resolution of $\mathbb{Z}$ by H-modules.

The isomorphisms of cohomology groups are proved similarly. By tensor-hom adjunction we have natural isomorphisms

$$\operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^n],\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G],A)) \simeq \operatorname{Hom}(\mathbb{Z}[G^n]\otimes_{\mathbb{Z}[G]}\mathbb{Z}[G],A) \simeq \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G^n],A)$$

that map φ to $(\alpha\mapsto\varphi(\alpha)(1))$ and commute with the left G-action. Indeed, we have $g\varphi = (\alpha\mapsto\varphi(\alpha g))\mapsto(\alpha\mapsto\varphi(\alpha g)(1)) = g(\alpha\mapsto\varphi(\alpha)(1))$. It follows that

$$H^n(G,\operatorname{CoInd}_H^G(A)) = \operatorname{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},\operatorname{CoInd}_H^G(A)) \simeq \operatorname{Ext}^n_{\mathbb{Z}[H]}(\mathbb{Z},A) = H^n(H,A)$$

for all $n\ge 0$. The last statement of the lemma follows from Lemma 29.3. $\square$

29.2 Dimension shifting

Dimension shifting is a technique that simplifies many proofs by exploiting the relationship between a G-module A and the cohomologically trivial G-modules $\operatorname{Ind}^G(A) := \mathbb{Z}[G]\otimes_{\mathbb{Z}} A$ and $\operatorname{CoInd}^G(A) := \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A)$. Recall that for any abelian group A, the G-action on $\mathbb{Z}[G]\otimes_{\mathbb{Z}} A$ is given by $g(z\otimes a) := gz\otimes a$ and the G-action on $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A)$ is $g\varphi := (z\mapsto \varphi(zg))$, as specified in Definitions 23.25 and 23.17, respectively.

When A is also a G-module it is convenient to redefine the G-action on $\operatorname{Ind}^G(A)$ and $\operatorname{CoInd}^G(A)$ in a way that will allow us to view A as a G-module quotient of $\operatorname{Ind}^G(A)$ and a G-submodule of $\operatorname{CoInd}^G(A)$.

Definition 29.6. For a G-module A we define the G-actions on $\operatorname{Ind}^G(A) := \mathbb{Z}[G]\otimes_{\mathbb{Z}} A$ by $g(z\otimes a) := gz\otimes ga$ and on $\operatorname{CoInd}^G(A) := \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A)$ by $g\varphi := (z\mapsto g\varphi(g^{-1}z))$.

Note that Definition 29.6 changes the meaning of $\operatorname{Ind}^G(A)$ and $\operatorname{CoInd}^G(A)$ when A is a G-module. This change is justified by the Lemma 29.7 below, which implies that we can still appeal to all the facts we have previously proved about induced and coinduced G-modules. If we ever need to refer to the old definition we will do so by using the notation A° to distinguish the abelian group A from the G-module A.

Lemma 29.7. *Let A be a G-module and let A° denote the corresponding abelian group. We have G-module isomorphisms*

$$\begin{aligned}\Phi\colon\ \operatorname{Ind}^G(A) &\xrightarrow{\sim} \operatorname{Ind}^G(A^\circ), & \Phi(g\otimes a) &:= g\otimes g^{-1}a\\ \Psi\colon\ \operatorname{CoInd}^G(A) &\xrightarrow{\sim} \operatorname{CoInd}^G(A^\circ), & \Psi(\varphi) &:= (g\mapsto g\varphi(g^{-1}))\end{aligned}$$

Proof. For all $g, h \in G$ and $a \in A$ we have

$$\Phi(g(h \otimes a)) = \Phi(gh \otimes ga) = gh \otimes (gh)^{-1}ga = gh \otimes h^{-1}a = g\,\Phi(h \otimes a),$$

so Φ is a G-morphism, and $\Phi^{-1}(g \otimes a) \coloneqq g \otimes ga$ is an inverse G-morphism. Similarly, for all $g, h \in G$ we have

$$\Psi(g\varphi) = \Psi(h \mapsto g\varphi(g^{-1}h)) = (h \mapsto hg\varphi(g^{-1}h^{-1})) = (h \mapsto \Psi(\varphi)(hg)) = g\Psi(\varphi),$$

and Ψ is its own inverse, since $\Psi(\Psi(\varphi)) = \Psi(g \mapsto g\varphi(g^{-1})) = (g \mapsto gg^{-1}\varphi(g)) = \varphi$. $\square$

Recall that the augmentation ideal $I_G \subseteq \mathbb{Z}[G]$ is the kernel of the augmentation map $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$ defined by $\sum n_g g \mapsto \sum n_g$ (see Definition 23.23), and we have an exact sequence of G-modules

$$0 \longrightarrow I_G \to \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{2}$$

where the G-action on $\mathbb{Z}$ is trivial. As an exact sequence of $\mathbb{Z}$-modules, this sequence splits. Indeed, we have a $\mathbb{Z}$-module isomorphism

$$\begin{aligned} \mathbb{Z}[G] &\simeq_{\mathbb{Z}} I_G \oplus \mathbb{Z} \\ z &\mapsto (z - \varepsilon(z),\, \varepsilon(z)) \\ z + n &\leftarrow\!\shortmid (z, n). \end{aligned}$$

This is not a G-module isomorphism because we are identifying $\mathbb{Z}$ with $1_G\mathbb{Z} \subseteq \mathbb{Z}[G]$, which is not a G-submodule. But $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, A)$ is a G-submodule of $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) = \operatorname{CoInd}^G(A)$.

Lemma 29.8. *Let A be a G-module. The map $\pi\colon \operatorname{Ind}^G(A) \to A$ defined by $z \otimes a \mapsto \varepsilon(z)a$ is a surjective morphism of G-modules with kernel $I_G \otimes A$.*

The map $\iota\colon A \to \operatorname{CoInd}^G(A)$ defined by $a \mapsto (z \mapsto \varepsilon(z)a)$ for $g \in G$ is an injective morphism of G-modules with cokernel isomorphic to $\operatorname{Hom}_{\mathbb{Z}}(I_G, A)$.

Proof. We have $g\pi(z \otimes a) = g\varepsilon(z)a = \varepsilon(z)ga = \varepsilon(gz)ga = \pi(g(z \otimes a))$, and π is clearly surjective (take $z = 1$) and $\mathbb{Z}$-linear, so it is a surjective morphism of G-modules. We now note that $\ker(\pi) = \ker(\varepsilon) \otimes A = I_G \otimes A$ as claimed.

It is clear that ι is an injective morphism of $\mathbb{Z}$-modules, and a morphism of G-modules, since $g\iota(a) = g(z \mapsto \varepsilon(z)a) = (z \mapsto g\varepsilon(g^{-1}z)a) = (z \mapsto \varepsilon(z)ga) = \iota(ga)$. In terms of the $\mathbb{Z}$-isomorphism $\mathbb{Z}[G] \simeq_{\mathbb{Z}} I_G \oplus \mathbb{Z}$ defined above, the image of ι consists precisely of the $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ supported on $\mathbb{Z}$, and it follows that $\operatorname{coker}(\iota) \simeq \operatorname{Hom}_{\mathbb{Z}}(I_G, A)$. $\square$

Lemma 29.8 gives us two exact sequences of G-modules associated to any G-module A.

$$0 \longrightarrow I_G \otimes_{\mathbb{Z}} A \longrightarrow \operatorname{Ind}^G(A) \xrightarrow{\pi} A \longrightarrow 0. \tag{3}$$

$$0 \longrightarrow A \xrightarrow{\iota} \operatorname{CoInd}^G(A) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(I_G, A) \longrightarrow 0. \tag{4}$$

The fact that $\operatorname{Ind}^G(A)$ and $\operatorname{CoInd}^G(A)$ have trivial homology/cohomology in degrees $n \geq 1$, and if G is finite, trivial Tate cohomology in all degrees $n \in \mathbb{Z}$ (see Corollary 23.28), yields isomorphisms in the long exact sequences in homology/cohomology arising from the short exact sequences above. The key point is that these isomorphisms allows us to relate homology/cohomology groups in different degrees. This is called *dimension shifting*.

Theorem 29.9 (Dimension shifting)**.** *Let A be a G-module and let H be any subgroup of G. If G is finite then for all $n \in \mathbb{Z}$ we have*

$$\hat{H}^{n+1}(H,A) \simeq \hat{H}^n(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \qquad \text{and} \qquad \hat{H}^{n-1}(H,A) \simeq \hat{H}^n(H, I_G \otimes_{\mathbb{Z}} A).$$

and whether G is finite or not, for all $n \ge 1$ we have

$$H^{n+1}(H,A) \simeq H^n(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \qquad \text{and} \qquad H_{n+1}(H,A) \simeq H_n(H, I_G \otimes_{\mathbb{Z}} A).$$

Proof. If G is finite then by Corollary 23.28 the coinduced G-module $\operatorname{CoInd}^G(A)$ and the induced G-module $\operatorname{Ind}^G(A)$ have trivial Tate cohomology groups for all degrees n. This remains true if we regard $\operatorname{CoInd}^G(A)$ and $\operatorname{Ind}^G(A)$ as H-modules rather than G-modules, because $\mathbb{Z}[G]$ is free as a $\mathbb{Z}[H]$-module (take a direct sum over cosets). If we now consider the long exact sequences in Tate cohomology (see Theorem 23.32) associated to the short exact sequences (4) and (3), we obtain the isomorphisms in the theorem. When G is not finite we instead apply Lemmas 23.19 and Lemmas 23.26. ☐

Remark 29.10. Dimension shifting provides an alternate way to define Tate cohomology (and standard cohomology and homology). Having defined $\hat{H}^0(G,A) = A^G/N_GA$ for all G-modules A, we put $A_1 := \operatorname{Hom}_{\mathbb{Z}}(I_G, A)$ and define $\hat{H}^1(G,A) := \hat{H}^0(G,A_1)$. Continuing in this fashion with $A_n := \operatorname{Hom}_{\mathbb{Z}}(I_G, A_{n-1})$ and $\hat{H}^n(G,A) := \hat{H}^{n-1}(G,A_1) = \cdots = \hat{H}^0(G,A_n)$, we obtain Tate cohomology groups for all $n \ge 0$; applying a similar strategy with $I_G \otimes_{\mathbb{Z}} A$ addresses $n < 0$.

Proposition 29.11. *Let G be a finite group and let A be a G-module. The Tate cohomology groups $\hat{H}^n(G,A)$ are all torsion groups of exponent dividing $\#G$.*

Proof. By Theorem 29.9, it suffices to prove the case $n = 0$. For any $a \in A^G$ we have $N_Ga = (\#G)a$, thus every element of $\hat{H}^0(G,A) = A^G/N_GA$ has order dividing $\#G$. ☐

Corollary 29.12. *Let G be a finite group and let A be a G-module. If multiplication by $\#G$ induces an isomorphism $A \to A$ then $\hat{H}^n(G,A) = 0$ for all $n \in \mathbb{Z}$. This applies, in particular, when A is the additive group of a ring in which $\#G = 1 + \cdots + 1$ is a unit, including all fields whose characteristic is not a prime divisor of $\#G$.*

Proof. If the G-module morphism $A \to A$ induced by multiplication by $\#G$ is an isomorphism, then multiplication by $\#G$ induces an isomorphism $\hat{H}^n(G,A) \to \hat{H}^n(G,A)$ for all $n \in \mathbb{Z}$ (consider the long exact sequence in Tate cohomology corresponding to the short exact sequence $0 \to 0 \to A \to A \to 0$). But Proposition 29.11 implies that multiplication by $\#G$ induces the zero morphism $\hat{H}^n(G,A) \to \hat{H}^n(G,A)$ for all $n \in \mathbb{Z}$, which is an isomorphism if and only if $\hat{H}^n(G,A) = 0$. ☐

Corollary 29.13. *Let G be a finite group and let A be a finitely generated G-module. Then $\hat{H}^n(G,A)$ is finite for all $n \in \mathbb{Z}$.*

Proof. Since G is finite, being finitely generated as a G-module is the same thing as being finitely generated as an abelian group, so we can assume A is a finitely generated abelian group. We now observe that I_G, $I_G \otimes_{\mathbb{Z}} A$ and $\operatorname{Hom}_{\mathbb{Z}}(I_G, A)$, are finitely generated abelian groups: if $\{x_i\}$ is a finite set of generators for I_G and $\{a_j\}$ is a finite set of generators for A then $\{x_i \otimes a_j\}$ is a finite set of generators for $I_G \otimes_{\mathbb{Z}} A$, and if we define f_{ij} by $f_{ij}(x_i) = a_j$ and $f_{ij}(x_k) = 0$ for $k \ne i$ then $\{f_{ij}\}$ is a finite set of generators for $\operatorname{Hom}_{\mathbb{Z}}(I_G, A)$.

By Theorem 29.9 it suffices to consider $n = 0$. The group $\hat{H}^0(A,G) = A^G/N_GA$ is finitely generated, since A^G is (it is a $\mathbb{Z}$-submodule of the noetherian $\mathbb{Z}$-module A), and $\hat{H}^n(G,A)$ is torsion, by Proposition 29.11, so it must be finite. □

Remark 29.14. Corollary 29.13 implies that if G is cyclic and A is a finitely generated G-module then the Herbrand quotient is always defined (see Definition 23.38).

29.3 Restriction and inflation

Recall that a morphism of G-modules $\phi\colon A \to B$ induces natural homomorphisms of homology and cohomology groups

$$\phi_n\colon H_n(G,A) \to H_n(G,B), \qquad \text{and} \qquad \phi^n\colon H^n(G,A) \to H^n(G,B).$$

Indeed, $H_n(G,\bullet)$ and $H^n(G,\bullet)$ are functors from the category of G-modules to the category of abelian groups, and $\phi_n := H_n(G,\phi)$ and $\phi^n := H^n(G,\phi)$; see Lemma 23.6 and Definition 23.20. If $\varphi\colon H \to G$ is a group homomorphism and A is a G-module, then we can give A the structure of an H-module by defining $ha := \varphi(h)a$, and if we write $H^n(H,A)$ or $H_n(H,A)$ we understand that we are regarding A as an H-module with the H-action induced by $\varphi\colon H \to G$ (if there is no φ in sight, it is inclusion).

The map φ induces a homomorphism from the standard resolution of $\mathbb{Z}$ by H-modules to the standard resolution of $\mathbb{Z}$ by G-modules, and thus induces natural homomorphisms

$$\varphi_n\colon H_n(H,\operatorname{Res}^G_H(A)) \to H_n(G,A), \qquad \text{and} \qquad \varphi^n\colon H^n(G,A) \to H^n(H,\operatorname{Res}^G_H(A)).$$

Definition 29.15. Let $\varphi\colon H \to G$ be a group homomorphism, let A be an H-module, and let B be a G-module. If $\phi\colon A \to B$ is a homomorphism of abelian groups that is also a morphism of H-modules $A \to B$ (where the H-module structure on B is induced by φ) then ϕ is *compatible* with φ. Similarly, if $\phi\colon B \to A$ is a homomorphism of abelian groups that is also a morphism of H-modules $B \to A$ then we say that ϕ is compatible with φ.

If $\phi\colon A \to B$ is compatible with $\varphi\colon H \to G$ then we obtain natural homomorphisms

$$H_n(H,A) \xrightarrow{\phi_n} H_n(H,B) \xrightarrow{\varphi_n} H_n(G,B),$$

and for $\phi\colon B \to A$ compatible with $\varphi\colon H \to G$ we have natural homomorphisms

$$H^n(G,B) \xrightarrow{\varphi^n} H^n(H,B) \xrightarrow{\phi^n} H^n(H,A).$$

In other words, if we consider the category of group modules with morphisms defined by compatible pairs (φ,ϕ) as above, we obtain functors from the category of group modules to the category of abelian groups. We now apply this functor to some particular morphisms.

Definition 29.16. Let A be a G-module and let H be a subgroup of G. The homomorphisms $\operatorname{Res}\colon H^n(G,A) \to H^n(H,A)$ and $\operatorname{CoRes}\colon H_n(H,A) \to H_n(G,A)$ induced by the inclusion $\varphi\colon H \hookrightarrow G$ and the compatible identity maps $\phi\colon A \rightleftharpoons A$ are *restriction* and *corestriction* maps, respectively.

If $\alpha\colon A \to B$ is a morphism of G-modules, the induced morphisms α^n and α_n in cohomology/homology will automatically commute with Res and CoRes, respectively, because we defined them functorially. We can thus view Res and CoRes as morphisms (natural transformations) of functors $\operatorname{Res}\colon H^n(G,\bullet) \to H^n(H,\bullet)$ and $\operatorname{CoRes}\colon H_n(H,\bullet) \to H_n(G,\bullet)$.[3]

[3]The $\bullet$ in $H^n(H,\bullet)$ is a G-module that is then viewed as an H-module, so here $H^n(H,\bullet)$ is a functor from the category of G-modules to the category of abelian groups (as it should be).

Example 29.17. In degree 0 we have

- Res: $A^G \to A^H$ is the inclusion map;
- CoRes: $A_H \to A_G$ is the quotient map;

Note that $H_0(G, A) = A_G = A/I_G A$ is naturally a quotient of $H_0(H, A) = A_H = A/I_H A$, since the $\mathbb{Z}[H]$-ideal I_H is contained in the $\mathbb{Z}[G]$-ideal I_G. If G is finite these maps induce corresponding maps on Tate cohomology groups in degrees 0 and -1.

When G is finite the restriction and corestriction maps can be extended to maps defined on Tate cohomology groups (of all degrees $n \in \mathbb{Z}$) via dimension shifting. However, it will be useful for us to explicitly define restriction on $H_0(G, A)$ and corestriction on $H^0(G, A)$ (and it will give us a more general result).

Definition 29.18. Let A be a G-module, let H be a finite index subgroup of G. Let $S \subseteq G$ be a set of left coset representatives for H, and let $N_{G/H} \in \mathbb{Z}[G]$ denote the sum of the elements of S, and let $N_{G/H}^{-1}$ denote the sum of their inverses. We define

$$\begin{aligned} \operatorname{Res}\colon H_0(G, A) &\to H_0(H, A) \\ a + I_G A &\mapsto N_{G/H}^{-1} a + I_H A. \end{aligned}$$

This map does not depend on the choice of S because $(gh)^{-1}a - g^{-1}a = (h^{-1}-1)g^{-1}a \in I_H A$ for all $g \in S, h \in H, a \in A$. If $\alpha\colon A \to B$ is a morphism of G-modules then

$$\alpha_0(\operatorname{Res}(a)) = N_{G/H}^{-1}\alpha_0(a) + I_H B = \operatorname{Res}(\alpha(a))$$

so Res is a morphism of homology functors $H_0(G, \bullet) \to H_0(H, \bullet)$. We also note that if G is finite then Res maps elements in the kernel of N_G to elements in the kernel of N_H, and thus also defines a morphism of Tate homology functors $\hat{H}_0(G, \bullet) \to \hat{H}_0(H, \bullet)$.

We also define

$$\begin{aligned} \operatorname{CoRes}\colon H^0(H, A) &\to H^0(G, A) \\ a &\mapsto N_{G/H} a. \end{aligned}$$

This map does not depend on the choice of S because $a \in H^0(H, A) = A^H$ is H-invariant. As above, CoRes is a morphism of cohomology functors $H^0(H, \bullet) \to H^0(G, \bullet)$, and when G is finite, a morphism of Tate cohomology functors $\hat{H}^0(H, \bullet) \to \hat{H}^0(G, \bullet)$.

We extend these maps to all degrees $n \geq 0$ as follows. We have the following commutative diagram with exact rows arising from the long exact sequences in homology induced by the exact sequence (3) applied to both G and H:

$$\begin{tikzcd}
0 \arrow[r] & H_1(G, A) \arrow[r] \arrow[d, dashed, "\exists!\,\mathrm{Res}"] & H_0(G, I_G \otimes_{\mathbb{Z}} A) \arrow[r] \arrow[d, "\mathrm{Res}"] & H_0(G, \operatorname{Ind}^G(A)) \arrow[d, "\mathrm{Res}"] \\
0 \arrow[r] & H_1(H, A) \arrow[r] & H_0(H, I_G \otimes_{\mathbb{Z}} A) \arrow[r] & H_0(H, \operatorname{Ind}^G(A))
\end{tikzcd}$$

The trivial groups on the left arise from the fact that the G-module $\operatorname{Ind}^G(A)$ has trivial homology in degrees $n \geq 1$ (both as a G-module and as an H-module) by Lemma 23.26, and therefore $H_1(G, \operatorname{Ind}^G(A)) = H_1(H, \operatorname{Ind}^G(A)) = 0$.

Having defined $\mathrm{Res}\colon H_1(G,A) \to H_1(H,A)$ for all G-modules A, for each $n \geq 2$ we proceed as above, except now we get trivial groups on both the left and the right and this is just dimension shifting (Theorem 29.9).

Extension of the corestriction map in cohomology to degrees $n > 0$ proceeds similarly: we use a commutative diagram arising from the long exact sequence in cohomology induced by the short exact sequence (4) to define $\mathrm{CoRes}\colon H^1(H,A) \to H^1(G,A)$, and use dimension shifting in degrees $n \geq 2$.

Remark 29.19. Having defined restriction and corestriction as morphisms of both homology and cohomology functors, we can view them both as morphisms of families of homology and cohomology functors, which we recall are δ-functors (see Definition 23.9), meaning that they naturally transform short exact sequences of G-modules to long exact sequences in homology and cohomology. Restriction and corestriction are both morphisms of δ-functors. Verifying this involves checking compatibility with the connecting homomorphisms obtained from the snake lemma; see the proof of Theorem 23.8 for an example of how this is done. In the case of Tate cohomology one also needs to check compatibility with the connecting homomorphism between cohomology groups in degrees -1 and 0; see the proof of Theorem 23.32 for an example.

Remark 29.20. It is easy to verify that the restriction and corestriction functors are both transitive with respect to the inclusion $H \subseteq G$. That is, if $H \subseteq K \subseteq G$ is a tower of subgroup inclusions and Res_H^G denotes $\mathrm{Res}\colon H^n(G,\bullet) \to H^n(H,\bullet)$ or $\mathrm{Res}\colon H_n(G,\bullet) \to H_n(H,\bullet)$ then $\mathrm{Res}_H^G = \mathrm{Res}_H^K \circ \mathrm{Res}_K^G$, and a similar statement holds for corestriction. We won't use this.

Proposition 29.21. *Let A be a G-module and let $H \leq G$ be a subgroup of finite index. The composition $\mathrm{CoRes} \circ \mathrm{Res}$ corresponds to the multiplication-by-$[G\!:\!H]$ endomorphism on $H^n(G,A)$ and $H_n(G,A)$ for all $n \geq 0$. When G is finite it induces multiplication-by-$[G\!:\!H]$ on $\hat{H}^n(G,A)$ for all $n \in \mathbb{Z}$.*

Proof. On $H^0(G,A) \simeq A^G$ we have the map $A^G \to A^H \to A^G$, where the first map is inclusion and the second map is $a \mapsto N_{G/H}a$, where $N_{G/H} \in \mathbb{Z}[G]$ is the sum of any set of left coset representatives for H. But for $a \in A^H$ multiplication by $N_{G/H}$ is the same as multiplication by $[G\!:\!H]$. When G is finite the induced map on $\hat{H}^0(G,A) = A^G/N_GA$ is also multiplication by $[G\!:\!H]$.

On $H_0(G,A) \simeq A_G$ we have $A_G \to A_H \to A_G$, the first map is $a + I_GA \mapsto N_{G/H}^{-1}a + I_HA$ and the second quotienting by I_GA. But $(N_{G/H}^{-1} - [G\!:\!H])a \in I_GA$, so this is also equivalent to multiplication by $[G\!:\!H]$.

If G is finite multiplication by $[G:H]$ on A^G induces multiplication by $[G:H]$ on $A^G/N_GA = \hat{H}^0(G,A)$ and multiplication by by $[G\!:\!H]$ on A_G restricts to multiplication by $[G\!:\!H]$ on the kernel of $N_G\colon A_G \to A^G$, which is $\hat{H}_0(G,A)$.

The proposition now follows by dimension shifting (Theorem 29.9). ∎

Definition 29.22. Let A be a G-module and let H be a normal subgroup of G. Then A^H and A_H are G-modules that are trivial as H-modules, hence G/H modules. The homomorphisms $\mathrm{Inf}\colon H^n(G/H, A^H) \to H^n(G,A)$ induced by the quotient $\varphi\colon G \twoheadrightarrow G/H$ and the compatible inclusion $\phi\colon A^H \hookrightarrow A$ are *inflation maps*. Replacing $\phi\colon A^H \hookrightarrow A$ with the quotient map $\phi\colon A \twoheadrightarrow A_H$ yields *coinflation maps* $\mathrm{CoInf}\colon H_n(G,A) \to H_n(G/H, A_H)$.

In degree zero inflation and coinflation are identity maps (note that $(A^H)^{G/H} = A^G$ and $(A_H)_{G/H} = A_G$). As with restriction and corestriction, if $\alpha\colon A \to B$ is a morphism of

G-modules, the induced morphisms α^n and α_n in cohomology and homology commute with Inf and CoInf (by functoriality), and we have morphisms of functors $\mathrm{Inf}\colon H^n(G/H, \bullet^H) \to H^n(G, \bullet)$ and $\mathrm{CoInf}\colon H_n(G, \bullet) \to H_n(G/H, \bullet_H)$. One can show that they are also morphisms of δ-functors (see Remark 29.19).

The restriction and inflation maps in cohomology are easy to understand explicitly. If $f\colon G^n \to A$ is an n-cocycle representing $\gamma \in H^n(G, A)$, restricting f to $H^n \subseteq G^n$ yields an n-cocycle representing $\mathrm{Res}(\gamma) \in H^n(H, A)$. If $f\colon (G/H)^n \to A^H$ represents $\gamma \in H^n(G/H, A^H)$, then composing f with the map $G^n \to (G/H)^n$ induced by the quotient map $G \to G/H$ yields an n-cocycle representing $\mathrm{Inf}(\gamma) \in H^n(G, A)$.

Theorem 29.23 (Inflation-Restriction). *Let H be a normal subgroup of G, let A be a G-module, and let $n \ge 1$. Suppose $H^i(H, A) = 0$ for $1 \le i < n$. Then the sequence*

$$0 \longrightarrow H^n(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^n(G, A) \xrightarrow{\mathrm{Res}} H^n(H, A)$$

is exact.

Proof. We proceed by induction on $n \ge 1$. In the base case $n = 1$, and the hypothesis is vacuously satisfied. Let $f\colon G/H \to A^H$ be a 1-cocycle representing an element of the kernel of Inf. The composition of f with $G \to G/H$ is a coboundary $(g \mapsto ga - a)$ with $a \in A^H$, so $f = (\bar{g} \mapsto \bar{g}a - \bar{g})$ is a coboundary and $\mathrm{Inf}\colon H^1(G/H, A^H) \to H^1(G, A)$ is injective. Restricting the composition of a cocycle $f\colon G/H \to A^H$ with $G \to G/H$ to H yields a constant 1-cocycle, which must be the zero map; therefore $\mathrm{im}\,\mathrm{Inf} \subseteq \ker \mathrm{Res}$. Now let $f\colon G \to A$ be a 1-cocycle representing an element in the kernel of Res. The restriction of f to H is then a coboundary, which must be of the form $h \mapsto ha - a$ for some $a \in A$. If we now define $\bar{f}\colon G^n \to A$ by $\bar{f}(g) := f(g) - ga + a$, then $\bar{f}$ vanishes on all $h \in H$ and

$$\bar{f}(gh) = g\bar{f}(h) + \bar{f}(g) = \bar{f}(g)$$

for all $g \in G$ and $h \in H$, so $\bar{f}$ factors through G/H. We also have

$$\bar{f}(hg) = h\bar{f}(g) + \bar{f}(h) = h\bar{f}(g),$$

so $\mathrm{im}\,\bar{f} \subseteq A^H$ and f represents an element in the image of Inf. This proves the base case.

For the inductive step we apply dimension shifting. We can assume the theorem holds in dimension n (for all G, $H \trianglelefteq G$, A) and need to show that it holds in dimension $n + 1$. By Theorem 29.9, we have $H^{i+1}(H, A) \simeq H^i(H, \mathrm{Hom}_{\mathbb{Z}}(I_G, A))$ for all $i > 0$, so if A satisfies the hypothesis of the theorem for $n + 1$ then $\mathrm{Hom}(I_G, A)$ satisfies it for n. Therefore

$$0 \longrightarrow H^n(G/H, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)^H) \xrightarrow{\mathrm{Inf}} H^n(G, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \xrightarrow{\mathrm{Res}} H^n(H, \mathrm{Hom}_{\mathbb{Z}}(I_G, A))$$

is exact, by the inductive hypothesis. It follows that the isomorphic sequence

$$0 \longrightarrow H^{n+1}(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^{n+1}(G, A) \xrightarrow{\mathrm{Res}} H^{n+1}(H, A)$$

is also exact. Here we are using the fact that $\mathrm{Hom}_{\mathbb{Z}}(I_G, A)^H \simeq \mathrm{Hom}_{\mathbb{Z}}(I_{G/H}, A^H)$ as G/H-modules and applying dimension shifting to the G/H-module A^H as well. $\square$

Remark 29.24. There is is an analog of Theorem 29.23 that yields an exact sequence

$$H_n(G, A) \xrightarrow{\mathrm{CoRes}} H_n(H, A) \xrightarrow{\mathrm{CoInf}} H_n(G/H, A_H) \longrightarrow 0.$$

in homology, but we will not use this.

29.4 Tate's theorem

Theorem 29.25. *Let G be a finite group and let A be a G-module. If $H^1(H,A)=0$ and $H^2(H,A)=0$ for every subgroup $H\le G$ then $\hat H^n(G,A)=0$ for all $n\in\mathbb{Z}$.*

Proof. If G is cyclic then $\hat H^n(G,A)=\hat H^{n+2}(G,A)$ for all n (by Theorem 23.37) and the theorem follows.

If G is solvable it has a subnormal series $1=H_0\trianglelefteq H_1\trianglelefteq\cdots\trianglelefteq H_n=G$ in which each factor group H_i/H_{i-1} is a non-trivial cyclic group. We proceed by induction on the length of the shortest such series. The base case is covered by the cyclic case, so we now assume that H is a normal subgroup of G with G/H cyclic and non-trivial. The hypothesis of the theorem is satisfied by H (since we are assuming it holds for G), so $\hat H^n(H,A)=0$ for all $n\in\mathbb{Z}$, by the inductive hypothesis. If we now consider the inflation-restriction sequence, for all $n\ge1$ we have an exact sequence

$$0\longrightarrow H^n(G/H,A^H)\xrightarrow{\text{Inf}}H^n(G,A)\xrightarrow{\text{Res}}H^n(H,A),$$

with $H^n(H,A)=0$. Therefore $H^n(G/H,A^H)\simeq H^n(G,A)$ for $n\ge1$, so $H^1(G/H,A^H)$ and $H^2(G/H,A^H)=0$ are both trivial and this implies that $\hat H^n(G/H,A^H)=0$ for all $n\in\mathbb{Z}$, since G/H is cyclic, by Theorem 23.37. Thus $H^n(G,A)=0$ for $n\ge1$. We also have $\hat H^0(G/H,A^H)=(A^H)^{G/H}/N_{G/H}A^H=0$ and $\hat H^0(H,A)=A^H/N_HA=0$, so

$$A^G=(A^H)^{G/H}=N_{G/H}A^H=N_{G/H}N_HA=N_GA,$$

and therefore $\hat H^0(G,A)=0$. We use dimension shifting to address $n<0$. By Theorem 29.9 we have $\hat H^n(H,I_G\otimes_{\mathbb{Z}}A)\simeq\hat H^{n-1}(H,A)$ for all $H\le G$, so $I_G\otimes_{\mathbb{Z}}A$ satisfies the hypothesis of the the theorem, and by what we have proved so far, $\hat H^n(G,I_G\otimes A)=0$ for all $n\ge0$. Therefore $\hat H^{-1}(G,A)=\hat H^0(G,I_G\otimes_{\mathbb{Z}}A)=0$. Repeating this argument addresses all $n<0$.

If G is not solvable, we can assume that $\hat H^n(H,A)=0$ for every solvable subgroup $H\le G$, since the hypothesis of the theorem holds for all such groups and we have now proved that the theorem holds for solvable groups. So let H by a p-Sylow subgroup of G (solvable, since it is a p-group), and consider the composition

$$H^n(G,A)\xrightarrow{\text{Res}}H^n(H,A)\xrightarrow{\text{CoRes}}H^n(G,A),$$

which by Proposition 29.21 is equivalent to multiplication by $[G:H]$, but also the zero map, since the middle group is trivial. So every element of $H^n(G,A)$ has order dividing $[G:H]$, and this implies that $H^n(G,A)$ contains no elements of order p. This holds for every p-Sylow subgroup $H\le G$, so $H^n(G,A)$ has no elements of prime order. But $H^n(G,A)$ is torsion, by Proposition 29.11, so it must be trivial. □

Theorem 29.26 (Tate's Theorem)**.** *Let G be a finite group and let A be a G-module such that $H^1(H,A)=0$ and $H^2(H,A)$ is cyclic of order $\#H$ for every subgroup $H\le G$, and let γ be a generator for $H^2(G,A)$. Then for all $n\in\mathbb{Z}$ there is an isomorphism*

$$\Phi_\gamma\colon\hat H^n(G,\mathbb{Z})\xrightarrow{\sim}\hat H^{n+2}(G,A)$$

that is uniquely determined by the choice of γ. The isomorphism Φ_γ is compatible with restriction and corestriction; that is, for any subgroup $H\le G$, for all $n\in\mathbb{Z}$ we have commutative diagrams

$$\begin{array}{ccc}
\hat H^n(G,\mathbb{Z}) & \xrightarrow{\Phi_\gamma} & \hat H^{n+2}(G,A) \\
\big\downarrow{\scriptstyle\mathrm{Res}} & & \big\downarrow{\scriptstyle\mathrm{Res}} \\
\hat H^n(H,\mathbb{Z}) & \xrightarrow{\Phi_{\mathrm{Res}(\gamma)}} & \hat H^{n+2}(H,A)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\hat H^n(G,\mathbb{Z}) & \xrightarrow{\Phi_\gamma} & \hat H^{n+2}(G,A) \\
{\scriptstyle\mathrm{CoRes}}\big\uparrow & & {\scriptstyle\mathrm{CoRes}}\big\uparrow \\
\hat H^n(H,\mathbb{Z}) & \xrightarrow{\Phi_{\mathrm{Res}(\gamma)}} & \hat H^{n+2}(H,A).
\end{array}$$

Proof. Let $\varphi\colon G^2\to A$ be a 2-cocycle representing γ and define the G-module $A(\varphi)$ as the direct sum of the abelian group A and the free abelian group with basis $\{x_g : g\in G-\{1\}\}$, where G acts on the image of $\iota\colon A\hookrightarrow A(\varphi)$ as usual and

$$gx_h \coloneqq x_{gh}-x_g+\varphi(g,h),$$

where $x_1\coloneqq\varphi(1,1)$. This is a group action because φ is a cocycle. Indeed, for $g_1,g_2,h\in G$:

$$\begin{aligned}
g_1g_2x_h-(g_1g_2)x_h &= g_1(x_{g_2h}-x_{g_2}+\varphi(g_2,h))-x_{g_1g_2h}+x_{g_1g_2}-\varphi(g_1g_2,h)\\
&= x_{g_1g_2h}-x_{g_1}+\varphi(g_1,g_2h)-x_{g_1g_2}+x_{g_1}-\varphi(g_1,g_2)+g_1\varphi(g_2,h)\\
&\quad -x_{g_1g_2h}+x_{g_1g_2}-\varphi(g_1g_2,h)\\
&= \varphi(g_1,g_2h)-\varphi(g_1,g_2)+g_1\varphi(g_2,h)-\varphi(g_1g_2,h)\\
&= d^2(\varphi)(g_1,g_2,h)=0.
\end{aligned}$$

The 2-cocycle $\iota\circ\varphi\colon G^2\to A\hookrightarrow A(\varphi)$ is the coboundary of the 1-cochain $\psi\colon G\to A(\varphi)$ defined by $g\mapsto x_g$, since

$$d^1(\psi)(g,h)=gx_h-x_{gh}+x_g=x_{gh}-x_g+\varphi(g,h)-x_{gh}+x_g=\varphi(g,h),$$

so γ lies in the kernel of the map $\iota^2\colon H^2(G,A)\to H^2(G,A(\varphi))$ induced by $\iota\colon A\hookrightarrow A(\varphi)$; it follows that ι^2 is the zero map, since $H^2(G,A)=\langle\gamma\rangle$.

We now define the map

$$\begin{aligned}
\phi\colon A(\varphi)&\to\mathbb{Z}[G]\\
a&\mapsto 0 \qquad (\text{for } a\in A)\\
x_g&\mapsto g-1 \quad (\text{for } g\in G-\{1\}).
\end{aligned}$$

with kernel A and image I_G (the augmentation ideal). We thus have a short exact sequence of G-modules

$$0\longrightarrow A\xrightarrow{\ \iota\ }A(\varphi)\xrightarrow{\ \phi\ }I_G\longrightarrow 0, \tag{5}$$

that is also a short exact sequence of H-modules for every subgroup $H\le G$.

For each subgroup $H\le G$ the long exact sequence in Tate cohomology induced by the short exact sequence $0\to I_G\to\mathbb{Z}[G]\xrightarrow{\varepsilon}\mathbb{Z}\to 0$ of H-modules has $\hat H^n(H,\mathbb{Z}[G])=0$ for all $n\in\mathbb{Z}$, since $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-module, so for all $n\in\mathbb{Z}$ we have

$$\hat H^n(H,\mathbb{Z})\simeq\hat H^{n+1}(H,I_G). \tag{6}$$

In particular,

$$H^2(H,I_G)\simeq H^1(H,\mathbb{Z})\simeq\mathrm{Hom}(H,\mathbb{Z})=0 \tag{7}$$

$$H^1(H,I_G)\simeq\hat H^0(H,\mathbb{Z})\simeq\mathbb{Z}^H/N_H\mathbb{Z}=\mathbb{Z}/(\#H)\mathbb{Z}, \tag{8}$$

since $\mathbb{Z}$ is a trivial H-module ($H^1(H,\mathbb{Z})\simeq\mathrm{Hom}(H,\mathbb{Z})$ because 1-coboundaries are trivial and 1-cocycles are homomorphisms, and $\mathrm{Hom}(H,\mathbb{Z})=0$ because H is finite).

For each subgroup $H \le G$, the long exact sequence in cohomology induced by (5) contains the exact sequence

$$H^1(H,A) \xrightarrow{\iota^1} H^1(H,A(\varphi)) \xrightarrow{\phi^1} H^1(H,I_G) \xrightarrow{\delta^1} H^2(H,A) \xrightarrow{\iota^2} H^2(H,A(\varphi)) \xrightarrow{\phi^2} H^2(H,I_G)$$

We have $H^1(H,A) = 0$, by hypothesis, and $H^2(H,I_G) \simeq H^1(H,\mathbb{Z}) = 0$ by (7). As shown above, ι^2 is the zero map, as is ϕ^2, so $H^2(H,A(\varphi)) = 0$. The connecting homomorphism δ^1 is surjective and therefore an isomorphism because $H^1(H,I_G)$ and $H^2(H,A)$ are both cyclic of order $\#H$, by (8) and the hypothesis of the theorem. It follows that ϕ^1 is the zero map, and it is also injective (since $H^1(H,A) = 0$), so $H^1(H,A(\varphi)) = 0$. By Theorem 29.25, we have $\hat{H}^n(G,A(\varphi)) = 0$ for all $n \in \mathbb{Z}$.

If we now consider the long exact sequence in Tate cohomology induced by (5), the terms $\hat{H}^n(G,A(\varphi))$ are all zero, so $\hat{H}^n(G,I_G) \simeq \hat{H}^{n+1}(G,A)$ for all $n \in \mathbb{Z}$. Combining this with (6), for every $n \in \mathbb{Z}$ we have isomorphims

$$\hat{H}^n(G,\mathbb{Z}) \xrightarrow{\hat{\delta}} \hat{H}^{n+1}(G,I_G) \xrightarrow{\hat{\delta}_\gamma} \hat{H}^{n+2}(G,A),$$

where $\hat{\delta}$ and $\hat{\delta}_\gamma$ are both connecting homomorphisms in long exact sequences in Tate cohomology: $\hat{\delta}$ appears in the long exact sequence induced by $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$, while $\hat{\delta}_\gamma$ appears in the long exact sequence induced by $0 \to A \to A(\varphi) \to I_G \to 0$. The isomorphism $\hat{\delta}$ is canonical, and $\hat{\delta}_\gamma$ depends only on the choice of γ (a different choice of the 2-cocycle φ representing γ would change $A(\varphi)$, but it would not change any of the maps in cohomology). We now define $\Phi_\gamma \coloneqq \hat{\delta}_\gamma \circ \hat{\delta}$.

Compatibility with restriction and corestriction follows from the fact that Φ_γ is defined as a composition of connecting homomorphisms in Tate cohomology and restriction and corestriction are both morphisms of δ-functors, as noted in Remark 29.19. $\square$

Remark 29.27. The isomorphism Φ_γ can also be defined using the cup product in Tate cohomology; see Problem Set 1 for details. For our purposes it is more useful to spell out the definition explicitly in terms of connecting homomorphisms. We will use the 2-cocycle φ constructed in the proof of Tate's theorem in our proof of local Artin reciprocity.

References

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