

12 The different and the discriminant

12.1 The different

We continue in our usual *AKLB* setup: A is a Dedekind domain, K is its fraction field, L/K is a finite separable extension, and B is the integral closure of A in L (a Dedekind domain with fraction field L). We would like to understand the primes that ramify in L/K . Recall that a prime $\mathfrak{q}|\mathfrak{p}$ of L is unramified if and only if $e_{\mathfrak{q}} = 1$ and B/\mathfrak{q} is a separable extension of A/\mathfrak{p} , equivalently, if and only if $B/\mathfrak{q}^{e_{\mathfrak{q}}}$ is a finite étale A/\mathfrak{p} algebra (by Theorem 4.40).¹ A prime \mathfrak{p} of K is unramified if and only if all the primes $\mathfrak{q}|\mathfrak{p}$ lying above it are unramified, equivalently, if and only if the ring $B/\mathfrak{p}B$ is a finite étale A/\mathfrak{p} algebra.²

Our main tools for studying ramification are the *different* $\mathcal{D}_{B/A}$ and *discriminant* $D_{B/A}$. The different is a B -ideal that is divisible by precisely the ramified primes \mathfrak{q} of L , and the discriminant is an A -ideal divisible by precisely the ramified primes \mathfrak{p} of K . Moreover, the valuation $v_{\mathfrak{q}}(\mathcal{D}_{B/A})$ will give us information about the ramification index $e_{\mathfrak{q}}$ (its exact value when \mathfrak{q} is tamely ramified).

Recall from Lecture 5 the trace pairing $L \times L \rightarrow K$ defined by $(x, y) \mapsto \mathrm{Tr}_{L/K}(xy)$; under our assumption that L/K is separable, it is a perfect pairing. An A -lattice M in L is a finitely generated A -module that spans L as a K -vector space (see Definition 5.9). Every A -lattice M in L has a *dual lattice* (see Definition 5.11)

$$M^* := \{x \in L : \mathrm{Tr}_{L/K}(xm) \in A \ \forall m \in M\},$$

which is an A -lattice in L isomorphic to the dual A -module $M^{\vee} := \mathrm{Hom}_A(M, A)$ (see Theorem 5.12). In our *AKLB* setting we have $M^{**} = M$, by Proposition 5.16.

Every fractional ideal I of B is finitely generated as a B -module, and therefore finitely generated as an A module (since B is finite over A). If I is nonzero, it necessarily spans L , since B does. It follows that every element of the group \mathcal{I}_B of nonzero fractional ideals of B is an A -lattice in L . We now show that \mathcal{I}_B is closed under the operation of taking duals.

Lemma 12.1. *Assume AKLB. If $I \in \mathcal{I}_B$ then $I^* \in \mathcal{I}_B$.*

Proof. The dual lattice I^* is a finitely generated A -module, thus to show that it is a finitely generated B -module it is enough to show it is closed under multiplication by elements of B . So consider any $b \in B$ and $x \in I^*$. For all $m \in I$ we have $\mathrm{Tr}_{L/K}((bx)m) = \mathrm{Tr}_{L/K}(x(bm)) \in A$, since $x \in I^*$ and $bm \in I$, so $bx \in I^*$ as desired. \square

Definition 12.2. *Assume AKLB. The different $\mathcal{D}_{L/K}$ of L/K (and the different $\mathcal{D}_{B/A}$ of B/A), is the inverse of B^* in \mathcal{I}_B . Explicitly, we have*

$$B^* := \{x \in L : \mathrm{Tr}_{L/K}(xb) \in A \text{ for all } b \in B\},$$

and we define

$$\mathcal{D}_{L/K} := \mathcal{D}_{B/A} := (B^*)^{-1} = (B : B^*) = \{x \in L : xB^* \subseteq B\}.$$

Note that $B \subseteq B^*$, since $\mathrm{Tr}_{L/K}(ab) \in A$ for $a, b \in B$ (by Corollary 4.52), and this implies $\mathcal{D}_{B/A} = (B^*)^{-1} \subseteq B^{-1} = B$. Thus the different is an ideal, not just a fractional ideal.

¹Note that $B/\mathfrak{q}^{e_{\mathfrak{q}}}$ is reduced if and only if $e_{\mathfrak{q}} = 1$; consider the image of a uniformizer in $B/\mathfrak{q}^{e_{\mathfrak{q}}}$.

²As usual, by a *prime* of A or K we mean a nonzero prime ideal of A , and similarly for B and L . The notation $\mathfrak{q}|\mathfrak{p}$ means that \mathfrak{q} is a prime of B lying above \mathfrak{p} (so $\mathfrak{p} = \mathfrak{q} \cap A$ and \mathfrak{q} divides $\mathfrak{p}B$).

The different respects localization and completion.

Proposition 12.3. *Assume AKLB and let S be a multiplicative subset of A . Then*

$$S^{-1}\mathcal{D}_{B/A} = \mathcal{D}_{S^{-1}B/S^{-1}A}.$$

Proof. This follows from the fact that inverses and duals are both compatible with localization, by Lemmas 3.1 and 5.15. \square

Proposition 12.4. *Assume AKLB and let $\mathfrak{q}|\mathfrak{p}$ be a prime of B . Then*

$$\mathcal{D}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}} = \mathcal{D}_{B/A}\hat{B}_{\mathfrak{q}},$$

where $\hat{A}_{\mathfrak{p}}$ and $\hat{B}_{\mathfrak{q}}$ are the completions of A and B at \mathfrak{p} and \mathfrak{q} , respectively.

Proof. Let $\hat{L} := L \otimes K_{\mathfrak{p}}$ be the base change of the finite étale K -algebra L to $K_{\mathfrak{p}}$. By (5) of Theorem 11.23, we have $\hat{L} \simeq \prod_{\mathfrak{q}|\mathfrak{p}} L_{\mathfrak{q}}$. Note that even though \hat{L} need not be a field, in general, it is a free $K_{\mathfrak{p}}$ -module of finite rank, and is thus equipped with a trace map that necessarily satisfies $\mathrm{Tr}_{\hat{L}/K_{\mathfrak{p}}}(x) = \sum_{\mathfrak{q}|\mathfrak{p}} \mathrm{Tr}_{L_{\mathfrak{q}}/K_{\mathfrak{p}}}(x)$ that defines a trace pairing on \hat{L} .

Now let $\hat{B} := B \otimes \hat{A}_{\mathfrak{p}}$; it is an $A_{\mathfrak{p}}$ -lattice in the $K_{\mathfrak{p}}$ -vector space \hat{L} . By Corollary 11.26, $\hat{B} \simeq \prod_{\mathfrak{q}|\mathfrak{p}} \hat{B}_{\mathfrak{q}} \simeq \bigoplus_{\mathfrak{q}|\mathfrak{p}} \hat{B}_{\mathfrak{q}}$, and therefore $\hat{B}^* \simeq \bigoplus_{\mathfrak{q}|\mathfrak{p}} \hat{B}_{\mathfrak{q}}^*$, by Corollary 5.13. It follows that $\hat{B}^* \simeq B^* \otimes_A \hat{A}_{\mathfrak{p}}$. In particular, B^* generates each fractional ideal $\hat{B}_{\mathfrak{q}}^* \in \mathcal{I}_{\hat{B}_{\mathfrak{q}}}$. Taking inverses, $\mathcal{D}_{B/A} = (B^*)^{-1}$ generates the $\hat{B}_{\mathfrak{q}}$ -ideal $(\hat{B}_{\mathfrak{q}}^*)^{-1} = \mathcal{D}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}}$. \square

12.2 The discriminant

Definition 12.5. Let S/R be a ring extension in which S is a free R -module of rank n . For any $x_1, \dots, x_n \in S$ we define the *discriminant*

$$\mathrm{disc}(x_1, \dots, x_n) := \mathrm{disc}_{S/R}(x_1, \dots, x_n) := \det[\mathrm{Tr}_{S/R}(x_i x_j)]_{i,j} \in R.$$

Note that we do not require x_1, \dots, x_n to be an R -basis for S , but if they satisfy a non-trivial R -linear relation then the discriminant will be zero (by linearity of the trace).

In our AKLB setup, we have in mind the case where $e_1, \dots, e_n \in B$ is a basis for L as a K -vector space, in which case $\mathrm{disc}(e_1, \dots, e_n) = \det[\mathrm{Tr}_{L/K}(e_i e_j)]_{i,j} \in A$. Note that we do not need to assume that B is a free A -module; L is certainly a free K -module. The fact that the discriminant lies in A when $e_1, \dots, e_n \in B$ follows immediately from Corollary 4.52.

Proposition 12.6. *Let L/K be a finite separable extension of degree n , and let Ω/K be a field extension for which there are distinct $\sigma_1, \dots, \sigma_n \in \mathrm{Hom}_K(L, \Omega)$. For any $e_1, \dots, e_n \in L$ we have*

$$\mathrm{disc}(e_1, \dots, e_n) = \det[\sigma_i(e_j)]_{i,j}^2,$$

and for any $x \in L$ we have

$$\mathrm{disc}(1, x, x^2, \dots, x^{n-1}) = \prod_{i < j} (\sigma_i(x) - \sigma_j(x))^2.$$

Such a field extension Ω/K always exists, since L/K is separable ($\Omega = K^{\mathrm{sep}}$ works).

Proof. For $1 \leq i, j \leq n$ we have $\mathbb{T}_{L/K}(e_i e_j) = \sum_{k=1}^n \sigma_k(e_i e_j)$, by Theorem 4.50. Therefore

$$\begin{aligned} \text{disc}(e_1, \dots, e_n) &= \det[\mathbb{T}_{L/K}(e_i e_j)]_{ij} \\ &= \det([\sigma_k(e_i)]_{ik} [\sigma_k(e_j)]_{kj}) \\ &= \det([\sigma_k(e_i)]_{ik} [\sigma_k(e_j)]_{jk}^t) \\ &= \det[\sigma_i(e_j)]_{ij}^2 \end{aligned}$$

since the determinant is multiplicative and $\det M = \det M^t$ for any matrix M .

Now let $x \in L$ and put $e_i := x^{i-1}$ for $1 \leq i \leq n$. Then

$$\text{disc}(1, x, x^2, \dots, x^{n-1}) = \det[\sigma_i(x^{j-1})]_{ij}^2 = \det[\sigma_i(x)^{j-1}]_{ij}^2 = \prod_{i < j} (\sigma_i(x) - \sigma_j(x))^2,$$

since $[\sigma_i(x)^{j-1}]_{ij}$ is a Vandermonde matrix (rows of the form z^0, \dots, z^{n-1} for some z); see [3, p. 258] for a proof of this standard fact. \square

Definition 12.7. For a polynomial $f(x) = \prod_i (x - \alpha_i)$, the *discriminant* of f is

$$\text{disc}(f) := \prod_{i < j} (\alpha_i - \alpha_j)^2.$$

Equivalently, if A is a Dedekind domain, $f \in A[x]$ is a monic separable polynomial, and α is the image of x in $A[x]/(f(x))$, then

$$\text{disc}(f) = \text{disc}(1, \alpha, \alpha^2, \dots, \alpha^{n-1}) \in A.$$

Example 12.8. $\text{disc}(x^2 + bx + c) = b^2 - 4c$ and $\text{disc}(x^3 + ax + b) = -4a^3 - 27b^2$.

Now assume $AKLB$ and let M be an A -lattice in L . Then M is a finitely generated A -module that contains a K -basis for L . We want to define the discriminant of M in a way that does not require us to choose a basis.

Let us first consider the case where M is a free A -lattice. If $e_1, \dots, e_n \in M \subseteq L$ and $e'_1, \dots, e'_n \in M \subseteq L$ are two A -bases for M , then

$$\text{disc}(e'_1, \dots, e'_n) = u^2 \text{disc}(e_1, \dots, e_n)$$

for some unit $u \in A^\times$; this follows from the fact that the change of basis matrix $P \in A^{n \times n}$ is invertible and its determinant is therefore a unit u . This unit gets squared because we need to apply the change of basis matrix twice in order to change $\mathbb{T}(e_i e_j)$ to $\mathbb{T}(e'_i e'_j)$. Explicitly, writing bases as row-vectors, let $e = (e_1, \dots, e_n)$ and $e' = (e'_1, \dots, e'_n)$ satisfy $e' = eP$. Then

$$\begin{aligned} \text{disc}(e') &= \det[\mathbb{T}_{L/K}(e'_i e'_j)]_{ij} \\ &= \det[\mathbb{T}_{L/K}((eP)_i (eP)_j)]_{ij} \\ &= \det[P^t [\mathbb{T}_{L/K}(e_i e_j)]_{ij} P] \\ &= (\det P^t) \text{disc}(e) (\det P) \\ &= (\det P)^2 \text{disc}(e), \end{aligned}$$

where we have used the linearity of $\mathbb{T}_{L/K}$ to go from the second equality to the third.

This actually gives us a basis independent definition when $A = \mathbb{Z}$. In this case B is always a free \mathbb{Z} -lattice, and the only units in \mathbb{Z} are $u = \pm 1$, so $u^2 = 1$.

Definition 12.9. Assume $AKLB$, let M be an A -lattice in L , and let $n := [L : K]$. The *discriminant* $D(M)$ of M is the A -module generated by $\{\text{disc}(x_1, \dots, x_n) : x_1, \dots, x_n \in M\}$.

Lemma 12.10. Assume $AKLB$ and let $M' \subseteq M$ be free A -lattices in L . The discriminants $D(M') \subseteq D(M)$ are nonzero principal fractional ideals. If $D(M') = D(M)$ then $M' = M$.

Proof. Let $e := (e_1, \dots, e_n)$ be an A -basis for M . Then $\text{disc}(e) \in D(M)$, and for any row vector $x := (x_1, \dots, x_n)$ with entries in M there is a matrix $P \in A^{n \times n}$ for which $x = eP$, and we then have $\text{disc}(x) = (\det P)^2 \text{disc}(e)$ as above. It follows that

$$D(M) = (\text{disc}(e))$$

is principal, and it is nonzero because e is a basis for L and the trace pairing is nondegenerate. If we now let $e' := (e'_1, \dots, e'_n)$ be an A -basis for M' then $D(M') = (\text{disc}(e'))$ is also a nonzero and principal. Our assumption that $M' \subseteq M$ implies that $e' = eP$ for some matrix $P \in A^{n \times n}$, and we have $\text{disc}(e') = (\det P)^2 \text{disc}(e)$. If $D(M') = D(M)$ then $\det P$ must be a unit, in which case P is invertible and $e = e'P^{-1}$. This implies $M \subseteq M'$, so $M' = M$. \square

Proposition 12.11. Assume $AKLB$ and let M be an A -lattice in L . Then $D(M) \in \mathcal{I}_A$.

Proof. The A -module $D(M) \subseteq K$ is nonzero because M contains a K -basis $e = (e_1, \dots, e_n)$ for L and $\text{disc}(e) \neq 0$ because the trace pairing is nondegenerate. To show that $D(M)$ is a finitely generated A -module (and thus a fractional ideal), we use the usual trick: make it a submodule of a noetherian module. So let N be the free A -lattice in L generated by e and then pick a nonzero $a \in A$ such that $M \subseteq a^{-1}N$ (write each generator for M in terms of the K -basis e and let a be the product of all the denominators that appear; note that M is finitely generated). We then have $D(M) \subseteq D(a^{-1}N)$, and $D(a^{-1}N)$ is a principal fractional ideal of A , hence a noetherian A -module (since A is noetherian), so its submodule $D(M)$ must be finitely generated. \square

Definition 12.12. Assume $AKLB$. The *discriminant* $D_{L/K}$ of L/K (and the *discriminant* $D_{B/A}$ of B/A) is the discriminant of B as an A -module:

$$D_{L/K} := D_{B/A} := D(B) \in \mathcal{I}_A,$$

which is an A -ideal, since $\text{disc}(x_1, \dots, x_n) = \det[T_{B/A}(x_i x_j)]_{i,j} \in A$ for all $x_1, \dots, x_n \in B$.

Example 12.13. Consider the case $A = \mathbb{Z}$, $K = \mathbb{Q}$, $L = \mathbb{Q}(i)$, $B = \mathbb{Z}[i]$. Then B is a free A -lattice with basis $(1, i)$ and we can compute $D_{L/K}$ in three ways:

- $\text{disc}(1, i) = \det \begin{bmatrix} T_{L/K}(1 \cdot 1) & T_{L/K}(1 \cdot i) \\ T_{L/K}(i \cdot 1) & T_{L/K}(i \cdot i) \end{bmatrix} = \det \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = -4$.
- The non-trivial automorphism of L/K fixes 1 and sends i to $-i$, so we could instead compute $\text{disc}(1, i) = \left(\det \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \right)^2 = (-2i)^2 = -4$.
- We have $B = \mathbb{Z}[i] = \mathbb{Z}[x]/(x^2 + 1)$ and can compute $\text{disc}(x^2 + 1) = -4$.

In every case the discriminant $D_{L/K}$ is the ideal $(-4) = (4)$.

Remark 12.14. If $A = \mathbb{Z}$ then B is the ring of integers of the number field L , and B is a free A -lattice, because it is a torsion-free module over a PID and therefore a free module. In this situation it is customary to define the *absolute discriminant* D_L of the number field L to be the *integer* $\text{disc}(e_1, \dots, e_n) \in \mathbb{Z}$, for any basis (e_1, \dots, e_n) of B , rather than the ideal it generates. As noted above, this integer is independent of the choice of basis because $u^2 = 1$ for all $u \in \mathbb{Z}^\times$; in particular, the sign of D_L is well defined (as we shall see, the sign of D_L carries information about L). In the example above, the absolute discriminant is $D_L = -4$.

Like the different, the discriminant respects localization.

Proposition 12.15. *Assume AKLB and let S be a multiplicative subset of A . Then*

$$S^{-1}D_{B/A} = D_{S^{-1}B/S^{-1}A}.$$

Proof. Let $x = s^{-1} \text{disc}(e_1, \dots, e_n) \in S^{-1}D_{B/A}$ for some $s \in S$ and $e_1, \dots, e_n \in B$. Then $x = s^{2n-1} \text{disc}(s^{-1}e_1, \dots, s^{-1}e_n)$ lies in $D_{S^{-1}B/S^{-1}A}$. This proves the forward inclusion.

Conversely, for any $e_1, \dots, e_n \in S^{-1}B$ we can choose a single $s \in S \subseteq A$ so that each se_i lies in B . We then have $\text{disc}(e_1, \dots, e_n) = s^{-2n} \text{disc}(se_1, \dots, se_n) \in S^{-1}D_{B/A}$, which proves the reverse inclusion. \square

Proposition 12.16. *Assume AKLB and let \mathfrak{p} be a prime of A . Then*

$$D_{B/A}\hat{A}_{\mathfrak{p}} = \prod_{\mathfrak{q}|\mathfrak{p}} D_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}}$$

where $\hat{A}_{\mathfrak{p}}$ and $\hat{B}_{\mathfrak{q}}$ are the completions of A and B at \mathfrak{p} and \mathfrak{q} , respectively.

Proof. After localizing at \mathfrak{p} we can assume A is a DVR and B is a free A -module of rank n . As in the proof of Proposition 12.4, we have a trace pairing on the finite étale $K_{\mathfrak{p}}$ -algebra $\hat{L} := L \otimes K_{\mathfrak{p}}$ and $\hat{B} := B \otimes \hat{A}_{\mathfrak{p}} \simeq \bigoplus_{\mathfrak{q}|\mathfrak{p}} \hat{B}_{\mathfrak{q}}$ is an $\hat{A}_{\mathfrak{p}}$ -lattice in the $K_{\mathfrak{p}}$ -vector space \hat{L} that is a direct sum of free $\hat{A}_{\mathfrak{p}}$ -modules, and thus a free $\hat{A}_{\mathfrak{p}}$ -module of rank $n = \sum e_{\mathfrak{q}}f_{\mathfrak{q}}$; see Corollary 11.26.

We can choose $\hat{A}_{\mathfrak{p}}$ bases for each $\hat{B}_{\mathfrak{q}}$ using elements in B ; this follows from weak approximation (Theorem 8.5) and the fact that B is dense in $\hat{B}_{\mathfrak{q}}$ (or see [2, Thm. 2.3]). From these bases we can construct an $\hat{A}_{\mathfrak{p}}$ -basis \hat{e} for the direct sum $\bigoplus_{\mathfrak{q}|\mathfrak{p}} \hat{B}_{\mathfrak{q}} \simeq \hat{B}$ whose elements each have nonzero projections to exactly one of the $\hat{B}_{\mathfrak{q}}$, along with a corresponding A -basis e for B obtained from \hat{e} as the union of these projections.

The matrix $[T_{\hat{L}/K_{\mathfrak{p}}}(\hat{e}_i\hat{e}_j)]$ is block diagonal; each block corresponds to a matrix whose determinant is the discriminant of the $\hat{A}_{\mathfrak{p}}$ -basis we chose for one of the $\hat{B}_{\mathfrak{q}}$. It follows that $D_{\hat{B}/\hat{A}_{\mathfrak{p}}} = \prod_{\mathfrak{q}|\mathfrak{p}} D_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}}$ (here we are using the fact that $\hat{B} \simeq \bigoplus_{\mathfrak{q}|\mathfrak{p}} \hat{B}_{\mathfrak{q}}$ is both an isomorphism of rings and an isomorphism of $\hat{A}_{\mathfrak{p}}$ -modules, hence it preserves traces to $\hat{A}_{\mathfrak{p}}$). We now observe that

$$\text{disc}_{B/A}(e_1, \dots, e_n) = \text{disc}_{(B \otimes A_{\mathfrak{p}})/\hat{A}_{\mathfrak{p}}}(e_1 \otimes 1, \dots, e_n \otimes 1)$$

generates $D_{B/A}$ as an A -ideal, and also generates $D_{\hat{B}/\hat{A}_{\mathfrak{p}}}$ as an $\hat{A}_{\mathfrak{p}}$ -ideal (note that \hat{B} is a free $\hat{A}_{\mathfrak{p}}$ -module, so $D_{\hat{B}/\hat{A}_{\mathfrak{p}}}$ is the principal ideal generated by the discriminant of any $\hat{A}_{\mathfrak{p}}$ -basis for \hat{B}). It follows that $D_{B/A}\hat{A}_{\mathfrak{p}} = D_{\hat{B}/\hat{A}_{\mathfrak{p}}} = \prod_{\mathfrak{q}|\mathfrak{p}} D_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}}$. \square

We now have two ideals associated to a finite separable extension of Dedekind domains B/A in the $AKLB$ setup. We have the different ideal $\mathcal{D}_{B/A}$, which is a fractional ideal of B , and the discriminant ideal $D_{B/A}$, which is a fractional ideal of A . We now relate these two ideals in terms of the ideal norm $N_{B/A}: \mathcal{I}_B \rightarrow \mathcal{I}_A$, which for $I \in \mathcal{I}_B$ is defined as $N_{B/A}(I) := [B : I]_A$, where $[B : I]_A$ is the module index (see Definitions 6.1 and 6.5).

Theorem 12.17. *Assume $AKLB$. Then $D_{B/A} = N_{B/A}(\mathcal{D}_{B/A})$.*

Proof. The different and discriminant are both compatible with localization, by Propositions 12.3 and 12.15, and the A -modules $D_{B/A}$ and $N_{B/A}(\mathcal{D}_{B/A})$ of A are both determined by the intersections of their localizations at maximal ideals (Proposition 2.6), so it suffices to prove that the theorem holds when we replace A by its localization A at a prime of A . Then A is a DVR and B is a free A -lattice in L ; let us fix an A -basis (e_1, \dots, e_n) for B .

The dual A -lattice

$$B^* = \{x \in L : T_{L/K}(xb) \in A \ \forall b \in B\} \in \mathcal{I}_B$$

is also a free A -lattice in L , with basis (e_1^*, \dots, e_n^*) uniquely determined by $T_{L/K}(e_i^* e_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function; see Corollary 5.14. If we write $e_i = \sum a_{ij} e_j^*$ in terms of the K -basis (e_1^*, \dots, e_n^*) for L then

$$T_{L/K}(e_i e_j) = T_{L/K} \left(\sum_k a_{ik} e_k^* e_j \right) = \sum_k a_{ik} T_{L/K}(e_k^* e_j) = \sum_k a_{ik} \delta_{kj} = a_{ij}.$$

It follows that $P := [T_{L/K}(e_i e_j)]_{ij}$ is the change-of-basis matrix from $e^* := (e_1^*, \dots, e_n^*)$ to $e := (e_1, \dots, e_n)$ (as row vectors we have $e = e^* P$). If we let ϕ denote the K -linear transformation with matrix P (or its transpose, if you prefer to work with column vectors), then ϕ is an isomorphism of free A -modules and

$$D_{B/A} = (\det[T_{L/K}(e_i e_j)]_{ij}) = (\det \phi) = [B^* : B]_A,$$

where $[B^* : B]_A$ is the module index (see Definition 6.1). Applying Corollary 6.8 yields

$$D_{B/A} = [B^* : B]_A = N_{B/A}((B : B^*)) = N_{B/A}((B^*)^{-1}) = N_{B/A}(\mathcal{D}_{B/A}).$$

(the last three equalities each hold by definition). □

12.3 Ramification

Having defined the different and discriminant ideals we now want to understand how they relate to ramification. Recall that in our $AKLB$ setup, if \mathfrak{p} is a prime of A then we can factor the B -ideal $\mathfrak{p}B$ as

$$\mathfrak{p}B = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_r^{e_r}.$$

The Chinese remainder theorem implies

$$B/\mathfrak{p}B \simeq B/\mathfrak{q}_1^{e_1} \times \cdots \times B/\mathfrak{q}_r^{e_r}.$$

This is a commutative A/\mathfrak{p} -algebra of dimension $\sum e_i f_i$, where $f_i = [B/\mathfrak{q}_i : A/\mathfrak{p}]$ is the residue degree (see Theorem 5.35). It is a product of fields if and only if we have $e_i = 1$ for all i , and it is a finite étale-algebra if and only if it is a product of fields that are separable extensions of A/\mathfrak{p} . The following lemma relates the discriminant to the property of being a finite étale algebra.

Lemma 12.18. *Let k be a field and let R be a commutative k -algebra with k -basis r_1, \dots, r_n . Then R is a finite étale k -algebra if and only if $\text{disc}(r_1, \dots, r_n) \neq 0$.*

Proof. By Theorem 5.20, R is a finite étale k -algebra if and only if the trace pairing on R is a perfect pairing, which is equivalent to being nondegenerate, since k is a field.

If the trace pairing is degenerate then for some nonzero $x \in R$ we have $\text{Tr}_{R/k}(xy) = 0$ for all $y \in R$. If we write $x = \sum_i x_i r_i$ with $x_i \in k$ then $\text{Tr}_{R/k}(xr_j) = \sum_i x_i \text{Tr}_{R/k}(r_i r_j) = 0$ for all r_j (take $y = r_j$), and this implies that the columns of the matrix $[\text{Tr}_{R/k}(r_i r_j)]_{ij}$ are linearly dependent and $\text{disc}(r_1, \dots, r_n) = \det[\text{Tr}_{R/k}(r_i r_j)]_{ij} = 0$.

Conversely, if $\text{disc}(r_1, \dots, r_n) = 0$ then the columns of $\det[\text{Tr}_{R/k}(r_i r_j)]_{ij}$ are linearly dependent and for some $x_i \in k$ not identically zero we must have $\sum_i x_i \text{Tr}_{R/k}(r_i r_j) = 0$ for all j . For $x := \sum_i x_i r_i$ and any $y = \sum_j y_j r_j \in R$ we have $\text{Tr}_{R/k}(xy) = \sum_j y_j \sum_i x_i \text{Tr}_{R/k}(r_i r_j) = 0$, which shows that the trace pairing is degenerate. \square

Theorem 12.19. *Assume AKLB, let \mathfrak{q} be a prime of B lying above a prime \mathfrak{p} of A such that B/\mathfrak{q} is a separable extension of A/\mathfrak{p} . The extension L/K is unramified at \mathfrak{q} if and only if \mathfrak{q} does not divide $\mathcal{D}_{B/A}$, and it is unramified at \mathfrak{p} if and only if \mathfrak{p} does not divide $D_{B/A}$.*

Proof. We first consider the different $\mathcal{D}_{B/A}$. By Proposition 12.4, the different is compatible with completion, so it suffices to consider the case that A and B are complete DVRs (complete K at \mathfrak{p} and L at \mathfrak{q} and apply Theorem 11.23). We then have $[L : K] = e_{\mathfrak{q}} f_{\mathfrak{q}}$, where $e_{\mathfrak{q}}$ is the ramification index and $f_{\mathfrak{q}}$ is the residue field degree, and $\mathfrak{p}B = \mathfrak{q}^{e_{\mathfrak{q}}}$.

Since B is a DVR with maximal ideal \mathfrak{q} , we must have $\mathcal{D}_{B/A} = \mathfrak{q}^m$ for some $m \geq 0$. By Theorem 12.17 we have

$$D_{B/A} = N_{B/A}(\mathcal{D}_{B/A}) = N_{B/A}(\mathfrak{q}^m) = \mathfrak{p}^{f_{\mathfrak{q}} m}.$$

Thus $\mathfrak{q} | \mathcal{D}_{B/A}$ if and only if $\mathfrak{p} | D_{B/A}$. Since A is a PID, B is a free A -module and we may choose an A -module basis e_1, \dots, e_n for B that is also a K -basis for L . Let $k := A/\mathfrak{p}$, and let \bar{e}_i be the reduction of e_i to the k -algebra $R := B/\mathfrak{p}B$. Then $(\bar{e}_1, \dots, \bar{e}_n)$ is a k -basis for R : it clearly spans, and we have $[R : k] = [B/\mathfrak{q}^{e_{\mathfrak{q}}} : A/\mathfrak{p}] = e_{\mathfrak{q}} f_{\mathfrak{q}} = [L : K] = n$.

Since B has an A -module basis, we may compute its discriminant as

$$D_{B/A} = (\text{disc}(e_1, \dots, e_n)).$$

Thus $\mathfrak{p} | D_{B/A}$ if and only if $\text{disc}(e_1, \dots, e_n) \in \mathfrak{p}$, equivalently, $\text{disc}(\bar{e}_1, \dots, \bar{e}_n) = 0$ (note that $\text{disc}(e_1, \dots, e_n)$ is a polynomial in the $\text{Tr}_{L/K}(e_i e_j)$ and $\text{Tr}_{R/k}(\bar{e}_i \bar{e}_j)$ is the trace of the multiplication-by- $\bar{e}_i \bar{e}_j$ map, which is the same as the reduction to $k = A/\mathfrak{p}$ of the trace of the multiplication-by- $e_i e_j$ map $\text{Tr}_{L/K}(e_i e_j) \in A$). By Lemma 12.18, $\text{disc}(\bar{e}_1, \dots, \bar{e}_n) = 0$ if and only if the k -algebra $B/\mathfrak{p}B$ is not finite étale, equivalently, if and only if \mathfrak{p} is ramified. Thus $\mathfrak{p} | D_{B/A}$ if and only if \mathfrak{p} is ramified. There is only one prime \mathfrak{q} above \mathfrak{p} , so we also have $\mathfrak{q} | \mathcal{D}_{B/A}$ if and only if \mathfrak{q} is ramified. \square

We now note an important corollary of Theorem 12.19.

Corollary 12.20. *Assume AKLB. Only finitely many primes of A (or B) ramify.*

Proof. A and B are Dedekind domains, so the ideals $D_{B/A}$ and $\mathcal{D}_{B/A}$ both have unique factorizations into prime ideals in which only finitely many primes appear. \square

12.4 The discriminant of an order

Recall from Lecture 6 that an order \mathcal{O} is a noetherian domain of dimension one whose conductor is nonzero (see Definitions 6.16 and 6.19), and the integral closure of an order is always a Dedekind domain. In our $AKLB$ setup, the orders with integral closure B are precisely the A -lattices in L that are rings (see Proposition 6.22); if $L = K(\alpha)$ with $\alpha \in B$, then $A[\alpha]$ is an example. The discriminant $D_{\mathcal{O}/A}$ of such an order \mathcal{O} is its discriminant $D(\mathcal{O})$ as an A -module. The fact that $\mathcal{O} \subseteq B$ implies that $D(\mathcal{O}) \subseteq D_{B/A}$ is an A -ideal.

If \mathcal{O} is an order of the form $A[\alpha]$, where $\alpha \in B$ generates $L = K(\alpha)$ with minimal polynomial $f \in A[x]$, then \mathcal{O} is a free A -lattice with basis $1, \alpha, \dots, \alpha^{n-1}$, where $n = \deg f$, and we may compute its discriminant as

$$D_{\mathcal{O}/A} = (\text{disc}(1, \alpha, \dots, \alpha^{n-1})) = (\text{disc}(f)),$$

which is a principal A -ideal contained in $D_{B/A}$. If B is also a free A -lattice, then as in the proof of Lemma 12.10 we have

$$D_{\mathcal{O}/A} = (\det P)^2 D_{B/A} = [B:\mathcal{O}]_A^2 D_{B/A},$$

where P is the matrix of the A -linear map $\phi: B \rightarrow \mathcal{O}$ that sends an A -basis for B to an A -basis for \mathcal{O} and $[B:\mathcal{O}]_A$ is the module index (a principal A -ideal).

In the important special case where $A = \mathbb{Z}$ and L is a number field, the integer $(\det P)^2$ is uniquely determined and it necessarily divides $\text{disc}(f)$, the generator of the principal ideal $D(\mathcal{O}) = D(A[\alpha])$. It follows that if $\text{disc}(f)$ is squarefree then we must have $B = \mathcal{O} = A[\alpha]$. More generally, any prime p for which $v_p(\text{disc}(f))$ is odd must be ramified, and any prime that does not divide $\text{disc}(f)$ must be unramified. Another useful observation that applies when $A = \mathbb{Z}$: the module index $[B:\mathcal{O}]_{\mathbb{Z}} = ([B:\mathcal{O}])$ is the principal ideal generated by the index of \mathcal{O} in B (as \mathbb{Z} -lattices), and we have the relation

$$D_{\mathcal{O}} = [B:\mathcal{O}]^2 D_B$$

between the absolute discriminant of the order \mathcal{O} and its integral closure B .

Example 12.21. Consider $A = \mathbb{Z}$, $K = \mathbb{Q}$ with $L = \mathbb{Q}(\alpha)$, where $\alpha^3 - \alpha - 1 = 0$. We can compute the absolute discriminant of $\mathbb{Z}[\alpha]$ as

$$\text{disc}(1, \alpha, \alpha^2) = \text{disc}(x^3 - x - 1) = -4(-1)^3 - 27(-1)^2 = -23.$$

The fact that -23 is squarefree immediately implies that 23 is the only prime of A that ramifies, and we have $D_{\mathbb{Z}[\alpha]} = -23 = [\mathcal{O}_L : \mathbb{Z}[\alpha]]^2 D_L$, which forces $[\mathcal{O}_L : \mathbb{Z}[\alpha]] = 1$, so $D_L = -23$ and $\mathcal{O}_L = \mathbb{Z}[\alpha]$.

More generally, we have the following theorem.

Theorem 12.22. *Assume $AKLB$ and let \mathcal{O} be an order with integral closure B and conductor \mathfrak{c} . Then $D_{\mathcal{O}/A} = N_{B/A}(\mathfrak{c}) D_{B/A}$.*

Proof. See Problem Set 6. □

In the example above the fact that the discriminant of $\mathbb{Z}[\alpha]$ was squarefree immediately told us the discriminant of $\mathbb{Q}(\alpha)$. But it will often happen that $\text{disc}(\mathbb{Z}[\alpha])$ is not squarefree; indeed this is necessarily the case if the discriminant of $\mathbb{Q}(\alpha)$ is not squarefree. Computing

the discriminant of a number field is not an easy task in general. The standard approach is to begin by factoring the discriminant of a given order $\mathbb{Z}[\alpha]$ and then for each prime divisor of this discriminant determine whether p divides the index of $\mathbb{Z}[\alpha]$ in the ring of integers and if so constructing a larger order \mathcal{O} for which this is not the case. This approach is due to Pohst and Zassenhaus and is described in [1, §6]. The details are somewhat involved, but let us note the following result due to Dedekind that can often be used to determine whether or not a prime dividing the discriminant of $\mathbb{Z}[\alpha]$ divides the discriminant of $\mathbb{Q}(\alpha)$.

Theorem 12.23 (Dedekind). *Let $K = \mathbb{Q}(\alpha)$ be a number field, let f be the minimal polynomial of α , let p be a prime, let*

$$\bar{f} = \prod_{i=1}^r \bar{f}_i^{e_i}$$

be the factorization of $\bar{f} := f \bmod p$ in $\mathbb{F}_p[x]$. Let f_i be any monic lift of \bar{f}_i to $\mathbb{Z}[x]$, let $u := \prod f_i$, let v be any monic lift of \bar{f}/\bar{g} to $\mathbb{Z}[x]$, and let $w = (uv - f)/p \in \mathbb{Z}[x]$. Then p divides $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$ if and only if $\bar{u}, \bar{v}, \bar{w}$ have no common factor in $\mathbb{F}_p[x]$.

Proof. See [1, Thm. 6.1.4]. □

12.5 Computing the discriminant and different

We conclude with a number of results that allow one to explicitly compute the discriminant and different in many cases.

Proposition 12.24. *Assume AKLB. If $B = A[\alpha]$ for some $\alpha \in L$ and $f \in A[x]$ is the minimal polynomial of α , then*

$$\mathcal{D}_{B/A} = (f'(\alpha))$$

is the B -ideal generated by $f'(\alpha)$.

Proof. See Problem Set 6. □

The assumption $B = A[\alpha]$ in Proposition 12.24 does not always hold, but if we want to compute the power of \mathfrak{q} that divides $\mathcal{D}_{B/A}$ we can complete L at \mathfrak{q} and K at $\mathfrak{p} = \mathfrak{q} \cap A$ so that A and B become complete DVRs, in which case $B = A[\alpha]$ does hold (by Lemma 10.12), so long as the residue field extension is separable (always true if K and L are global fields, since the residue fields are then finite, hence perfect). The following definition and proposition give an alternative approach.

Definition 12.25. Assume AKLB and let $\alpha \in B$ have minimal polynomial $f \in A[x]$. The *different* of α is defined by

$$\delta_{B/A}(\alpha) := \begin{cases} f'(\alpha) & \text{if } L = K(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 12.26. *Assume AKLB. Then $\mathcal{D}_{B/A} = (\delta_{B/A}(\alpha) : \alpha \in B)$.*

Proof. See [4, Thm. III.2.5]. □

We can now more precisely characterize the ramification information given by the different ideal.

Theorem 12.27. Assume $AKLB$ and let \mathfrak{q} be a prime of L lying above $\mathfrak{p} = \mathfrak{q} \cap A$ for which the residue field extension $(B/\mathfrak{q})/(A/\mathfrak{p})$ is separable. Then

$$e_{\mathfrak{q}} - 1 \leq v_{\mathfrak{q}}(\mathcal{D}_{B/A}) \leq e_{\mathfrak{q}} - 1 + v_{\mathfrak{q}}(e_{\mathfrak{q}}),$$

and the lower bound is an equality if and only if \mathfrak{q} is tamely ramified.

Proof. See Problem Set 6. □

We also note the following proposition, which shows how the discriminant and different behave in a tower of extensions.

Proposition 12.28. Assume $AKLB$ and let M/L be a finite separable extension and let C be the integral closure of A in M . Then

$$\mathcal{D}_{C/A} = \mathcal{D}_{C/B} \cdot \mathcal{D}_{B/A}$$

(where the product on the right is taken in C), and

$$D_{C/A} = (D_{B/A})^{[M:L]} N_{B/A}(D_{C/B}).$$

Proof. See [5, Prop. III.8]. □

If $M/L/K$ is a tower of finite separable extensions, we note that the primes \mathfrak{p} of K that ramify are precisely those that divide either $D_{L/K}$ or $N_{L/K}(D_{M/L})$.

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