## Description

These problems are related to Lectures 12–15. Your solutions should be written up in latex and submitted as a pdf-file to Gradescope by midnight on the date due.

**Instructions:** Solve Problem 0, then pick any combination of problems that sum to 100 points. Collaboration is permitted/encouraged, but you must identify your collaborators (including any LLMs you consulted) and any references you consulted outside the course syllabus. Include this information after the **Collaborators/Sources** prompt at the end of the problem set (if there are none, you should enter "none", do not leave it blank). Note that each student is expected to write their own solutions; it is fine to discuss problems with others, but your writing must be your own.

## Problem 0.

These are warm up problems that do not need to be turned in.

(a) Prove that a cubic field $K$ is Galois if and only if $D_K$ is a perfect square.

(b) Prove that our two definitions of a lattice $\Lambda$ in $V \simeq \mathbb{R}^n$ are equivalent: $\Lambda$ is a $\mathbb{Z}$-submodule generated by an $\mathbb{R}$-basis for $V$ if and only if it is a discrete cocompact subgroup of $V$.

(c) Let $n \in \mathbb{Z}_{>0}$ and assume $n^2 - 1$ is squarefree. Prove that $n + \sqrt{n^2-1}$ is the fundamental unit of $\mathbb{Q}(\sqrt{n^2-1})$.

## Problem 1. Classification of global fields (66 points)

Let $K$ be a field and let $M_K$ be the set of places of $K$ (equivalence classes of nontrivial absolute values). We say that $K$ has a (strong) *product formula* if $M_K$ is nonempty for each $v \in M_K$ there is an absolute value $|\ |_v$ in its equivalence class and a positive real number $m_v$ such that for all $x \in K^\times$ we have

$$\prod_{v \in M_K} |x|_v^{m_v} = 1,$$

where all but finitely many factors in the product are equal to 1. Equivalently, if we fix *normalized absolute values* $\|\ \|_v := |x|_v^{m_v}$ for each $v \in M_K$, then for all $x \in K^\times$ we have

$$\prod_{v \in M_K} \|x\|_v = 1,$$

with $\|x\|_v = 1$ for all but finitely many $v \in M_K$.

**Definition.** A field $K$ is a *global field* if it has a product formula and the completion $K_v$ of $K$ at each place $v \in M_K$ is a local field.

In Lectures 10 and 13 we proved every finite extension of $\mathbb{Q}$ and $\mathbb{F}_q(t)$ is a global field. In this problem you will prove the converse, a result due to Artin and Whaples [1].

Let $K$ be a global field with normalized absolute values $\| \|_v$ for $v \in M_K$ that satisfy the product formula. As we defined in lecture, an $M_K$-*divisor* is a sequence of positive real numbers $c = (c_v)$ indexed by $v \in M_K$ with all but finitely many $c_v = 1$ such that for each $v \in M_K$ there is an $x \in K_v^\times$ for which $c_v = \|x\|_v$. For each $M_K$-divisor $c$ we define the set

$$L(c) := \{x \in K : \|x\|_v \leq c_v \text{ for all } v \in M_K\}.$$

**(a)** Let $E/F$ be a finite Galois extension. Prove $E$ is a global field if and only if $F$ is.

**(b)** Extend your proof of (a) to all finite extensions $E/F$.

**(c)** Prove that $M_K$ is infinite but contains only finitely many archimedean places.

**(d)** Assume $K$ has an archimedean place. Prove that $L(c)$ is finite for every $M_K$-divisor $c$ (we proved this in class for number fields, but here $K$ is a global field as defined above).

**(e)** Extend your proof of (d) to the case where $K$ has no archimedean places.

**(f)** Prove that if $M_K$ contains an archimedean place then $K$ is a finite extension of $\mathbb{Q}$ (hint: show $\mathbb{Q} \subseteq K$ and use (d) to show that $K/\mathbb{Q}$ is a finite extension).

**(g)** Prove that if $M_K$ does not contain an archimedean place then $K$ is a finite extension of $\mathbb{F}_q(t)$ for some finite field $\mathbb{F}_q$ (hint: by choosing an appropriate $M_K$-divisor $c$, show that $L(c)$ is a finite field $k \subseteq K$ and that every $t \in K - k$ is transcendental over $k$; then show that $K$ is a finite extension of $k(t)$).

**(h)** In your proofs of (a)-(g) above, where did you use the fact that the completions of $K$ are local fields? Show that if $K$ has a product formula and $K_v$ is a local field for any place $v \in M_K$ then $K_v$ is a local field for every place $v \in M_K$ (so we could weaken our definition of a global field to only require one $K_v$ to be a local field). Are there fields with a product formula for which no completion is a local field?

## Problem 2. Finiteness of global class groups (66 points)

A commutative ring $R$ with finite quotients by all nonzero ideals is a *finite quotient domain*; to rule out trivial cases we further assume $R$ is not a field. For such $R$ we define the *absolute ideal norm* $\mathrm{N}_R(I) := \#R/I$ for each nonzero $R$-ideal $I$, let $\mathrm{N}_R((0)) := 0$, and put $\mathrm{N}_R(r) := \mathrm{N}_R((r))$ for $r \in R$.

Recall our standard $AKLB$ assumption: $A$ is a Dedekind domain, $K$ is its fraction field, $L/K$ is finite separable, and $B$ is the integral closure of $A$ in $L$.

**(a)** Assume $AKLB$. Show that if $A$ is a finite quotient domain then so is $B$, and we have the identity $\mathrm{N}_B(\alpha) = \mathrm{N}_A(\mathrm{N}_{L/K}(\alpha))$ for all $\alpha \in B$.

**Definition.** A PID is *basic* if it is a finite quotient domain and $\exists c_1, c_2 \in \mathbb{Z}_{>0}$ such that

(i) $\#\{x \in A : \mathrm{N}_A(x) \leq c_1 m\} \geq m$ for all integers $m$,

(ii) $\mathrm{N}_A(x + y) \leq c_2(\mathrm{N}_A(x) + \mathrm{N}_A(y))$ for all $x, y \in A$.

**(b)** Show that $\mathbb{Z}$ is a basic PID but $\mathbb{Z}[\sqrt{5}]$ is not. Is $\mathbb{Z}[i]$ a basic PID?

**(c)** Show that $\mathbb{F}_q[t]$ is a basic PID.

**Definition.** A *global Dedekind domain* (over $A$) is a Dedekind domain that is free of finite rank as a module over a subring $A$ that is a basic PID.

**(d)** Assume $AKLB$. Show that if $A$ is a basic PID then $B$ is a global Dedekind domain. Conclude that every global field is the fraction field of a global Dedekind domain.

**(e)** Let $B$ be a global Dedekind domain over a basic PID $A$ and let $e_1, \ldots e_n$ be an $A$-basis for $B$. Prove there exists a homogeneous $f \in A[x_1, \ldots, x_n]$ of degree $n$ such that for any $\alpha \in B$,

$$\mathrm{N}_{L/K}(\alpha) = f(a_1, \ldots, a_n),$$

where $\alpha = a_1 e_1 + \cdots a_n e_n$ with $a_i \in A$. Prove there exists $c \in \mathbb{Z}_{>0}$ such that

$$\mathrm{N}_B(\alpha) \leq c \max(\mathrm{N}_A(a_1), \ldots, \mathrm{N}_A(a_n))^n$$

for all $\alpha = a_1 e_1 + \cdots a_n e_n \in B$.

**(f)** Let $B$ be a global Dedekind domain. Prove that there is a real number $m$ such that for every nonzero $B$-ideal $I$ there is a nonzero $\alpha \in I$ for which $\mathrm{N}_B(\alpha) \leq m \mathrm{N}_B(I)$.

**(g)** Prove that the ideal class group of a global Dedekind domain over $\mathbb{Z}$ or $\mathbb{F}_q(t)$ is finite (in fact all global Dedekind domains are global Dedekind domains over one of these basic PIDs, but you are not asked to prove this).

**Remark.** Given (d) and the terminology we have chosen, you might wonder if the fraction field of a global Dedekind domain is necessarily a global field. The answer is yes! You can find a proof of this in [2, §4], but I urge you to wait until you have solved this problem before reading it (this problem is adapted from [2, §2-3] and [3, §3]).

## Problem 3. Some applications of the Minkowski bound (33 points)

For a number field $K$, let

$$m_K := \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{|D_K|}$$

denote the Minkowski constant and let $h_K := \#\,\mathrm{cl}\,\mathcal{O}_K$ denote the class number. You may wish to use a computer to help with some of the calculations involved in this problem, but if you do so, please describe your computations (preferably in words or pseudo-code).

**(a)** Prove that if $\mathcal{O}_K$ contains no prime ideals $\mathfrak{p}$ of norm $\mathrm{N}(\mathfrak{p}) \leq m_K$ other than inert primes, then $h_K = 1$, and show that when $K$ is an imaginary quadratic field the converse also holds.

**(b)** Let $K$ be an imaginary quadratic field. Show that if $h_K = 1$ then $|D_K|$ is a power of 2 or a prime congruent to 3 mod 4, and then determine all imaginary quadratic fields $K$ of class number one with $|D_K| < 200$ (this is in fact all of them).

**(c)** Prove that there are no totally real cubic fields of discriminant less than 20 and that every totally real cubic field $K$ with $D_K < M$ can be written as $K = \mathbb{Q}(\alpha)$, where $\alpha$ is an algebraic integer with minimal polynomial $x^3 + ax^2 + bx + c$ whose coefficients satisfy $|a| < \sqrt{M} + 2$, $|b| < 2\sqrt{M} + 1$, and $|c| < \sqrt{M}$.

**(d)** Determine all totally real cubic fields $K$ that are ramified only at a prime $p < 10$ and give a defining polynomial for each field that arises. You may find the Sage function `pari.polredabs` useful: given a monic irreducible polynomial in $\mathbb{Z}[x]$ it will output another monic irreducible polynomial that defines the same number field but may have smaller discriminant (it will never be larger).

## Problem 4. Minkowski's lemma and sums of four squares (33 points)

Minkowski's lemma (for $\mathbb{Z}^n$) states that *if $S \subseteq \mathbb{R}^n$ is a symmetric convex set of volume $\mu(S) > 2^n$ then $S$ contains a nonzero element of $\mathbb{Z}^n$.*

Here *symmetric* means that $S$ is closed under negation, and *convex* means that for all $x, y \in S$ the set $\{tx + (1-t)y : t \in [0,1]\}$ lies in $S$.

**(a)** Prove that for any measurable $S \subseteq \mathbb{R}^n$ with measure $\mu(S) > 1$ there exist distinct $s, t \in S$ such that $s - t \in \mathbb{Z}^n$, then prove Minkowski's lemma.

**(b)** Prove that Minkowski's lemma is tight in the following sense: show that it is false if either of the words "symmetric" or "convex" is removed, or if the strict inequality $\mu(S) > 2^n$ is weakened to $\mu(S) \geq 2^n$ (give three explicit counter examples).

**(c)** Prove that one can weaken the inequality $\mu(S) > 2^n$ in Minkowski's lemma to $\mu(S) \geq 2^n$ if $S$ is assumed to be compact.

You will now use Minkowski's lemma to prove a theorem of Lagrange, which states that every positive integer is a sum of four integer squares. Let $p$ be an odd prime.

**(d)** Show that $x^2 + y^2 = a$ has a solution $(m, n)$ in $\mathbb{F}_p^2$ for every $a \in \mathbb{F}_p$.

**(e)** Let $V$ be the $\mathbb{F}_p$-span of $\{(m, n, 1, 0), (-n, m, 0, 1)\}$ in $\mathbb{F}_p^4$, where $m^2 + n^2 = -1$. Prove that $V$ is *isotropic*, meaning that $v_1^2 + v_2^2 + v_3^2 + v_4^2 = 0$ for all $v \in V$.

**(f)** Use Minkowski's lemma to prove that $p$ is a sum of four squares.

**(g)** Prove that every positive integer is the sum of four squares.

## Problem 5. Unit groups of real quadratic fields (66 points)

A (simple) *continued fraction* is a (possibly infinite) expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

with $a_n \in \mathbb{Z}$ and $a_n > 0$ for $n > 0$. They are more compactly written as $(a_0; a_1, a_2, \ldots)$. For any $t \in \mathbb{R}_{>0}$ the *continued fraction expansion* of $t$ is defined recursively via

$$t_0 := t, \qquad a_n := \lfloor t_n \rfloor, \qquad t_{n+1} := 1/(t_n - a_n),$$

where the sequence $a(t) := (a_0; a_1, a_2, \ldots)$ terminates at $a_n$ if $t_n = a_n$, in which case we say that $a(t) = (a_0; a_1, \ldots, a_n)$ is *finite*, and otherwise call $a(t) = (a_0; a_1, a_2, \ldots)$ *infinite*. If $a(t)$ is infinite and there exists $\ell \in \mathbb{Z}_{>0}$ such that $a_{n+\ell} = a_n$ for all sufficiently large $n$, we say that $a(t)$ is *periodic* and call the least such integer $\ell := \ell(t)$ the *period* of $a(t)$.

For an infinite continued fraction $a(t) := (a_0; a_1, a_2, \ldots)$, we define $P_n, Q_n \in \mathbb{Z}_{\geq 0}$ via

$$\begin{aligned} P_{-2} &= 0, & P_{-1} &= 1, & P_n &= a_n P_{n-1} + P_{n-2}; \\ Q_{-2} &= 1, & Q_{-1} &= 0, & Q_n &= a_n Q_{n-1} + Q_{n-2}. \end{aligned}$$

Note that $P_{n+2} > P_{n+1} \geq P_n > 0$ and $Q_{n+1} > Q_n \geq Q_{n-1} > 0$ for all $n \geq 1$.

**(a)** Prove that $a(t)$ is finite if and only if $t \in \mathbb{Q}$, in which case $t = a(t)$.

**(b)** Prove that if $a(t)$ is infinite then for all $n \geq 0$ we have

$$P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2} = \pm 1 \qquad \text{and} \qquad t = \frac{t_n P_{n-1} + P_{n-2}}{t_n Q_{n-1} + Q_{n-2}}.$$

**(c)** Prove that if $a(t)$ is infinite then $P_n/Q_n = (a_0; a_1, \ldots, a_n)$ with $|Q_n^2 t - P_n Q_n| < 1$ and $a(t_n) = (a_n; a_{n+1}, a_{n+2}, \ldots)$, for all $n \geq 0$. Conclude that $t = \lim_{n\to\infty} P_n/Q_n = a(t)$.

**(d)** Prove that if $a(t)$ is infinite then for any $P, Q \in \mathbb{Z}_{>0}$ with $|tQ - P| < 1/(2Q)$ we have $P/Q = P_n/Q_n$ for some $n \geq 0$.

**(e)** Prove that $a(t)$ is periodic if and only if $t$ is a real quadratic irrational.
(Hint: show that if $At^2 + Bt + C = 0$ with $A, B, C \in \mathbb{Z}$ then $A_n t_n^2 + B_n t_n + C_n = 0$ with $A_n, B_n, C_n \in \mathbb{Z}$ and $B_n^2 - 4A_nC_n = B^2 - 4AC$ and $|A_n|, |C_n| \leq 2|At| + |A| + |B|$).

**Definition.** Let $t$ be a real quadratic irrational with Galois conjugate $t'$ and continued fraction $a(t) = (a_0; a_1, a_2, \ldots)$ of period $\ell = \ell(t)$. We say that $t$ is *purely periodic* if we have $a_i = a_{\ell+i}$ for all $i \geq 0$ and call $t$ *reduced* if $t > 1$ and $-1 < t' < 0$,

**(f)** Prove that a real quadratic irrational $t$ is reduced if and only if it is purely periodic.

Now let $D > 0$ be a squarefree integer that is not congruent to $1 \bmod 4$ and let $K = \mathbb{Q}(\sqrt{D})$. As shown on previous problem sets, $\mathcal{O}_K = \mathbb{Z}[\sqrt{D}]$, and it is clear that $(\mathcal{O}_K^\times)_{\rm tors} = \{\pm 1\}$. Every $\alpha = x + y\sqrt{D} \in \mathcal{O}_K^\times$ has $N(\alpha) = \pm 1$, and $(x, y)$ is thus an (integer) solution to the *Pell equation*

$$X^2 - DY^2 = \pm 1 \tag{1}$$

**(g)** Prove that if $(x_1, y_1)$ and $(x_2, y_2)$ are solutions to (1) with $x_1, y_1, x_2, y_2 \in \mathbb{Z}_{>0}$ then $x_1 + y_1\sqrt{D} < x_2 + y_2\sqrt{D}$ if and only if $x_1 < x_2$ and $y_1 \leq y_2$. Conclude that the fundamental unit $\epsilon = x + y\sqrt{D}$ of $\mathcal{O}_K^\times$ is the unique solution $(x, y)$ to (1) with $x, y > 0$ and $x$ minimal.

**(h)** Let $a(\sqrt{D}) = (a_0; a_1, a_2, \ldots)$, define $t_n, P_n, Q_n$ as above, and let $\ell := \ell(\sqrt{D})$. Prove that $(P_{k\ell-1}, Q_{k\ell-1})$ is a solution to (1) for all $k \geq 0$ and that $\epsilon = P_{\ell-1} + Q_{\ell-1}\sqrt{D}$.

**(i)** Compute the fundamental unit $\epsilon$ for each of the real quadratic fields $\mathbb{Q}(\sqrt{19})$, $\mathbb{Q}(\sqrt{570})$, and $\mathbb{Q}(\sqrt{571})$; in each case give the period $\ell(\sqrt{D})$ as well as $\epsilon$.

### Problem 6. $S$-class groups and $S$-unit groups (33 points)

Let $K$ be a number field with ring of integers $\mathcal{O}_K$, and let $S$ be a finite set of places of $K$ including all archimedean places. Define the *ring of $S$-integers* $\mathcal{O}_{K,S}$ as the set

$$\mathcal{O}_{K,S} := \{x \in K : v_{\mathfrak{p}}(x) \geq 0 \text{ for all } \mathfrak{p} \notin S\}.$$

**(a)** Prove that $\mathcal{O}_{K,S}$ is a Dedekind domain containing $\mathcal{O}_K$ with the same fraction field.

**(b)** Define a natural homomorphism between $\operatorname{cl} \mathcal{O}_{K,S}$ and $\operatorname{cl} \mathcal{O}_K$ (it is up to you to determine which direction it should go) and use it to prove that $\operatorname{cl} \mathcal{O}_{K,S}$ is finite.

**(c)** Prove that there is a finite set $S$ for which $\mathcal{O}_{K,S}$ is a PID and give an explicit upper bound on $\#S$ that depends only on $n = [K : \mathbb{Q}]$ and $|\operatorname{disc} \mathcal{O}_K|$.

**(d)** Prove the *$S$-unit theorem*: $\mathcal{O}_{K,S}^{\times}$ is a finitely generated abelian group of rank $\#S-1$.

### Problem 7. Survey (1 point)

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

| | Interest | Difficulty | Time Spent |
|---|---|---|---|
| Problem 1 | | | |
| Problem 2 | | | |
| Problem 3 | | | |
| Problem 4 | | | |
| Problem 5 | | | |
| Problem 6 | | | |

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

**Collaborators/Sources**:

## References


[1] Emil Artin and George Whaples, *Axiomatic characterization of fields by the product formula for valuations*, Bull. Amer. Math. Soc. **51** (1945), 469–492.

[2] Alexander Stasinski, *A uniform proof of the finiteness of the class group of a global field*, American Mathematical Monthly **1228** (2021), 239–249.

[3] Richard G. Swan and E. Graham Evans, *$K$-theory of finite groups and orders*, Lecture Notes in Mathematics **149**, Springer, 1970.