

21 Class field theory: ray class groups and ray class fields

In the previous lecture we proved that every abelian extension L of \mathbb{Q} is contained in a cyclotomic field $\mathbb{Q}(\zeta_m)$. The isomorphism $\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$ then allows us to view $\text{Gal}(L/\mathbb{Q})$ as a quotient of $(\mathbb{Z}/m\mathbb{Z})^\times$. We would like to replace the base field \mathbb{Q} with an arbitrary number field K , but we need analogs of the cyclotomic fields $\mathbb{Q}(\zeta_m)$ and the abelian Galois groups $(\mathbb{Z}/m\mathbb{Z})^\times$. These analogs are *ray class fields*, and their Galois groups are isomorphic to *ray class groups*. Ray class fields are not, in general, cyclotomic extensions of K ; their construction is rather more complicated. Before defining them, let us first recall some properties of the Artin map we defined in Lecture 7.

21.1 The Artin map

Let L/K be a finite Galois extension of global fields, and let \mathfrak{p} be a prime of K . Recall that the Galois group $\text{Gal}(L/K)$ acts on the set $\{\mathfrak{q}|\mathfrak{p}\}$ (primes \mathfrak{q} of L lying above \mathfrak{p}) and the stabilizer of $\mathfrak{q}|\mathfrak{p}$ is the decomposition group $D_{\mathfrak{q}} \subseteq \text{Gal}(L/K)$. By Proposition 7.9, we have a surjective homomorphism

$$\begin{aligned} \pi_{\mathfrak{q}}: D_{\mathfrak{q}} &\rightarrow \text{Gal}(\mathbb{F}_{\mathfrak{q}}/\mathbb{F}_{\mathfrak{p}}) \\ \sigma &\mapsto \bar{\sigma} := (\bar{\alpha} \mapsto \overline{\sigma(\alpha)}), \end{aligned}$$

where $\alpha \in \mathcal{O}_L$ is any lift of $\bar{\alpha} \in \mathbb{F}_{\mathfrak{q}} := \mathcal{O}_L/\mathfrak{q}$ to \mathcal{O}_L and $\overline{\sigma(\alpha)}$ is the reduction of $\sigma(\alpha) \in \mathcal{O}_L$ to $\mathbb{F}_{\mathfrak{q}}$; kernel of $\pi_{\mathfrak{q}}$ is the inertia group $I_{\mathfrak{q}}$. If \mathfrak{q} is unramified then $I_{\mathfrak{q}}$ is trivial and $\pi_{\mathfrak{q}}$ is an isomorphism. The *Artin symbol* (Definition 7.20) is defined by

$$\left(\frac{L/K}{\mathfrak{q}}\right) := \sigma_{\mathfrak{q}} := \pi_{\mathfrak{q}}^{-1}(x \mapsto x^{\#\mathbb{F}_{\mathfrak{p}}}),$$

where $(x \mapsto x^{\#\mathbb{F}_{\mathfrak{p}}}) \in \text{Gal}(\mathbb{F}_{\mathfrak{q}}/\mathbb{F}_{\mathfrak{p}})$ is the Frobenius automorphism, a canonical generator for the cyclic group $\text{Gal}(\mathbb{F}_{\mathfrak{q}}/\mathbb{F}_{\mathfrak{p}})$. Equivalently, $\sigma_{\mathfrak{q}}$ is the unique element of $\text{Gal}(L/K)$ for which

$$\sigma_{\mathfrak{q}}(x) \equiv x^{\#\mathbb{F}_{\mathfrak{p}}} \pmod{\mathfrak{q}}$$

for all $x \in \mathcal{O}_L$. For $\mathfrak{q}|\mathfrak{p}$ the Frobenius elements $\sigma_{\mathfrak{q}}$ are all conjugate (they form the Frobenius class $\text{Frob}_{\mathfrak{p}}$), and when L/K is abelian they coincide, in which case we may write $\sigma_{\mathfrak{p}}$ instead of $\sigma_{\mathfrak{q}}$ (or use $\text{Frob}_{\mathfrak{p}} = \{\sigma_{\mathfrak{p}}\}$ to denote $\sigma_{\mathfrak{p}}$), and we may write the Artin symbol as

$$\left(\frac{L/K}{\mathfrak{p}}\right) := \sigma_{\mathfrak{p}}.$$

Now assume L/K is abelian, let \mathfrak{m} be an \mathcal{O}_K -ideal divisible by every ramified prime of K , and let $\mathcal{I}_K^{\mathfrak{m}}$ denote the subgroup of fractional ideals $I \in \mathcal{I}_K$ for which $v_{\mathfrak{p}}(I) = 0$ for all $\mathfrak{p}|\mathfrak{m}$. The Artin map (Definition 7.23) is the homomorphism

$$\begin{aligned} \psi_{L/K}^{\mathfrak{m}}: \mathcal{I}_K^{\mathfrak{m}} &\rightarrow \text{Gal}(L/K) \\ \prod_{\mathfrak{p}/\mathfrak{m}} \mathfrak{p}^{n_{\mathfrak{p}}} &\mapsto \prod_{\mathfrak{p}/\mathfrak{m}} \left(\frac{L/K}{\mathfrak{p}}\right)^{n_{\mathfrak{p}}}. \end{aligned}$$

A key ingredient of class field theory that we will prove in this lecture is surjectivity of the Artin map $\psi_{L/K}^{\mathfrak{m}}$. This allows us to identify $\text{Gal}(L/K)$ with the quotient $\mathcal{I}_K^{\mathfrak{m}}/\ker \psi_{L/K}^{\mathfrak{m}}$.

Every $\mathfrak{p} \in \ker \psi_{L/K}^m$ is unramified and has the property that the Frobenius elements $\sigma_{\mathfrak{q}}$ are trivial for all $\mathfrak{q}|\mathfrak{p}$, meaning that all the residue field extensions $\mathbb{F}_{\mathfrak{q}}/\mathbb{F}_{\mathfrak{p}}$ are trivial. This implies that \mathfrak{p} splits completely in L (it is unramified and primes above it have residue degree one). Conversely, every prime $\mathfrak{p} \in \mathcal{I}_K^m$ that splits completely in L lies in $\ker \psi_{L/K}^m$.

Proposition 21.1. *Let $K \subseteq L \subseteq M$ be a tower of finite abelian extension of global fields and let \mathfrak{m} be an \mathcal{O}_K -ideal divisible by all primes \mathfrak{p} of K that ramify in M . We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{I}_K^m & \xrightarrow{\psi_{M/K}^m} & \text{Gal}(M/K) \\ & \searrow \psi_{L/K}^m & \downarrow \text{res} \\ & & \text{Gal}(L/K) \end{array}$$

where the vertical map is the homomorphism $\sigma \rightarrow \sigma|_L$ induced by restriction.

Proof. It suffices to check commutativity at primes $\mathfrak{p} \nmid \mathfrak{m}$, which are necessarily unramified. The proposition then follows from Proposition 7.22. \square

21.2 Class field theory for \mathbb{Q}

We now specialize to $K = \mathbb{Q}$. The Kronecker-Weber theorem tells us that every abelian extension L/K lies in a cyclotomic field $\mathbb{Q}(\zeta_m)$. Each $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ is determined by its action on ζ_m , and we have an isomorphism

$$\omega: \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^\times$$

defined by $\sigma(\zeta_m) = \zeta_m^{\omega(\sigma)}$. The primes p that ramify in $\mathbb{Q}(\zeta_m)$ are precisely those that divide m (by Corollary 10.18). For each prime $p \nmid m$ the Frobenius element σ_p is the unique $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ for which $\sigma(x) \equiv x^p \pmod{\mathfrak{q}}$ for any (equivalently, all) $\mathfrak{q}|\mathfrak{p}$. Thus $\omega(\sigma_p) = p \pmod{m}$, and it follows that the Artin map induces an inverse isomorphism $(\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$: for every integer a coprime to m we have $(a) \in \mathcal{I}_{\mathbb{Q}}^m$ and

$$\omega^{-1}(\bar{a}) = \left(\frac{\mathbb{Q}(\zeta_m)/\mathbb{Q}}{(a)} \right),$$

where $\bar{a} = a \pmod{m}$. As you showed on Problem Set 4, the surjectivity of the Artin map follows immediately, since a ranges over all integers coprime to m .

Now let L be a subfield of $\mathbb{Q}(\zeta_m)$. We cannot apply ω to $\text{Gal}(L/\mathbb{Q})$, since $\text{Gal}(L/\mathbb{Q})$ is a quotient of $\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, not a subgroup, but the Artin map $\mathcal{I}_{\mathbb{Q}}^m \rightarrow \text{Gal}(L/\mathbb{Q})$ is available; notice that the modulus m works for L as well as $\mathbb{Q}(\zeta_m)$, since any primes that ramify in L also ramify in $\mathbb{Q}(\zeta_m)$ and therefore divide m . By Proposition 21.1, the Artin map factors through the surjective homomorphism $\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \rightarrow \text{Gal}(L/\mathbb{Q})$ induced by restriction and thus induces a surjective homomorphism $(\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \text{Gal}(L/\mathbb{Q})$.

To sum up, we can now say the following about abelian extensions of \mathbb{Q} :

- **Existence:** for each integer m we have a *ray class field* $\mathbb{Q}(\zeta_m)$: an abelian extension ramified only at $p|m$ with Galois group isomorphic to the *ray class group* $(\mathbb{Z}/m\mathbb{Z})^\times$.
- **Completeness:** every abelian extension of \mathbb{Q} lies in a ray class field $\mathbb{Q}(\zeta_m)$.

- **Reciprocity:** if L is an abelian extension of \mathbb{Q} contained in the ray class field $\mathbb{Q}(\zeta_m)$, the Artin map $\mathcal{I}_{\mathbb{Q}}^m \rightarrow \text{Gal}(L/\mathbb{Q})$ induces a surjective homomorphism from the ray class group $(\mathbb{Z}/m\mathbb{Z})^\times$ to $\text{Gal}(L/\mathbb{Q})$, letting us view $\text{Gal}(L/\mathbb{Q})$ as a quotient of $(\mathbb{Z}/m\mathbb{Z})^\times$.

All of these statements will be made more precise; in particular, we will refine the first two statements so that ray class fields are uniquely determined by the modulus m , and we will give an explicit description of the kernel of the Artin map that allows us to identify $\text{Gal}(L/\mathbb{Q})$ with a quotient of $(\mathbb{Z}/m\mathbb{Z})^\times$. But let us first consider how to generalize these statements to number fields other than \mathbb{Q} and define the terms *ray class field*, and *ray class group*. In order to do so, we first need to make the role of the integer m more precise by introducing the notion of a *modulus*.

21.3 Moduli and ray class groups

Recall that for a global field K we use M_K to denote its set of places (equivalence classes of absolute values). We generically denote places by the symbol v , but for finite places, those arising from a discrete valuation associated to a prime \mathfrak{p} of K (a nonzero prime ideal of \mathcal{O}_K), we may write \mathfrak{p} in place of v . We write $v|\infty$ to indicate that v is an infinite place (one not arising from a prime of K); recall that when K is a number field all infinite places are archimedean, and they may be real ($K_v \simeq \mathbb{R}$) or complex ($K_v \simeq \mathbb{C}$).

Definition 21.2. Let K be a number field. A *modulus* (or *cycle*) \mathfrak{m} for K is a function $M_K \rightarrow \mathbb{Z}_{\geq 0}$ with finite support such that for $v|\infty$ we have $\mathfrak{m}(v) \leq 1$ with $\mathfrak{m}(v) = 0$ unless v is a real place. We view \mathfrak{m} as a formal product $\prod v^{\mathfrak{m}(v)}$ over M_K , which we may factor as

$$\mathfrak{m} = \mathfrak{m}_0 \mathfrak{m}_\infty, \quad \mathfrak{m}_0 := \prod_{\mathfrak{p}|\infty} \mathfrak{p}^{\mathfrak{m}(\mathfrak{p})}, \quad \mathfrak{m}_\infty := \prod_{v|\infty} v^{\mathfrak{m}(v)},$$

where \mathfrak{m}_0 is an \mathcal{O}_K -ideal and \mathfrak{m}_∞ represents a subset of the real places of K ; we use $\#\mathfrak{m}_\infty$ to denote the number of real places in the support of \mathfrak{m} . If \mathfrak{m} and \mathfrak{n} are moduli for K we say that \mathfrak{m} *divides* \mathfrak{n} and write $\mathfrak{m}|\mathfrak{n}$ if $\mathfrak{m}(v) \leq \mathfrak{n}(v)$ for all $v \in M_K$. We define the product modulus $\mathfrak{m}\mathfrak{n}$ by $\mathfrak{m}\mathfrak{n}(v) := \mathfrak{m}(v) + \mathfrak{n}(v)$ for $v \nmid \infty$ and $\mathfrak{m}\mathfrak{n}(v) := \max(\mathfrak{m}(v) + \mathfrak{n}(v), 1)$ for $v|\infty$; we also define $\text{gcd}(\mathfrak{m}, \mathfrak{n})(v) := \min(\mathfrak{m}(v), \mathfrak{n}(v))$ and $\text{lcm}(\mathfrak{m}, \mathfrak{n})(v) := \max(\mathfrak{m}(v), \mathfrak{n}(v))$. The zero function is the *trivial modulus*, with $\mathfrak{m}_0 = (1)$ and $\#\mathfrak{m}_\infty = 0$.

We use \mathcal{I}_K to denote the ideal class group of \mathcal{O}_K and define the following notation:¹

- a fractional ideal $I \in \mathcal{I}_K$ is *coprime to* \mathfrak{m} (or *prime to* \mathfrak{m}) if $v_{\mathfrak{p}}(I) = 0$ for all $\mathfrak{p}|\mathfrak{m}_0$.
- $\mathcal{I}_K^{\mathfrak{m}} \subseteq \mathcal{I}_K$ is the subgroup of fractional ideals coprime to \mathfrak{m} .
- $K^{\mathfrak{m}} \subseteq K^\times$ is the subgroup of elements $\alpha \in K^\times$ for which $(\alpha) \in \mathcal{I}_K^{\mathfrak{m}}$.
- $K^{\mathfrak{m},1} \subseteq K^{\mathfrak{m}}$ is the subgroup of elements $\alpha \in K^{\mathfrak{m}}$ with $v_{\mathfrak{p}}(\alpha - 1) \geq v_{\mathfrak{p}}(\mathfrak{m}_0)$ for all $\mathfrak{p}|\mathfrak{m}_0$ and $\alpha_v > 0$ for $v|\mathfrak{m}_\infty$ (here α_v is the image of α under $K \hookrightarrow K_v \simeq \mathbb{R}$).
- $\mathcal{R}_K^{\mathfrak{m}} \subseteq \mathcal{I}_K^{\mathfrak{m}}$ is the subgroup of principal fractional ideals $(\alpha) \in \mathcal{I}_K^{\mathfrak{m}}$ with $\alpha \in K^{\mathfrak{m},1}$.

The groups $\mathcal{R}_K^{\mathfrak{m}}$ are called *rays* or *ray groups*.

¹This notation varies from author to author; there is no universally accepted notation for these objects (in particular, the modulus \mathfrak{m} may appear as a subscript rather than a superscript). Things will improve when we come to the adelic/idelic formulation of class field theory where there is more consistency.

Definition 21.3. Let \mathfrak{m} be a modulus for a number field K . The *ray class group* for the modulus \mathfrak{m} is the quotient

$$\mathrm{Cl}_K^{\mathfrak{m}} := \mathcal{I}_K^{\mathfrak{m}} / \mathcal{R}_K^{\mathfrak{m}}.$$

A finite abelian extension L/K that is unramified at all places² not in the support of \mathfrak{m} for which the kernel of the Artin map $\psi_{L/K}^{\mathfrak{m}}: \mathcal{I}_K^{\mathfrak{m}} \rightarrow \mathrm{Gal}(L/K)$ is equal to the ray group $\mathcal{R}_K^{\mathfrak{m}}$ is a *ray class field* for the modulus \mathfrak{m} .

When \mathfrak{m} is the trivial modulus, the ray class group is the same as the usual class group $\mathrm{Cl}_K := \mathrm{cl}(\mathcal{O}_K)$, but in general the class group Cl_K is a quotient of the ray class group $\mathrm{Cl}_K^{\mathfrak{m}}$ (as we will prove shortly). While not immediately apparent from the definition, we will see that ray class fields are uniquely determined by \mathfrak{m} , so it makes sense to speak of *the* ray class field for the modulus \mathfrak{m} (assuming existence).

Remark 21.4. The definitions above make sense for any global field, but in our ideal-theoretic treatment of class field theory we will mostly restrict our attention to number fields. Our adelic/idelic formulation of class field theory will address all global fields.

Remark 21.5. If $\mathfrak{m}(v) = 1$ for every real place v of K then $\mathrm{Cl}_K^{\mathfrak{m}}$ is a *narrow ray class group*. The narrow ray class group with $\mathfrak{m}_0 = (1)$ is the *narrow class group*; the usual class group $\mathrm{Cl}_K = \mathrm{cl} \mathcal{O}_K$ is sometimes called the *wide class group* to distinguish the two. Note that the wide class group is a quotient of the narrow class group, thus smaller in general; this terminology can be confusing, but the thing to remember is that narrow equivalence is *stronger* than ordinary equivalence, so there are *more* narrow equivalence classes, in general. Of course for number fields with no real places (imaginary quadratic fields, in particular) there is no distinction.

Example 21.6. For $K = \mathbb{Q}$ with the modulus $\mathfrak{m} = (5)$ we have $K^{\mathfrak{m}} = \{a/b : a, b \not\equiv 0 \pmod{5}\}$ and $K^{\mathfrak{m},1} = \{a/b : a \equiv b \pmod{5}\}$. Thus

$$\begin{aligned} \mathcal{I}_K^{\mathfrak{m}} &= \{(1), (1/2), (2), (1/3), (2/3), (3/2), (3), (1/4), (3/4), (4/3), (4), (1/6), (6), \dots\}, \\ \mathcal{R}_K^{\mathfrak{m}} &= \{(1), (2/3), (3/2), (1/4), (4), (6), (1/6), (2/7), (7/2), \dots\}. \end{aligned}$$

You might not have expected $(2/3) \in \mathcal{R}_K^{\mathfrak{m}}$, since $2/3 \notin K^{\mathfrak{m},1}$, but note that $-2/3 \in K^{\mathfrak{m},1}$ and $(-2/3) = (2/3)$. The ray class group is

$$\mathrm{Cl}_K^{\mathfrak{m}} = \mathcal{I}_K^{\mathfrak{m}} / \mathcal{R}_K^{\mathfrak{m}} = \{[(1)], [(2)]\} \simeq (\mathbb{Z}/5\mathbb{Z})^{\times} / \{\pm 1\},$$

which is isomorphic to the Galois group of the totally real subfield $\mathbb{Q}(\zeta_5)^+$ of $\mathbb{Q}(\zeta_5)$, which is the ray class field for this modulus. If we change the modulus to $\mathfrak{m} = (5)_{\infty}$ we instead get $\mathcal{R}_K^{\mathfrak{m}} = \{(1), (6), (1/6), (2/7), (7/2), \dots\}$, $\mathrm{Cl}_K^{\mathfrak{m}} \simeq (\mathbb{Z}/5\mathbb{Z})^{\times}$, and the ray class field is $\mathbb{Q}(\zeta_5)$.

Lemma 21.7. *Let A be a Dedekind domain and let \mathfrak{a} be an A -ideal. Every ideal class in $\mathrm{cl}(A)$ contains an A -ideal coprime to \mathfrak{a} .*

Proof. Let I be a nonzero fractional ideal of A . For each prime $\mathfrak{p} | \mathfrak{a}$ we can pick $\pi_{\mathfrak{p}} \in \mathfrak{p}$ such that $v_{\mathfrak{q}}(\pi_{\mathfrak{p}}) = v_{\mathfrak{q}}(\mathfrak{p})$ for all $\mathfrak{q} | \mathfrak{a}$, by Corollary 3.17. If we then put $\alpha := \prod_{\mathfrak{p} | \mathfrak{a}} \pi_{\mathfrak{p}}^{-v_{\mathfrak{p}}(I)}$, then $v_{\mathfrak{p}}(\alpha I) = 0$ for all $\mathfrak{p} | \mathfrak{a}$; thus αI is coprime to \mathfrak{a} and $[\alpha I] = [I]$.

²Archimedean places of K are unramified in L except for real places v with a complex place w of L above them. But if L is unramified at all $\mathfrak{p} \nmid \mathfrak{m}_0$ (necessary for $\psi_{L/K}^{\mathfrak{m}}$ to be defined), and $\ker \psi_{L/K}^{\mathfrak{m}} = \mathcal{R}_K^{\mathfrak{m}}$, then L will necessarily be unramified at all infinite places $v \nmid \mathfrak{m}_{\infty}$; so in the definition of a ray class field it is enough for L to be unramified away from \mathfrak{m}_0 .

Now let S be the finite set of primes \mathfrak{p} for which $v_{\mathfrak{p}}(\alpha I) < 0$ and pick $\pi_{\mathfrak{p}} \in \mathfrak{p}$ such that $v_{\mathfrak{q}}(\pi_{\mathfrak{p}}) = v_{\mathfrak{q}}(\mathfrak{p})$ for all $\mathfrak{q} \in S$ and $\mathfrak{q} \nmid \mathfrak{a}$ (again using Corollary 3.17). If we now put $a := \prod_{\mathfrak{p} \in S} \pi_{\mathfrak{p}}^{-v_{\mathfrak{p}}(\alpha I)} \in A$, then $v_{\mathfrak{p}}(a\alpha I) \geq 0$ for all \mathfrak{p} and $v_{\mathfrak{p}}(a\alpha I) = 0$ for all $\mathfrak{p} \nmid \mathfrak{a}$. Thus $a\alpha I$ is an A -ideal coprime to \mathfrak{a} and $[a\alpha I] = [I]$. \square

Theorem 21.8. *Let \mathfrak{m} be a modulus for a number field K . We have an exact sequence*

$$1 \longrightarrow \mathcal{O}_K^\times \cap K^{\mathfrak{m},1} \longrightarrow \mathcal{O}_K^\times \longrightarrow K^{\mathfrak{m}}/K^{\mathfrak{m},1} \longrightarrow \text{Cl}_K^{\mathfrak{m}} \longrightarrow \text{Cl}_K \longrightarrow 1$$

and a canonical isomorphism

$$K^{\mathfrak{m}}/K^{\mathfrak{m},1} \simeq \{\pm 1\}^{\#\mathfrak{m}_\infty} \times (\mathcal{O}_K/\mathfrak{m}_0)^\times.$$

Proof. Let us consider the composition of the maps $K^{\mathfrak{m},1} \subseteq K^{\mathfrak{m}}$ and $\alpha \mapsto (\alpha)$:

$$K^{\mathfrak{m},1} \xrightarrow{f} K^{\mathfrak{m}} \xrightarrow{g} \mathcal{I}_K^{\mathfrak{m}}.$$

The kernel of f is trivial, the kernel of $g \circ f$ is $\mathcal{O}_K^\times \cap K^{\mathfrak{m},1}$ (since $(\alpha) = (1) \iff \alpha \in \mathcal{O}_K^\times$), the kernel of g is \mathcal{O}_K^\times , the cokernel of f is $K^{\mathfrak{m}}/K^{\mathfrak{m},1}$, the cokernel of $g \circ f$ is $\text{Cl}_K^{\mathfrak{m}} = \mathcal{I}_K^{\mathfrak{m}}/\mathcal{R}_K^{\mathfrak{m}}$ (by definition), and the cokernel of g is Cl_K (by Lemma 21.7). Applying the snake lemma (see [2, Lemma 5.13], for example) to the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 & \longrightarrow & K^{\mathfrak{m},1} & \xrightarrow{f} & K^{\mathfrak{m}} & \longrightarrow & K^{\mathfrak{m}}/K^{\mathfrak{m},1} \longrightarrow 1 \\ & & \downarrow g \circ f & & \downarrow g & & \downarrow \pi \\ 1 & \longrightarrow & \mathcal{I}_K^{\mathfrak{m}} & \xrightarrow{\sim} & \mathcal{I}_K^{\mathfrak{m}} & \longrightarrow & 1 \end{array}$$

yields the exact sequence $\ker g \circ f \rightarrow \ker g \rightarrow \ker \pi \rightarrow \text{coker } g \circ f \rightarrow \text{coker } g \rightarrow \text{coker } \pi$:

$$1 \longrightarrow \mathcal{O}_K^\times \cap K^{\mathfrak{m},1} \longrightarrow \mathcal{O}_K^\times \longrightarrow K^{\mathfrak{m}}/K^{\mathfrak{m},1} \longrightarrow \text{Cl}_K^{\mathfrak{m}} \longrightarrow \text{Cl}_K \longrightarrow 1,$$

where the initial 1 follows from the fact that f is injective (and $\ker \pi = \text{coker } f$).

We can write each $\alpha \in K^{\mathfrak{m}}$ as $\alpha = a/b$ with $a, b \in \mathcal{O}_K$ chosen so that (a) and (b) are both coprime to \mathfrak{m}_0 . We now define the homomorphism

$$\begin{aligned} \varphi: K^{\mathfrak{m}} &\rightarrow \left(\prod_{v|\mathfrak{m}_\infty} \{\pm 1\} \right) \times (\mathcal{O}_K/\mathfrak{m}_0)^\times \\ \alpha &\mapsto \left(\prod_{v|\mathfrak{m}_\infty} \text{sgn}(\alpha_v) \right) \times \bar{\alpha}, \end{aligned}$$

where $\bar{\alpha} = \bar{a}\bar{b}^{-1} \in (\mathcal{O}_K/\mathfrak{m}_0)^\times$ (here \bar{a}, \bar{b} are the images of $a, b \in \mathcal{O}_K$ in $\mathcal{O}_K/\mathfrak{m}_0$, and they both lie in $(\mathcal{O}_K/\mathfrak{m}_0)^\times$ because (a) and (b) are coprime to \mathfrak{m}_0). The ring $(\mathcal{O}_K/\mathfrak{m}_0)^\times$ is isomorphic to $\prod_{\mathfrak{p}|\mathfrak{m}_0} (\mathcal{O}_K/\mathfrak{p}^{\mathfrak{m}(\mathfrak{p})})^\times$, by the Chinese remainder theorem, and weak approximation (Theorem 8.5) implies that φ is surjective. The kernel of φ is clearly $K^{\mathfrak{m},1}$, thus φ induces an isomorphism $K^{\mathfrak{m}}/K^{\mathfrak{m},1} \simeq \{\pm 1\}^{\#\mathfrak{m}_\infty} \times (\mathcal{O}_K/\mathfrak{m}_0)^\times$. This isomorphism does not depend on the choices of a and b ; every choice yields the same image $\bar{\alpha}$ of $\alpha = a/b$ in $(\mathcal{O}_K/\mathfrak{m}_0)^\times$. \square

Corollary 21.9. Let K be a number field and let \mathfrak{m} be a modulus for K . The ray class group $\text{Cl}_K^{\mathfrak{m}}$ is a finite abelian group whose cardinality $h_K^{\mathfrak{m}} := \#\text{Cl}_K^{\mathfrak{m}}$ is given by

$$h_K^{\mathfrak{m}} = \frac{\phi(\mathfrak{m})h_K}{[\mathcal{O}_K^{\times} : \mathcal{O}_K^{\times} \cap K^{\mathfrak{m},1}]},$$

where $h_K := \#\text{Cl}_K$ and $\phi(\mathfrak{m}) := \#(K^{\mathfrak{m}}/K^{\mathfrak{m},1}) = \phi(\mathfrak{m}_{\infty})\phi(\mathfrak{m}_0)$, with

$$\phi(\mathfrak{m}_{\infty}) = 2^{\#\mathfrak{m}_{\infty}}, \quad \phi(\mathfrak{m}_0) = \#(\mathcal{O}_K/\mathfrak{m}_0)^{\times} = N(\mathfrak{m}_0) \prod_{\mathfrak{p}|\mathfrak{m}_0} (1 - N(\mathfrak{p})^{-1}).$$

In particular, h_K divides $h_K^{\mathfrak{m}}$ and $h_K^{\mathfrak{m}}$ divides $h_K\phi(\mathfrak{m})$.

Proof. The exact sequence implies $\phi(\mathfrak{m})/[\mathcal{O}_K^{\times} : \mathcal{O}_K^{\times} \cap K^{\mathfrak{m},1}] = h_K^{\mathfrak{m}}/h_K$, and that both sides of this equality are integers. \square

Computing the ray class number $h_K^{\mathfrak{m}}$ is not a trivial problem, but there are algorithms for doing so; see [1], which considers this problem in detail.

21.4 Polar density

We now want to prove the surjectivity of the Artin map for finite abelian extensions L/K of number fields (as noted in §21.2, we already know this for $K = \mathbb{Q}$). In order to do so we first introduce a new way to measure the density of a set of primes that is defined in terms of a generalization of the Dedekind zeta function. Throughout this section and the next, all number fields are assumed to lie in some fixed algebraic closure of \mathbb{Q} .

Definition 21.10. Let K be a number field and let S be a set of primes of K . The *partial Dedekind zeta function* associated to S is the complex function

$$\zeta_{K,S}(s) := \prod_{\mathfrak{p} \in S} (1 - N(\mathfrak{p})^{-s})^{-1},$$

which converges to a holomorphic function on $\text{Re}(s) > 1$ (by the same argument we used for $\zeta_K(s)$ in Lecture 18).

If S is finite then $\zeta_{K,S}(s)$ is certainly holomorphic (and nonzero) on a neighborhood of 1. If S contains all but finitely many primes of K then it differs from $\zeta_K(s)$ by a holomorphic factor and therefore extends to a meromorphic function with a simple pole at $s = 1$, by Theorem 19.12.

Between these two extremes the function $\zeta_{K,S}(s)$ may or may not extend to a function that is meromorphic on a neighborhood of 1, but if it does, or more generally, if some power of it does, then we can use the order of the pole at 1 (or the absence of a pole) to measure the density of S .

Definition 21.11. If for some integer $n \geq 1$ the function $\zeta_{K,S}^n$ extends to a meromorphic function on a neighborhood of 1, the *polar density* of S is defined by

$$\rho(S) := \frac{m}{n}, \quad m = -\text{ord}_{s=1} \zeta_{K,S}^n(s)$$

(so m is the order of the pole at $s = 1$, if one is present). Note that if $\zeta_{K,S}^{n_1}$ and $\zeta_{K,S}^{n_2}$ both extend to a meromorphic function on a neighborhood of 1 then we necessarily have

$$n_2 \text{ord}_{s=1} \zeta_{K,S}^{n_1}(s) = \text{ord}_{s=1} \zeta_{K,S}^{n_1 n_2}(s) = n_1 \text{ord}_{s=1} \zeta_{K,S}^{n_2}(s),$$

which implies that $\rho(S)$ does not depend on the choice of n . We will show below that (whenever it is defined) $\rho(S)$ is a rational number in the interval $[0, 1]$.

In Lecture 17 we encountered two other notions of density, the *Dirichlet density*

$$d(S) := \lim_{s \rightarrow 1^+} \frac{\sum_{\mathfrak{p} \in S} N(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}} N(\mathfrak{p})^{-s}} = \lim_{s \rightarrow 1^+} \frac{\sum_{\mathfrak{p} \in S} N(\mathfrak{p})^{-s}}{\log \frac{1}{s-1}},$$

(the equality of the two expressions for $d(S)$ follows from the fact that $\zeta_K(s)$ has a simple pole at $s = 1$, see Problem Set 9), and the *natural density*

$$\delta(S) := \lim_{x \rightarrow \infty} \frac{\#\{\mathfrak{p} \in S : N(\mathfrak{p}) \leq x\}}{\#\{\mathfrak{p} : N(\mathfrak{p}) \leq x\}}.$$

On Problem Set 9 you proved that if S has a natural density then it has a Dirichlet density and the two coincide. We now show that the same is true of the polar density.

Proposition 21.12. *Let S be a set of primes of a number field K . If S has a polar density then it has a Dirichlet density and the two are equal. In particular, $\rho(S) \in [0, 1]$ whenever it is defined.*

Proof. Suppose S has polar density $\rho(S) = m/n$. By taking the Laurent series expansion of $\zeta_{K,S}^n(s)$ at $s = 1$ and factoring out the leading nonzero term we can write

$$\zeta_{K,S}^n(s) = \frac{a}{(s-1)^m} \left(1 + \sum_{r \geq 1} a_r (s-1)^r \right),$$

for some $a \in \mathbb{C}^\times$. We must have $a \in \mathbb{R}_{>0}$, since $\zeta_{K,S}(s) \in \mathbb{R}_{>0}$ for $s \in \mathbb{R}_{>1}$ and therefore $\lim_{s \rightarrow 1^+} (s-1)^m \zeta_{K,S}^n(s)$ is a positive real number. Taking logs of both sides yields

$$n \sum_{\mathfrak{p} \in S} N(\mathfrak{p})^{-s} \sim m \log \frac{1}{s-1} \quad (\text{as } s \rightarrow 1^+),$$

which implies that S has Dirichlet density $d(S) = m/n$ (note that $\log(a) = O(1)$ plays no role, since $-m \log(s-1) \rightarrow \infty$ as $s \rightarrow 1^+$). \square

Corollary 21.13. *Let S be a set of primes of a number field K . If S has both a polar density and a natural density then the two coincide.*

We should note that not every set of primes with a natural density has a polar density, since the later is always a rational number while the former need not be.

Recall that a degree-1 prime in a number field K is a prime with residue field degree 1 over \mathbb{Q} , equivalently, a prime \mathfrak{p} whose absolute norm $N(\mathfrak{p}) = [\mathcal{O}_K : \mathfrak{p}] = \#\mathbb{F}_{\mathfrak{p}}$ is prime.

Proposition 21.14. *Let S and T denote sets of primes in a number field K , let \mathcal{P} be the set of all primes of K , and let \mathcal{P}_1 be the set of degree-1 primes of K . The following hold:*

- (a) *If S is finite then $\rho(S) = 0$; if $\mathcal{P} - S$ is finite then $\rho(S) = 1$.*
- (b) *If $S \subseteq T$ both have polar densities, then $\rho(S) \leq \rho(T)$.*
- (c) *If two sets S and T have finite intersection and any two of the sets S , T , and $S \cup T$ have polar densities then so does the third and $\rho(S \cup T) = \rho(S) + \rho(T)$.*

(d) We have $\rho(\mathcal{P}_1) = 1$, and $\rho(S \cap \mathcal{P}_1) = \rho(S)$ whenever S has a polar density.

Proof. We first note that for any finite set S , the function $\zeta_{K,S}(s)$ is a finite product of nonvanishing entire functions and therefore holomorphic and nonzero everywhere (including at $s = 1$). If the symmetric difference of S and T is finite, then $\zeta_{K,S}(s)f(s) = \zeta_{K,T}(s)g(s)$ for some nonvanishing functions $f(s)$ and $g(s)$ holomorphic on \mathbb{C} . Thus if S and T differ by a finite set, then $\rho(S) = \rho(T)$ whenever either set has a polar density.

Part (a) follows, since $\rho(\emptyset) = 0$ and $\rho(\mathcal{P}) = 1$ (note that $\zeta_{K,\mathcal{P}}(s) = \zeta_K(s)$, and $\text{ord}_{s=1}\zeta_K(s) = -1$, by Theorem 19.12).

Part (b) follows from the analogous statement for Dirichlet density proved on Problem Set 9.

For (c) we may assume S and T are disjoint (by the argument above), in which case $\zeta_{K,S \cup T}^n(s) = \zeta_{K,S}(s)^n \zeta_{K,T}^n(s)$ for all $n \geq 1$, and the claim follows.

For (d), let $\mathcal{P}_2 := \mathcal{P} - \mathcal{P}_1$ so that $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2$. For each rational prime p there are at most $n := [K : \mathbb{Q}]$ (in fact $n/2$) primes $\mathfrak{p}|p$ in \mathcal{P}_2 , each of which has absolute norm $N(\mathfrak{p}) \geq p^2$. It follows by comparison with $\zeta^n(2s)$ that the product defining $\zeta_{K,\mathcal{P}_2}(s)$ converges absolutely to a holomorphic function on $\text{Re}(s) > 1/2$ and is therefore holomorphic (and nonvanishing, since it is an Euler product) on a neighborhood of 1; thus $\rho(\mathcal{P}_2) = 0$ and $\rho(\mathcal{P}_1) = 1$. We therefore have $\rho(S \cap \mathcal{P}_2) = 0$, so $\rho(S) = \rho(S \cap \mathcal{P}_1)$ whenever $\rho(S)$ exists, by (c). \square

For a Galois extension of number fields L/K , let $\text{Spl}(L/K)$ denote the set of primes of K that split completely in L . When K is clear from context we may just write $\text{Spl}(L)$.

Theorem 21.15. *Let L/K be a Galois extension of number fields of degree n . Then*

$$\rho(\text{Spl}(L)) = 1/n.$$

Proof. Let S be the set of degree-1 primes of K that split completely in L ; it suffices to show $\rho(S) = 1/n$, by Proposition 21.14. Recall that \mathfrak{p} splits completely in L if and only if both the ramification index $e_{\mathfrak{p}}$ and residue field degree $f_{\mathfrak{p}}$ are equal to 1. Let T be the set of primes \mathfrak{q} of L that lie above some $\mathfrak{p} \in S$. For each $\mathfrak{q} \in T$ lying above $\mathfrak{p} \in S$ we have $N_{L/K}(\mathfrak{q}) = \mathfrak{p}^{f_{\mathfrak{p}}} = \mathfrak{p}$, so $N(\mathfrak{q}) = N(N_{L/K}(\mathfrak{q})) = N(\mathfrak{p})$, thus \mathfrak{q} is a degree-1 prime, since \mathfrak{p} is.

On the other hand, if \mathfrak{q} is any unramified degree-1 prime of L and $\mathfrak{p} = \mathfrak{q} \cap \mathcal{O}_K$, then $N(\mathfrak{q}) = N(N_{L/K}(\mathfrak{q})) = N(\mathfrak{p}^{f_{\mathfrak{p}}})$ is prime, so we must have $f_{\mathfrak{p}} = 1$, and $e_{\mathfrak{p}} = 1$ since \mathfrak{q} is unramified, which implies that \mathfrak{p} is a degree-1 prime that splits completely in L and is thus an element of S . Only finitely many primes ramify, so all but finitely many of the degree-1 primes in L lie in T , thus $\rho(T) = 1$, by Proposition 21.14. Each $\mathfrak{p} \in S$ has exactly n primes $\mathfrak{q} \in T$ lying above it (since \mathfrak{p} splits completely), and we have

$$\zeta_{L,T}(s) = \prod_{\mathfrak{q} \in T} (1 - N(\mathfrak{q})^{-s})^{-1} = \prod_{\mathfrak{q} \in T} (1 - N(N_{L/K}(\mathfrak{q}))^{-s})^{-1} = \prod_{\mathfrak{p} \in S} (1 - N(\mathfrak{p})^{-s})^{-n} = \zeta_{K,S}^n(s).$$

It follows that $\rho(S) = \frac{1}{n}\rho(T) = \frac{1}{n}$ as desired. \square

Corollary 21.16. *If L/K is a finite extension of number fields with Galois closure M/K of degree n , then $\rho(\text{Spl}(L)) = \rho(\text{Spl}(M)) = 1/n$.*

Proof. A prime \mathfrak{p} of K splits completely in L if and only if it splits completely in all the conjugates of L in M ; the Galois closure M is the compositum of the conjugates of L , so \mathfrak{p} splits completely in L if and only if it splits completely in M . \square

Corollary 21.17. *Let L/K be a finite Galois extension of number fields with Galois group $G := \text{Gal}(L/K)$ and let H be a normal subgroup of G . The set S of primes for which $\text{Frob}_{\mathfrak{p}} \subseteq H$ has polar density $\rho(S) = \#H/\#G$.*

Proof. Let $F = L^H$; then F/K is Galois (since H is normal) and $\text{Gal}(F/K) \simeq G/H$. For each unramified prime \mathfrak{p} of K , the Frobenius class $\text{Frob}_{\mathfrak{p}}$ lies in H if and only if every $\sigma_{\mathfrak{q}} \in \text{Frob}_{\mathfrak{p}}$ acts trivially on $L^H = F$, which occurs if and only if \mathfrak{p} splits completely in F . By Theorem 21.15, the density of this set of primes is $1/[F : K] = \#H/\#G$. \square

If S and T are sets of primes whose symmetric difference is finite, then either $\rho(S) = \rho(T)$ or neither set has a polar density. Let us write $S \sim T$ to indicate that two sets of primes have finite symmetric difference (this is clearly an equivalence relation), and partially order sets of primes by defining $S \lesssim T \Leftrightarrow S \sim S \cap T$ (in other words, $S - T$ is finite). If S and T have polar densities, then $S \lesssim T$ implies $\rho(S) \leq \rho(T)$, by Proposition 21.14.

Theorem 21.18. *If L/K and M/K are two finite Galois extensions of number fields then*

$$\begin{aligned} L \subseteq M &\iff \text{Spl}(M) \lesssim \text{Spl}(L) \iff \text{Spl}(M) \subseteq \text{Spl}(L), \\ L = M &\iff \text{Spl}(M) \sim \text{Spl}(L) \iff \text{Spl}(M) = \text{Spl}(L), \end{aligned}$$

and the map $L \mapsto \text{Spl}(L)$ is an injection from the set of finite Galois extensions of K (inside some fixed algebraic closure) to sets of primes of K that have a positive polar density.

Proof. The implications $L \subseteq M \Rightarrow \text{Spl}(M) \subseteq \text{Spl}(L) \Rightarrow \text{Spl}(M) \lesssim \text{Spl}(L)$ are clear, so it suffices to show that $\text{Spl}(M) \lesssim \text{Spl}(L) \Rightarrow L \subseteq M$.

A prime \mathfrak{p} of K splits completely in the compositum LM if and only if it splits completely in both L and M : the forward implication is clear and for the reverse, note that if \mathfrak{p} splits completely in both L and M then it certainly splits completely in $L \cap M$, so we may assume $K = L \cap M$; we then have $\text{Gal}(LM/K) \simeq \text{Gal}(L/K) \times \text{Gal}(M/K)$, and if the decomposition subgroups of all primes above \mathfrak{p} are trivial in both $\text{Gal}(L/K)$ and $\text{Gal}(M/K)$ then the same applies in $\text{Gal}(LM/K)$. Thus $\text{Spl}(LM) = \text{Spl}(L) \cap \text{Spl}(M)$.

It follows that $\text{Spl}(M) \lesssim \text{Spl}(L) \Rightarrow \text{Spl}(LM) \sim \text{Spl}(M)$. By Theorem 21.15, we have $\rho(\text{Spl}(M)) = 1/[M : K]$ and $\rho(\text{Spl}(LM)) = 1/[LM : K]$, thus $\text{Spl}(LM) \sim \text{Spl}(M)$ implies

$$[LM : K] = \rho(\text{Spl}(LM))^{-1} = \rho(\text{Spl}(M))^{-1} = [M : K],$$

in which case $LM = M$ and $L \subseteq M$. This proves $\text{Spl}(M) \lesssim \text{Spl}(L) \Rightarrow L \subseteq M$, so the three conditions in the first line of biconditionals are all equivalent, and this immediately implies the second line of biconditionals. The last statement of the theorem is clear, since $\text{Spl}(L)$ has positive polar density, by Theorem 21.15. \square

21.5 Ray class fields and Artin reciprocity

As a special case of Corollary 21.16, if F/K is a finite extension of number fields in which all but finitely many primes split completely, then $[F : K] = 1$ and therefore $F = K$. We will use this fact to prove that the Artin map is surjective.

Theorem 21.19. *Let L/K be an abelian extension of number fields and \mathfrak{m} a modulus divisible by all ramified primes. Then the Artin map $\psi_{L/K}^{\mathfrak{m}}: \mathcal{I}_K^{\mathfrak{m}} \rightarrow \text{Gal}(L/K)$ is surjective.*

Proof. Let $H \subseteq \text{Gal}(L/K)$ be the image of $\psi_{L/K}^{\mathfrak{m}}$ and let $F = L^H$ be its fixed field, which we note is a Galois extension of K , since H is normal (because $\text{Gal}(L/K)$ is abelian). For each prime $\mathfrak{p} \in \mathcal{I}_K^{\mathfrak{m}}$ the automorphism $\psi_{L/K}^{\mathfrak{m}}(\mathfrak{p}) \in H$ acts trivially on $F = L^H$, therefore \mathfrak{p} splits completely in F . The group $\mathcal{I}_K^{\mathfrak{m}}$ contains all but finitely many primes \mathfrak{p} of K , so the polar density of the set of primes of K that split completely in F is 1. Thus $[F : K] = 1$ and $H = \text{Gal}(L/K)$, by Corollary 21.16. \square

We now show that the kernel of the Artin map $\psi_{L/K}^{\mathfrak{m}}$ uniquely determines the field L .

Theorem 21.20. *Let \mathfrak{m} be a modulus for a number field K and let L and M be finite abelian extensions of K unramified at all primes not in the support of \mathfrak{m} . If $\ker \psi_{L/K}^{\mathfrak{m}} = \ker \psi_{M/K}^{\mathfrak{m}}$ then $L = M$. In particular, ray class fields are unique whenever they exist.*

Proof. Let S be the set of primes of K that do not divide \mathfrak{m} . Each prime \mathfrak{p} in S is unramified in both L and M , and \mathfrak{p} splits completely in L (resp. M) if and only if it lies in the kernel of $\psi_{L/K}^{\mathfrak{m}}$ (resp. $\psi_{M/K}^{\mathfrak{m}}$). If $\ker \psi_{L/K}^{\mathfrak{m}} = \ker \psi_{M/K}^{\mathfrak{m}}$ then

$$\text{Spl}(L) \sim (S \cap \ker \psi_{L/K}^{\mathfrak{m}}) = (S \cap \ker \psi_{M/K}^{\mathfrak{m}}) \sim \text{Spl}(M),$$

and therefore $L = M$, by Theorem 21.18. \square

Theorem 21.19 implies that we have an exact sequence

$$1 \rightarrow \ker \psi_{L/K}^{\mathfrak{m}} \rightarrow \mathcal{I}_K^{\mathfrak{m}} \rightarrow \text{Gal}(L/K) \rightarrow 1.$$

One of the key results of class field theory is that for a suitable choice of the modulus \mathfrak{m} , we have $\mathcal{R}_K^{\mathfrak{m}} \subseteq \ker \psi_{L/K}^{\mathfrak{m}}$. This implies that the Artin map induces an isomorphism between $\text{Gal}(L/K)$ and a quotient of the ray class group $\text{Cl}_K^{\mathfrak{m}} = \mathcal{I}_K^{\mathfrak{m}}/\mathcal{R}_K^{\mathfrak{m}}$. When L is the ray class field for the modulus \mathfrak{m} , the Artin map allows us to relate subfields of L to quotients of the ray class group $\text{Cl}_K^{\mathfrak{m}} \simeq \text{Gal}(L/K)$ in a way that we will make more precise in the next lecture; this is known as *Artin reciprocity*.

References

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