27 Local class field theory

In this lecture we give a brief overview of local class field theory. Recall that a local field is a locally compact field whose topology is induced by a nontrivial absolute value (Definition 9.1). As we proved in Theorem 9.9, every local field is isomorphic to one of the following:

- $\mathbb{R}$ or $\mathbb{C}$ (archimedean, characteristic 0);
- finite extension of $\mathbb{Q}_p$ (nonarchimedean, characteristic 0);
- finite extension of $\mathbb{F}_q((t))$ (nonarchimedean, characteristic $p > 0$).

In the nonarchimedean cases, the ring of integers of a local field is a complete DVR with finite residue field.

The goal of local class field theory is to classify all finite abelian extensions of a given local field K. Rather than considering each finite abelian extension L/K individually, we will treat them all at once, by working in the maximal abelian extension of K inside a fixed separable closure K^{sep}.

Definition 27.1. Let K be field with separable closure K^{sep}. The field

$$K^{\text{ab}} := \bigcup_{\substack{L \subseteq K^{\text{sep}} \\ L/K \text{ finite abelian}}} L$$

is the *maximal abelian extension of* K (in K^{sep}). We also define

$$K^{\text{unr}} := \bigcup_{\substack{L \subseteq K^{\text{sep}} \\ L/K \text{ finite unramified}}} L,$$

the *maximal unramified extension of* K (in K^{sep}).

The field K^{ab} contains the field K^{unr}; this is obvious in the archimedean case, where we have $K = K^{\text{unr}}$ is $\mathbb{R}$ or $\mathbb{C}$ and $K^{\text{ab}} = K^{\text{sep}} = \mathbb{C}$ (note that the extension $\mathbb{C}/\mathbb{R}$ is ramified). In the nonarchimedean case the inclusion $K^{\text{unr}} \subseteq K^{\text{ab}}$ follows from Theorem 10.15, which implies that K^{unr} is isomorphic to the algebraic closure of the residue field of K, which is an abelian extension because it is pro-cyclic (every finite extension of the residue field is cyclic because the residue field is finite). We thus have a tower of field extensions

$$K \subseteq K^{\text{unr}} \subseteq K^{\text{ab}} \subseteq K^{\text{sep}}.$$

By Theorem 26.22, the Galois group $\text{Gal}(K^{\text{ab}}/K)$ is the profinite group

$$\text{Gal}(K^{\text{ab}}/K) \simeq \varprojlim_{L} \text{Gal}(L/K),$$

where L ranges over the finite extensions of K in K^{ab}, ordered by inclusion (note that every finite extension of K in K^{ab} is normal because every open subgroup of the abelian group $\text{Gal}(K^{\text{ab}}/K)$ is a normal subgroup).

Like all Galois groups, the profinite group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is a totally disconnected compact group; see Problem Set 11. By Theorem 26.23, we have the Galois correspondence

$$\begin{aligned} \{\text{ extensions of } K \text{ in } K^{\mathrm{ab}}\} &\longleftrightarrow \{\text{ closed subgroups of } \mathrm{Gal}(K^{\mathrm{ab}}/K)\} \\ L &\longmapsto \mathrm{Gal}(K^{\mathrm{ab}}/L) \\ (K^{\mathrm{ab}})^H &\longleftarrow H. \end{aligned}$$

Finite abelian extensions L/K correspond to open subgroups of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ (which must have finite index since $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is compact).

When K is an archimedean local field its abelian extensions are easy to understand; either $K = \mathbb{R}$, in which case $\mathbb{C}$ is the unique nontrivial abelian extension, or $K = \mathbb{C}$ and there are no nontrivial abelian extensions.

Now suppose K is a nonarchimedean local field with ring of integers $\mathcal{O}_K$, maximal ideal $\mathfrak{p}$, and residue field $\mathbb{F}_\mathfrak{p} := \mathcal{O}_K/\mathfrak{p}$. If L/K is a finite unramified extension with residue field $\mathbb{F}_\mathfrak{q} := \mathcal{O}_L/\mathfrak{q}$, Theorem 10.15 gives us a canonical isomorphism

$$\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{p}) = \langle x \mapsto x^{\#\mathbb{F}_\mathfrak{p}} \rangle,$$

between the Galois group of L/K and the Galois group of the residue field extension $\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{p}$. The group $\mathrm{Gal}(\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{p})$ is generated by the Frobenius automorphism $x \to x^{\#\mathbb{F}_\mathfrak{p}}$, and we use $\mathrm{Frob}_{L/K} \in \mathrm{Gal}(L/K)$ to denote the corresponding element of $\mathrm{Gal}(L/K)$; note that $\mathrm{Frob}_{L/K}$ is an element, not just a conjugacy class, because $\mathrm{Gal}(L/K)$ is abelian. Every finite unramified extension of local fields L/K thus comes equipped with a canonical generator $\mathrm{Frob}_{L/K}$ for its Galois group (which is necessarily cyclic).

In this local unramified setting, the Artin map is very easy to understand. The ideal group $\mathcal{I}_K$ is the infinite cyclic group generated by the prime ideal $\mathfrak{p}$, and the Artin map

$$\begin{aligned} \psi_{L/K} \colon \mathcal{I}_K &\to \mathrm{Gal}(L/K) \\ \mathfrak{p} &\mapsto \mathrm{Frob}_{L/K}, \end{aligned}$$

corresponds to the quotient map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, where $n := [L : K]$. We can extend the Artin map to $K^\times$ by defining $\psi_{L/K}(x) := \psi_{L/K}((x))$; this map sends every uniformizer π to the Frobenius element $\mathrm{Frob}_{L/K}$; note that since $\mathcal{O}_K$ is a DVR, hence a PID, every ideal in $\mathcal{I}$ is of the form (x) for some $x \in K^\times$, so defining the Artin map on $K^\times$ rather than $\mathcal{I}_K$ does not lose any information when K is a local field.

27.1 Local Artin reciprocity

Local class field theory is based on the existence of a continuous homomorphism

$$\theta_K \colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

known as the *local Artin homomorphism* (or *local reciprocity map*), which is described by the following theorem.

Theorem 27.2 (Local Artin Reciprocity)**.** *Let K be a local field. There is a unique continuous homomorphism*

$$\theta_K \colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

with the property that for each finite extension L/K in K^{ab}, the homomorphism

$$\theta_{L/K} \colon K^\times \to \mathrm{Gal}(L/K)$$

given by composing θ_K with the natural map $\mathrm{res}_{L/K} \colon \mathrm{Gal}(K^{\mathrm{ab}}/K) \twoheadrightarrow \mathrm{Gal}(L/K)$ satisfies:

- *if K is nonarchimedean and L/K is unramified then $\theta_{L/K}(\pi) = \mathrm{Frob}_{L/K}$ for every uniformizer π of $\mathcal{O}_K$;*
- *$\theta_{L/K}$ is surjective with kernel $\mathrm{N}_{L/K}(L^\times)$, inducing $K^\times/\mathrm{N}_{L/K}(L^\times) \simeq \mathrm{Gal}(L/K)$.*

The natural map $\mathrm{res}_{L/K}\colon \mathrm{Gal}(K^{\mathrm{ab}}/K) \twoheadrightarrow \mathrm{Gal}(L/K)$ can be viewed as any of

- the map induced by restriction $\sigma \mapsto \sigma_{|_L}$ (note that $\sigma(L) = L$ because L/K is Galois);
- the quotient map $\mathrm{Gal}(K^{\mathrm{ab}}/K) \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)/\mathrm{Gal}(K^{\mathrm{ab}}/L)$;
- the projection coming from $\mathrm{Gal}(K^{\mathrm{ab}}/K) = \varprojlim_L \mathrm{Gal}(L/K) \subseteq \prod_L \mathrm{Gal}(L/K)$ (where L ranges over finite extensions of K in K^{ab}).

These are equivalent descriptions of the same surjective homomorphism of topological groups (where the finite group $\mathrm{Gal}(L/K)$ has the discrete topology).

We will not have time to prove this theorem, but we would like to understand exactly what it says. The homomorphisms $\theta_{L/K}$ form a compatible system, in the sense that if $L_1 \subseteq L_2$ then $\theta_{L_1/K} = \mathrm{res}_{L_2/L_1} \circ \theta_{L_2/K}$, where res_{L_2/L_1} is the natural map from $\mathrm{Gal}(L_2/K)$ to $\mathrm{Gal}(L_1/K) = \mathrm{Gal}(L_2/K)/\mathrm{Gal}(L_2/L_1)$. Indeed, the maps res_{L_2/L_1} are precisely the maps that appear in the inverse system defining $\varprojlim_L \mathrm{Gal}(L/K) \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K)$.

It is first worth contrasting local Artin reciprocity with the more complicated global version of Artin reciprocity that we saw in Lecture 21:

- There is no modulus $\mathfrak{m}$; working in K^{ab} addresses all abelian extensions of K at once.
- The ray class groups $\mathrm{Cl}_K^{\mathfrak{m}}$ are replaced by quotients of $K^\times$.
- The Takagi group $\mathrm{N}_{L/K}(\mathcal{I}_L^{\mathfrak{m}})\mathcal{R}_K^{\mathfrak{m}} \subseteq \mathcal{I}_K^{\mathfrak{m}}$ is replaced by $\mathrm{N}_{L/K}(L^\times) \subseteq K^\times$.

27.2 Norm groups

Definition 27.3. A *norm group* of a local field K is a subgroup of the form

$$\mathrm{N}(L^\times) := \mathrm{N}_{L/K}(L^\times) \subseteq K^\times,$$

for some finite abelian extension L/K.

Remark 27.4. Removing the word abelian does not change the definition above. If L/K is any finite extension (not necessarily Galois), then $\mathrm{N}(L^\times) = \mathrm{N}(F^\times)$, where F is the maximal abelian extension of K in L; this result is known as the NORM LIMITATION THEOREM (see [1, Theorem III.3.5]). So we could have defined norm groups more generally. This is not relevant to classifying the abelian extension of K, but it demonstrates a key limitation of local class field theory (which extends to global class field theory): norm groups tell us nothing about nonabelian extensions of K.

Theorem 27.2 implies that the Galois group of any finite abelian extension L/K of a local fields is canonically isomorphic to the quotient $K^\times/\mathrm{N}_{L/K}(L^\times)$. In order to understand the finite abelian extensions of a local field K, we just need to understand its norm groups.

Corollary 27.5. *The map $L \mapsto \mathrm{N}(L^\times)$ defines an inclusion reversing bijection between the finite abelian extensions L/K in K^{ab} and the norm groups in $K^\times$ which satisfies*

$$(a)\ \mathrm{N}((L_1L_2)^\times) = \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times) \qquad \text{and} \qquad (b)\ \mathrm{N}((L_1 \cap L_2)^\times) = \mathrm{N}(L_1^\times)\mathrm{N}(L_2^\times).$$

In particular, every norm group of K has finite index in $K^\times$, and every subgroup of $K^\times$ that contains a norm group is a norm group.

Here we write L_1L_2 for the compositum of L_1 and L_2 inside K^{ab} (the intersection of all subfields of K^{ab} that contain both L_1 and L_2).

Proof. We first note that if $L_1 \subseteq L_2$ are two extensions of K then transitivity of the field norm (Corollary 4.52) implies

$$\mathrm{N}_{L_2/K} = \mathrm{N}_{L_1/K} \circ \mathrm{N}_{L_2/L_1},$$

and therefore $\mathrm{N}(L_2^\times) \subseteq \mathrm{N}(L_1^\times)$; the map $L \mapsto \mathrm{N}(L^\times)$ thus reverses inclusions.

This immediately implies $\mathrm{N}((L_1L_2)^\times) \subseteq \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times)$, since $L_1, L_2 \subseteq L_1L_2$. For the reverse inclusion, let us consider the commutative diagram

$$\begin{tikzcd} K^\times \arrow[r, "\theta_{L_1L_2/K}"] \arrow[dr, "\theta_{L_1/K}\times\theta_{L_2/K}"'] & \mathrm{Gal}(L_1L_2/K) \arrow[d, hook, "\mathrm{res}\times\mathrm{res}"] \\ & \mathrm{Gal}(L_1/K)\times\mathrm{Gal}(L_2/K) \end{tikzcd}$$

By Theorem 27.2, each $x \in \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times) \subseteq K^\times$ lies in the kernel of $\theta_{L_1/K}$ and $\theta_{L_2/K}$, hence in the kernel of $\theta_{L_1L_2/K}$ (by the diagram), and therefore in $\mathrm{N}((L_1L_2)^\times)$, again by Theorem 27.2. This proves (a).

We now show that $L \mapsto \mathrm{N}(L^\times)$ is a bijection; it is surjective by definition, so we just need to show it is injective. If $\mathrm{N}(L_2^\times) = \mathrm{N}(L_1^\times)$ then by (a) we have

$$\mathrm{N}((L_1L_2)^\times) = \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times) = \mathrm{N}(L_1^\times) = \mathrm{N}(L_2^\times),$$

and Theorem 27.2 implies $\mathrm{Gal}(L_1L_2/K) \simeq \mathrm{Gal}(L_1/K) \simeq \mathrm{Gal}(L_2/K)$, which forces $L_1 = L_2$; thus $L \mapsto \mathrm{N}(L^\times)$ is injective.

We now prove (b). The field $L_1 \cap L_2$ is the largest extension of K that lies in both L_1 and L_2, while $\mathrm{N}(L_1^\times)\mathrm{N}(L_2^\times)$ is the smallest subgroup of $K^\times$ containing both $\mathrm{N}(L_1^\times)$ and $\mathrm{N}(L_2^\times)$; they therefore correspond under the inclusion reversing bijection $L \mapsto \mathrm{N}(L^\times)$ and we have $\mathrm{N}((L_1 \cap L_2)^\times) = \mathrm{N}(L_1^\times)\mathrm{N}(L_2^\times)$ as desired.

The fact that every norm group has finite index in $K^\times$ follows immediately from the isomorphism $\mathrm{Gal}(L/K) \simeq K^\times/N_{L/K}(L^\times)$ given by Theorem 27.2, since $\mathrm{Gal}(L/K)$ is finite.

Finally, let us prove that every subgroup of $K^\times$ that contains a norm group is a norm group. Suppose $\mathrm{N}(L^\times) \subseteq H \subseteq K^\times$, for some finite abelian L/K, and subgroup H of $K^\times$, and put $F := L^{\theta_{L/K}(H)}$. We have a commutative diagram

$$\begin{tikzcd} K^\times \arrow[r, "\theta_{L/K}"] \arrow[dr, "\theta_{F/K}"'] & \mathrm{Gal}(L/K) \arrow[d, "\mathrm{res}"] \\ & \mathrm{Gal}(F/K) \end{tikzcd}$$

in which $\mathrm{Gal}(L/F) = \theta_{L/K}(H)$ is precisely the kernel of the map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(F/K)$ induced by restriction. It follows from Theorem 27.2 that

$$H = \ker \theta_{F/K} = \mathrm{N}(F^\times)$$

is a norm group as claimed. $\square$

Lemma 27.6. *Let L/K be any extension of local fields. If $\mathrm{N}(L^\times)$ has finite index in $K^\times$ then it is open.*

Proof. The lemma is clear if K is archimedean (either $L = K$ and $\mathrm{N}(L^\times) = K^\times$, or $L \simeq \mathbb{C}$, $K \simeq \mathbb{R}$, and $[K^\times : \mathrm{N}(L^\times)] = [\mathbb{R}^\times : \mathbb{R}_{>0}] = 2$), so assume K is nonarchimedean. Suppose $[K^\times : \mathrm{N}(L^\times)] < \infty$. The unit group $\mathcal{O}_L^\times$ is compact, so $\mathrm{N}(\mathcal{O}_L^\times)$ is compact (since $\mathrm{N}\colon L^\times \to K^\times$ is continuous), thus closed in the Hausdorff space $K^\times$. For any $\alpha \in L$,

$$\alpha \in \mathcal{O}_L^\times \iff |\alpha| = 1 \iff |\mathrm{N}_{L/K}(\alpha)| = 1 \iff \mathrm{N}_{L/K}(\alpha) \in \mathcal{O}_K^\times,$$

and therefore

$$\mathrm{N}(\mathcal{O}_L^\times) = \mathrm{N}(L^\times) \cap \mathcal{O}_K^\times.$$

It follows that $\mathrm{N}(\mathcal{O}_L^\times)$ is the kernel of the homomorphism $\mathcal{O}_K^\times \hookrightarrow K^\times \twoheadrightarrow K^\times/\mathrm{N}(L^\times)$ and therefore $[\mathcal{O}_K^\times : \mathrm{N}(\mathcal{O}_L^\times)] \le [K^\times : \mathrm{N}(L^\times)] < \infty$. Thus $\mathrm{N}(\mathcal{O}_L^\times)$ is a closed subgroup of finite index in $\mathcal{O}_K^\times$, hence open (its complement is a finite union of closed cosets, hence closed), and $\mathcal{O}_K^\times$ is open[1] in $K^\times$, so $\mathrm{N}(\mathcal{O}_L^\times)$ is open in $K^\times$, and therefore $\mathrm{N}(L^\times)$ is open in $K^\times$, since $\mathrm{N}(L^\times)$ is a union of cosets of the open subgroup $\mathrm{N}(\mathcal{O}_L^\times)$. $\square$

Remark 27.7. If K is a local field of characteristic zero then one can show that in fact every finite index subgroup of $K^\times$ is open (whether it is a norm group or not), but this is not true in positive characteristic.

27.3 The main theorems of local class field theory

Corollary 27.5 implies that all norm groups of K have finite index in $K^\times$, and Lemma 27.6 then implies that all norm groups are finite index open subgroups of $K^\times$. The existence theorem of local class field theory states that the converse also holds.

Theorem 27.8 (Local Existence Theorem)**.** *Let K be a local field and let H be a finite index open subgroup of $K^\times$. There is a unique extension L/K in K^{ab} with $\mathrm{N}_{L/K}(L^\times) = H$.*

The local Artin homomorphism $\theta_K\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is not an isomorphism; indeed, it cannot be, because $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is compact and $K^\times$ is not. However, the local existence theorem implies that after taking profinite completions the local Artin homomorphism becomes an isomorphism.

Theorem 27.9 (Main Theorem of Local Class Field Theory)**.** *Let K be a local field. The local Artin homomorphism induces a canonical isomorphism*

$$\widehat{\theta}_K\colon \widehat{K^\times} \xrightarrow{\sim} \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

of profinite groups.

Proof. The Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is a profinite group, isomorphic to the inverse limit

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \varprojlim_L \mathrm{Gal}(L/K), \tag{1}$$

where L ranges over the finite extensions of K in K^{ab} ordered by inclusion; see Theorem 26.22. It follows from Lemma 27.6, Theorem 27.8, and the definition of the profinite completion, that

$$\widehat{K^\times} \simeq \varprojlim_L K^\times/\mathrm{N}(L^\times), \tag{2}$$

[1] Recall that in a nonarchimedean local field, $|K^\times|$ is discrete in $\mathbb{R}_{>0}$ and we can always pick $\epsilon > 0$ so that $\mathcal{O}_K^\times = \{x \in K^\times : 1-\epsilon < |x| < 1+\epsilon\}$, which is clearly open in the metric topology induced by $|\;|$.

where L ranges over finite abelian extensions of K (in K^{sep}). By local Artin reciprocity (Theorem 27.2), for each finite abelian extension L/K we have an isomorphism

$$\theta_{L/K}\colon K^\times/\mathrm{N}(L^\times) \xrightarrow{\sim} \mathrm{Gal}(L/K),$$

and these isomorphisms commute with inclusion maps between finite abelian extensions of K. We thus have an isomorphism of the inverse systems appearing in (1) and (2). The isomorphism is canonical because the Artin homomorphism θ_K is unique and the isomorphisms in (1) and (2) are both canonical. $\square$

In view of Theorem 27.9, we would like to better understand the profinite group $\widehat{K^\times}$. If K is archimedean then $\widehat{K^\times}$ is either trivial or the cyclic group of order 2, so let us assume that K is nonarchimedean. If we pick a uniformizer π for the maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$, then we can uniquely write each $x \in K^\times$ in the form $u\pi^{v(x)}$, with $u \in \mathcal{O}_K^\times$ and $v(x) \in \mathbb{Z}$. This defines an isomorphism

$$\begin{aligned} K^\times &\xrightarrow{\sim} \mathcal{O}_K^\times \times \mathbb{Z} \\ x &\longmapsto (x/\pi^{v(x)}, v(x)). \end{aligned}$$

Taking profinite completions (which commutes with products), we obtain an isomorphism

$$\widehat{K^\times} \simeq \mathcal{O}_K^\times \times \widehat{\mathbb{Z}},$$

since the unit group

$$\mathcal{O}_K^\times \simeq \mathbb{F}_\mathfrak{p}^\times \times (1+\mathfrak{p}) \simeq \mathbb{F}_\mathfrak{p}^\times \times \varprojlim_n \mathcal{O}_K/(1+\mathfrak{p}^n)$$

is already profinite (hence isomorphic to its profinite completion, by Corollary 26.19). Note that the isomorphism $\widehat{K^\times} \simeq \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}$ is far from canonical; it depends on our choice of π, and there are uncountably many π to choose from.

We have a commutative diagram of exact sequences of topological groups

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_K^\times \arrow[r] \arrow[d, "\wr"] & K^\times \arrow[r, "v"] \arrow[d, "\theta_K"] & \mathbb{Z} \arrow[r] \arrow[d, hook, "\phi"] & 0 \\
1 \arrow[r] & \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}}) \arrow[r] & \mathrm{Gal}(K^{\mathrm{ab}}/K) \arrow[r, "\mathrm{res}"] & \mathrm{Gal}(K^{\mathrm{unr}}/K) \arrow[r] & 1
\end{tikzcd}$$

in which the bottom row is the profinite completion of the top row. The map ϕ on the right is given by

$$\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}} \simeq \mathrm{Gal}(\overline{\mathbb{F}}_\mathfrak{p}/\mathbb{F}_\mathfrak{p}) \simeq \mathrm{Gal}(K^{\mathrm{unr}}/K),$$

and sends 1 to the sequence of Frobenius elements $(\mathrm{Frob}_{L/K})$ in the profinite group

$$\mathrm{Gal}(K^{\mathrm{unr}}/K) \simeq \varprojlim_L \mathrm{Gal}(L/K) \subseteq \prod_L \mathrm{Gal}(L/K),$$

where L ranges over finite unramified extensions of K; here we are using the canonical isomorphisms $\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{p})$ given by Theorem 10.15. The Frobenius element $\phi(1)$ is a *topological generator* for $\mathrm{Gal}(K^{\mathrm{unr}}/K)$, meaning that it generates a dense subset.

Remark 27.10. The Frobenius element $\phi(1) \in \mathrm{Gal}(K^{\mathrm{unr}}/K)$ corresponds to the Frobenius automorphism $x \mapsto x^{\#\mathbb{F}_\mathfrak{p}}$ of $\mathrm{Gal}(\overline{\mathbb{F}}_\mathfrak{p}/\mathbb{F}_\mathfrak{p})$; both are canonical topological generators of the Galois groups in which they reside, and both are sometimes referred to as the *arithmetic Frobenius.* There is another obvious generator for $\mathrm{Gal}(K^{\mathrm{unr}}/K) \simeq \mathrm{Gal}(\overline{\mathbb{F}}_\mathfrak{p}/\mathbb{F}_\mathfrak{p})$, namely $\phi(-1)$, which is called the *geometric Frobenius* (for reasons we won't explain here).

The group $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}}) \simeq \mathcal{O}_K^\times$ corresponds to the inertia subgroup of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. The top sequence splits (but not canonically), hence so does the bottom, and we have

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}}) \times \mathrm{Gal}(K^{\mathrm{unr}}/K) \simeq \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}.$$

For each choice of a uniformizer $\pi \in \mathcal{O}_K$ we get a decomposition $K^{\mathrm{ab}} = K_\pi K^{\mathrm{unr}}$ corresponding to $K^\times = \mathcal{O}_K^\times \pi^{\mathbb{Z}}$. The field K_π is the subfield of K^{ab} fixed by $\theta_K(\pi) \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$. Equivalently, K_π is the compositum of all the totally ramified finite extensions L/K in K^{ab} for which $\pi \in \mathrm{N}(L^\times)$.

Example 27.11. Let $K = \mathbb{Q}_p$ and pick $\pi = p$. The decomposition $K^{\mathrm{ab}} = K_\pi K^{\mathrm{unr}}$ is

$$\mathbb{Q}_p^{\mathrm{ab}} = \bigcup_n \mathbb{Q}_p(\zeta_{p^n}) \cdot \bigcup_{m \perp p} \mathbb{Q}_p(\zeta_m),$$

where the first union on the RHS is fixed by $\theta_K(p)$ and the second is fixed by $\theta_K(\mathcal{O}_K^\times)$.

Constructing the local Artin homomorphism is the difficult part of local class field theory. However, assuming the local existence theorem, it is easy to show that the local Artin homomorphism is unique if it exists.

Proposition 27.12. *Let K be a local field and assume every finite index open subgroup of $K^\times$ is a norm group. There is at most one homomorphism $\theta\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ of topological groups that has the properties given in Theorem 27.2.*

Proof. The proposition is clear when K is archimedean, so assume it is nonarchimedean. Let $\mathfrak{p} = (\pi)$ be the maximal ideal of $\mathcal{O}_K$, and for each integer $n \geq 0$ let $K_{\pi,n}/K$ be the finite abelian extension given by Theorem 27.8 corresponding to the finite index subgroup $(1+\mathfrak{p}^n)\langle\pi\rangle$ of $K^\times$; here $1+\mathfrak{p}^n$ and $\langle\pi\rangle$ denote subgroups of $K^\times$, with $1+\mathfrak{p}^0 := \mathcal{O}_K^\times$, and we note that $K^\times \simeq \mathcal{O}_K^\times\langle\pi\rangle$.

Suppose $\theta\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is a continuous homomorphism as in Theorem 27.2. Then $\theta(\pi)$ fixes $K_\pi := \bigcup_n K_{\pi,n}$, since $\pi \in \mathrm{N}(K_{\pi,n}) = \ker\theta_{K_{\pi,n}/K}$. We also know that $\theta_{L/K}(\pi) = \mathrm{Frob}_{L/K}$ for all finite unramified extensions L/K, which uniquely determines the action of $\theta(\pi)$ on K^{unr}, and hence on $K^{\mathrm{ab}} = K_\pi K^{\mathrm{unr}}$.

Now suppose $\theta'\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is another continuous homomorphism as in Theorem 27.2. By the argument above we must have $\theta'(\pi) = \theta(\pi)$ for every uniformizer π of $\mathcal{O}_K$, and $K^\times$ is generated by its subset of uniformizers: if we fix one uniformizer π, every $x \in K^\times$ can be written as $u\pi^n = (u\pi)\pi^{n-1}$ for some $u \in \mathcal{O}_K^\times$ and $n \in \mathbb{Z}$, and $u\pi$ is another uniformizer). It follows that $\theta(x) = \theta'(x)$ for all $x \in K^\times$ and therefore $\theta = \theta'$ is unique. $\square$

Remark 27.13. One approach to proving local class field theory uses the theory of formal groups due to Lubin and Tate to explicitly construct the fields $K_\pi = \bigcup_n K_{\pi,n}$ used in the proof of Proposition 27.12, along with a continuous homomorphism $\theta_\pi\colon \mathcal{O}_K^\times \to \mathrm{Gal}(K_\pi/K)$ that extends uniquely to a continuous homomorphism $\theta\colon K^\times \to \mathrm{Gal}(K_\pi K^{\mathrm{unr}}/K)$. One then shows that $K^{\mathrm{ab}} = K_\pi K^{\mathrm{unr}}$ (using the Hasse-Arf Theorem), and that θ does not depend on the choice of π; see [1, §I.2-4] for details.

27.4 Finite abelian extensions

Local class field theory gives us canonical bijections between the following sets:

(1) finite-index open subgroups of $K^\times$ (which are necessarily normal);

(2) open subgroups of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ (which are necessarily normal and of finite index);

(3) finite extensions of K in K^{ab} (which are necessarily normal).

The bijection from (1) to (2) is induced by the isomorphism $\widehat{K^\times} \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K)$ given by Theorem 27.9 and is inclusion preserving. The bijection from (2) to (3) follows from Galois theory (for infinite extensions), and is inclusion reversing, while the bijection from (3) to (1) is via the map $L \mapsto \mathrm{N}(L^\times)$, which is also inclusion reversing.

References

[1] J.S. Milne, *Class field theory*, version 4.02, 2013.

[2] Jean-Pierre Serre, *Local fields*, Springer, 1979.