25 The ring of adeles, strong approximation

25.1 Introduction to adelic rings

Recall that we have a canonical injection

\[ \mathbb{Z} \hookrightarrow \hat{\mathbb{Z}} := \lim_{\rightarrow} \mathbb{Z}/n\mathbb{Z} \simeq \prod_p \mathbb{Z}_p, \]

that embeds \( \mathbb{Z} \) into the product of its nonarchimedean completions. Each of the rings \( \mathbb{Z}_p \) is compact, hence \( \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \) is compact (by Tychonoff’s theorem). If we consider the analogous product \( \prod_p \mathbb{Q}_p \) of the completions of \( \mathbb{Q} \), each of the local fields \( \mathbb{Q}_p \) is locally compact (as is \( \mathbb{Q}_\infty = \mathbb{R} \)), but the product \( \prod_p \mathbb{Q}_p \) is not locally compact.

To see where the problem arises, recall that for any family of topological spaces \( (X_i)_{i \in I} \) (where the index set \( I \) is any set), the product topology on \( X := \prod X_i \) is defined as the weakest topology that makes all the projection maps \( \pi_i : X \to X_i \) continuous; it is thus generated by open sets of the form \( \pi_i^{-1}(U_i) \) with \( U_i \subseteq X_i \) open. Every open set in \( X \) is a (possibly empty or infinite) union of open sets of the form

\[ \prod_{i \in S} U_i \times \prod_{i \notin S} X_i, \]

with \( S \subseteq I \) finite and each \( U_i \subseteq X_i \) open (these sets form a basis for the topology on \( X \)). In particular, every open \( U \subseteq X \) satisfies \( \pi_i(U) = X_i \) for all but finitely many \( i \in I \). Unless all but finitely many of the \( X_i \) are compact, the space \( X \) cannot possibly be locally compact for the simple reason that no compact set \( C \) in \( X \) contains a nonempty open set (if it did then we would have \( \pi_i(C) = X_i \) compact for all but finitely many \( i \in I \)). Recall that to be locally compact means that for every \( x \in X \) there is an open \( U \) and compact \( C \) such that \( x \in U \subseteq C \).

To address this issue we want to take the product of the fields \( \mathbb{Q}_p \) (or more generally, the completions of any global field) in a different way, one that yields a locally compact topological ring. This is the motivation of the restricted product, a topological construction that was invented primarily for the purpose of solving this number-theoretic problem.

25.2 Restricted products

This section is purely about the topology of restricted products; readers already familiar with restricted products should feel free to skip to the next section.

**Definition 25.1.** Let \( (X_i) \) be a family of topological spaces indexed by \( i \in I \), and let \( (U_i) \) be a family of open sets \( U_i \subseteq X_i \). The **restricted product** \( \prod(X_i, U_i) \) is the topological space

\[ \prod(X_i, U_i) := \{ (x_i) : x_i \in U_i \text{ for almost all } i \in I \} \subseteq \prod X_i \]

with the basis of open sets

\[ \mathcal{B} := \left\{ \prod V_i : V_i \subseteq X_i \text{ is open for all } i \in I \text{ and } V_i = U_i \text{ for almost all } i \in I \right\}, \]

where *almost all* means all but finitely many.
For each \( i \in I \) we have a projection map \( \pi_i : \prod(X_i, U_i) \to X_i \) defined by \( (x_i) \mapsto x_i \); each \( \pi_i \) is continuous, since if \( W_i \) is an open subset of \( X_i \), then \( \pi_i^{-1}(W_i) \) is the union of all basic open sets \( \prod V_i \in \mathcal{B} \) with \( V_i = W_i \), which is an open set.

As sets, we always have
\[
\prod U_i \subseteq \prod(X_i, U_i) \subseteq \prod X_i,
\]
but in general the restricted product topology on \( \prod(X_i, U_i) \) is not the same as the subspace topology it inherits from \( \prod X_i \); it has more open sets. For example, \( \prod U_i \) is an open set in \( \prod(X_i, U_i) \), but unless \( U_i = X_i \) for almost all \( i \) (in which case \( \prod(X_i, U_i) = \prod X_i \) ), it is not open in \( \prod X_i \), and it is not open in the subspace topology on \( \prod(X_i, U_i) \) because it does not contain the intersection of \( \prod(X_i, U_i) \) with any basic open set in \( \prod X_i \).

Thus the restricted product is a strict generalization of the direct product; the two coincide if and only if \( U_i = X_i \) for almost all \( i \). This is automatically true whenever the index set \( I \) is finite, so only infinite restricted products are of independent interest.

**Remark 25.2.** The restricted product does not depend on any particular \( U_i \). Indeed,
\[
\prod(X_i, U_i) = \prod(X_i, U'_i)
\]
whenever \( U'_i = U_i \) for almost all \( i \); note that the two restricted products are not merely isomorphic, they are identical, both as sets and as topological spaces. It is thus enough to specify the \( U_i \) for all but finitely many \( i \in I \).

Each \( x \in X := \prod(X_i, U_i) \) determines a (possibly empty) finite set
\[
S(x) := \{ i \in I : x_i \notin U_i \}.
\]
Given any finite \( S \subseteq I \), let us define
\[
X_S := \{ x \in X : S(x) \subseteq S \} = \prod_{i \in S} X_i \times \prod_{i \notin S} U_i.
\]
Notice that \( X_S \in \mathcal{B} \) is an open set, and we can view it as a topological space in two ways, both as a subspace of \( X \) or as a direct product of certain \( X_i \) and \( U_i \). Restricting the basis \( \mathcal{B} \) for \( X \) to a basis for the subspace \( X_S \) yields
\[
\mathcal{B}_S := \left\{ \prod V_i : V_i \subseteq \pi_i(X_S) \text{ is open and } V_i = U_i = \pi_i(X_S) \text{ for almost all } i \in I \right\},
\]
which is the standard basis for the product topology, so the two topologies on \( X_S \) coincide.

We have \( X_S \subseteq X_T \) whenever \( S \subseteq T \), thus if we partially order the finite subsets \( S \subseteq I \) by inclusion, the family of topological spaces \( \{ X_S : S \subseteq I \text{ finite} \} \) with inclusion maps \( \{ i_{ST} : X_S \hookrightarrow X_T | S \subseteq T \} \) forms a direct system, and we have a corresponding direct limit
\[
\lim_{\rightarrow S} X_S := \prod X_S / \sim,
\]
which is the quotient of the coproduct space (disjoint union) \( \prod X_S \) by the equivalence relation \( x \sim i_{ST}(x) \) for all \( x \in S \subseteq T \).\(^1\) This direct limit is canonically isomorphic to the restricted product \( X \), which gives us another way to define the restricted product; before proving this let us recall the general definition of a direct limit of topological spaces.

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\(^1\)The topology on \( \prod X_S \) is the weakest topology that makes the injections \( X_S \hookrightarrow \prod X_S \) continuous; its open sets are disjoint unions of open sets in the \( X_S \). The topology on \( \prod X_S / \sim \) is the weakest topology that makes the quotient map \( \prod X_S \twoheadrightarrow \prod X_S / \sim \) continuous; its open sets are images of open sets in \( \prod X_S \).
Definition 25.3. A **direct system** (or inductive system) in a category is a family of objects \( \{X_i : i \in I\} \) indexed by a directed set \( I \) (see Definition 8.7) and a family of morphisms \( \{f_{ij} : X_i \to X_j : i \leq j\} \) such that each \( f_{ii} \) is the identity and \( f_{ik} = f_{jk} \circ f_{ij} \) for all \( i \leq j \leq k \).

Definition 25.4. Let \((X_i, f_{ij})\) be a direct system of topological spaces. The **direct limit** (or inductive limit) of \((X_i, f_{ij})\) is the quotient space

\[
X = \lim_{\to} X_i := \prod_{i \in I} X_i / \sim,
\]

where \(x_i \sim f_{ij}(x_i)\) for all \(i \leq j\). It is equipped with continuous maps \(\phi_i : X_i \to X\) that are compositions of the inclusion maps \(X_i \hookrightarrow \prod X_i\) and quotient maps \(\prod X_i \to \prod X_i / \sim\) and satisfy \(\phi_i = \phi_j \circ f_{ij}\) for \(i \leq j\).

The topological space \(X = \lim_{\to} X_i\) has the universal property that if \(Y\) is another topological space with continuous maps \(\psi_i : X_i \to Y\) that satisfy \(\psi_i = \psi_j \circ f_{ij}\) for \(i \leq j\), then there is a unique continuous map \(X \to Y\) for which all of the diagrams commute (this universal property defines the direct limit in any category with coproducts).

We now prove that that \(\prod (X_i, U_i) \simeq \lim_{\to} X_S\) as claimed above.

Proposition 25.5. Let \((X_i)\) be a family of topological spaces indexed by \(i \in I\), let \((U_i)\) be a family of open sets \(U_i \subseteq X_i\), and let \(X := \prod (X_i, U_i)\) be the corresponding restricted product. For each finite \(S \subseteq I\) define

\[
X_S := \prod_{i \in S} X_i \times \prod_{i \notin S} U_i \subseteq X,
\]

and inclusion maps \(i_{ST} : X_S \hookrightarrow X_T\), and let \(\lim_{\to} X_S\) be the corresponding direct limit.

There is a canonical homeomorphism of topological spaces

\[
\varphi : X \xrightarrow{\sim} \lim_{\to} X_S
\]

that sends \(x \in X\) to the equivalence class of \(x \in X_{S(x)} \subseteq \prod X_S\) in \(\lim_{\to} X_S := \prod X_S / \sim\), where \(S(x) := \{i \in I : x \notin U_i\}\).

Proof. To prove that the map \(\varphi : X \to \lim_{\to} X_S\) is a homeomorphism, we need to show that it is (1) a bijection, (2) continuous, and (3) an open map.

(1) For each equivalence class \(C \in \lim_{\to} X_S := \prod X_S / \sim\), let \(S(C)\) be the intersection of all the sets \(S\) for which \(C\) contains an element of \(\prod X_S\) in \(X_S\). Then \(S(x) = S(C)\) for all \(x \in C\), and \(C\) contains a unique element for which \(x \in X_{S(x)} \subseteq \prod X_S\) (distinct \(x, y \in X_S\) cannot be equivalent). Thus \(\varphi\) is a bijection.

(2) Let \(U\) be an open set in \(\lim_{\to} X_S = \prod X_S / \sim\). The inverse image \(V\) of \(U\) in \(\prod X_S\) is open, as are the inverse images \(\bar{V}_S\) of \(V\) under the canonical injections \(i : X_S \hookrightarrow \prod X_S\). The union of the \(V_S\) in \(X\) is equal to \(\varphi^{-1}(U)\) and is an open set in \(X\); thus \(\varphi\) is continuous.
(3) Let $U$ be an open set in $X$. Since the $X_S$ form an open cover of $X$, we can cover $U$ with open sets $U_S := U \cap X_S$, and then $\bigsqcup U_S$ is an open set in $\bigsqcup X_S$. Moreover, for each $x \in \bigsqcup U_S$, if $y \sim x$ for some $y \in \bigsqcup X_S$ then $y$ and $x$ must correspond to the same element in $U$; in particular, $y \in \bigsqcup U_S$, so $\bigsqcup U_S$ is a union of equivalence classes in $\bigsqcup X_S$. It follows that its image in $\lim X_S = \bigsqcup X_S/\sim$ is open. □

Proposition 25.5 gives us another way to construct the restricted product $\prod(X_i, U_i)$: rather than defining it as a subset of $\prod X_i$ with a modified topology, we can instead construct it as a limit of direct products that are subspaces of $\prod X_i$.

We now specialize to the case of interest, where we are forming a restricted product using a family $(X_i)_{i \in I}$ of locally compact spaces and a family of open subsets $(U_i)$ that are almost all compact. Under these conditions the restricted product $\prod(X_i, U_i)$ is locally compact, even though the product $\prod X_i$ is not unless the index set $I$ is finite.

Proposition 25.6. Let $(X_i)_{i \in I}$ be a family of locally compact topological spaces and let $(U_i)_{i \in I}$ be a corresponding family of open subsets $U_i \subseteq X_i$ almost all of which are compact. Then the restricted product $X := \bigsqcup(X_i, U_i)$ is locally compact.

Proof. We first note that for each finite set $S \subseteq I$ the topological space

$$X_S := \prod_{i \in S} X_i \times \prod_{i \notin S} U_i$$

can be viewed as a finite product of locally compact spaces, since all but finitely many $U_i$ are compact, and the product of these is compact (by Tychonoff’s theorem), hence locally compact. A finite product of locally compact spaces is locally compact, since we can construct compact neighborhoods as products of compact neighborhoods in each factor (in a finite product, products of open sets are open and products of compact sets are compact); thus the $X_S$ are locally compact, and they cover $X$ (since each $x \in X$ lies in $X_{S(x)}$). It follows that $X$ is locally compact, since each $x \in X_S$ has a compact neighborhood $x \in U \subseteq C \subseteq X_S$ that is also a compact neighborhood in $X$ (the image of $C$ under the inclusion map $X_S \to X$ is certainly compact, and $U$ is open in $X$ because $X_S$ is open in $X$).

25.3 The ring of adeles

Recall that for a global field $K$ (a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$), we use $M_K$ to denote the set of places of $K$ (equivalence classes of absolute values), and for any $v \in M_K$ we use $K_v$ to denote the corresponding local field (the completion of $K$ with respect to $v$). When $v$ is nonarchimedean we use $\mathcal{O}_v$ to denote the valuation ring of $K_v$, and for archimedean $v$ we define $\mathcal{O}_v := K_v$.$^2$

Definition 25.7. Let $K$ be a global field. The *adele ring* of $K$ is the restricted product

$$\mathbb{A}_K := \prod_{v \in M_K} (K_v, \mathcal{O}_v),$$

which we may view as a subset (but not a subspace!) of $\prod_v K_v$; indeed

$$\mathbb{A}_K = \left\{ (a_v) \in \prod_k K_v : a_v \in \mathcal{O}_v \text{ for almost all } v \right\}.$$  

$^2$Per Remark 25.2, as far as the topology goes it doesn’t matter how we define $\mathcal{O}_v$ at the finite number of archimedean places, but we would like each $\mathcal{O}_v$ to be a topological ring, which motivates this choice.

$^3$In French one writes *adèle*, but it is common practice to omit the accent when writing in English.
For each \( a \in \mathbb{A}_K \) we use \( a_v \) to denote its projection in \( K_v \); we make \( \mathbb{A}_K \) a ring by defining addition and multiplication component-wise.

For each finite set of places \( S \) we have the subring of \( S \)-adeles

\[
\mathbb{A}_{K,S} := \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v,
\]

which is a direct product of topological rings. By Proposition 25.5, \( \mathbb{A}_K \cong \varprojlim \mathbb{A}_{K,S} \) is the direct limit of the \( S \)-adele rings, which makes it clear that \( \mathbb{A}_K \) is also a topological ring.\(^4\)

The canonical embeddings \( K \hookrightarrow K_v \) induce a canonical embedding

\[
K \hookrightarrow \mathbb{A}_K \quad x \mapsto (x, x, x, \ldots).
\]

Note that for each \( x \in K \) we have \( x \in \mathcal{O}_v \) for all but finitely many \( v \). The image of \( K \) in \( \mathbb{A}_K \) is the subring of principal adeles (which of course is also a field).

We extend the normalized absolute value \( \| \cdot \|_v \) of \( K_v \) (see Definition 13.17) to \( \mathbb{A}_K \) via

\[
\|a\|_v := \|a_v\|_v,
\]

and define the adelic absolute value (or adelic norm)

\[
\|a\| := \prod_{v \in M_K} \|a\|_v \in \mathbb{R} \geq 0
\]

which we note converges to zero unless \( \|a\|_v = 1 \) for all but finitely many \( v \), in which case it is effectively a finite product.\(^5\) For \( \|a\| \neq 0 \) this is equal to the size of the \( M_K \)-divisor \( (\|a\|_v) \) we defined in Lecture 15 (see Definition 15.1). For any nonzero principal adele \( a \), we have \( a \in K^\times \) and \( \|a\| = 1 \), by the product formula (Theorem 13.21).

**Example 25.8.** For \( K = \mathbb{Q} \) the adele ring \( \mathbb{A}_\mathbb{Q} \) is the union of the rings

\[
\mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p
\]

where \( S \) varies over finite sets of primes (but note that the topology is the restricted product topology, not the subspace topology in \( \prod_{p \leq \infty} \mathbb{Q}_p \)). We can also write \( \mathbb{A}_\mathbb{Q} \) as

\[
\mathbb{A}_\mathbb{Q} = \left\{ a \in \prod_{p \leq \infty} \mathbb{Q}_p : \|a\|_p \leq 1 \text{ for almost all } p \right\}.
\]

**Proposition 25.9.** The adele ring \( \mathbb{A}_K \) of a global field \( K \) is locally compact and Hausdorff.

**Proof.** Local compactness follows from Proposition 25.6, since the local fields \( K_v \) are all locally compact and all but finitely many \( \mathcal{O}_v \) are valuation rings of a nonarchimedean local field, hence compact (\( \mathcal{O}_v = \{ x \in K_v : \|x\|_v \leq 1 \} \) is a closed ball). The product space \( \prod_v K_v \) is Hausdorff, since each \( K_v \) is Hausdorff, and the topology on \( \mathbb{A}_K \subseteq \prod K_v \) is finer than the subspace topology, so \( \mathbb{A}_K \) is also Hausdorff. \( \square \)

\(^4\)By definition it is a topological space that is also a ring; to be a topological ring is a stronger condition (the ring operations must be continuous), but this property is preserved by direct limits so all is well.

\(^5\)For \( v \mid \infty \), if \( \|a\|_v < 1 \) then \( \|a\|_v \leq 1/2 \), since \( \|a\|_v := q^{-v(a_v)} \) for some prime power \( q \).
Proposition 25.9 implies that the additive group of $\mathbb{A}_K$ (which is sometimes denoted $\mathbb{A}^+_K$ to emphasize that we are viewing it as a group rather than a ring) is a locally compact group, and therefore has a Haar measure that is unique up to scaling, by Theorem 13.14. Each of the completions $K_v$ is a local field with a Haar measure $\mu_v$, which we normalize as follows:

- $\mu_v(\mathcal{O}_v) = 1$ for all nonarchimedean $v$;
- $\mu_v(S) = \mu_\mathbb{R}(S)$ for $K_v \simeq \mathbb{R}$, where $\mu_\mathbb{R}(S)$ is the Lebesgue measure on $\mathbb{R}$;
- $\mu_v(S) = 2\mu_\mathbb{C}(S)$ for $K_v \simeq \mathbb{C}$, where $\mu_\mathbb{C}(S)$ is the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$.

Note that the normalization of $\mu_v$ at the archimedean places is consistent with the measure $\mu$ on $K_\mathbb{R} \simeq \mathbb{R}^r \times \mathbb{C}^s \simeq \mathbb{R}^n$ induced by the canonical inner product on $K_\mathbb{R} \subseteq K_\mathbb{C}$ that we defined in Lecture 14 (see §14.2).

We now define a measure $\mu$ on $\mathbb{A}_K$ as follows. We take as a basis for the $\sigma$-algebra of measurable sets all sets of the form $\prod_v B_v$, where each $B_v$ is a measurable set in $K_v$ with $\mu_v(B_v) < \infty$ such that $B_v = \mathcal{O}_v$ for almost all $v$ (the $\sigma$-algebra is then generated by taking countable intersections, unions, and complements in $\mathbb{A}_K$). We then define

$$\mu \left( \prod_v B_v \right) := \prod_v \mu_v(B_v).$$

It is easy to verify that $\mu$ is a Radon measure, and it is clearly translation invariant since each of the Haar measures $\mu_v$ is translation invariant and addition is defined component-wise; note that for any $x \in \mathbb{A}_K$ and measurable set $B = \prod_v B_v$ the set $x + B = \prod_v (x_v + B_v)$ is also measurable, since $x_v + B_v = \mathcal{O}_v$ whenever $x_v \in \mathcal{O}_v$ and $B_v = \mathcal{O}_v$, and this applies to almost all $v$. It follows from uniqueness of the Haar measure (up to scaling) that $\mu$ is a Haar measure on $\mathbb{A}_K$ which we henceforth adopt as our normalized Haar measure on $\mathbb{A}_K$.

We now want to understand the behavior of the adele ring $\mathbb{A}_K$ under base change. Note that the canonical embedding $K \hookrightarrow \mathbb{A}_K$ makes $\mathbb{A}_K$ a $K$-vector space, and if $L/K$ is any finite separable extension of $K$ (also a $K$-vector space), we may consider the tensor product $\mathbb{A}_K \otimes_K L$, which is also an $L$-vector space. As a topological $K$-vector space, the topology on $\mathbb{A}_K \otimes L$ is just the product topology on $[L : K]$ copies of of $\mathbb{A}_K$ (this applies whenever we take a tensor product of topological vector spaces, one of which has finite dimension).

**Proposition 25.10.** Let $L$ be a finite separable extension of a global field $K$. There is a natural isomorphism of topological rings

$$\mathbb{A}_L \simeq \mathbb{A}_K \otimes_K L$$

that makes the following diagram commute

$$
\begin{array}{ccc}
L & \sim & K \otimes_K L \\
\downarrow & & \downarrow \\
\mathbb{A}_L & \sim & \mathbb{A}_K \otimes_K L
\end{array}
$$
Proof. On the RHS the tensor product \( \mathbb{A}_K \otimes_K L \) is isomorphic to the restricted product 
\[
\prod_{v \in M_K} (K_v \otimes_K L, \mathcal{O}_v \otimes_\mathcal{O}_K \mathcal{O}_L).
\]
Explicitly, each element of \( \mathbb{A}_K \otimes_K L \) is a finite sum of elements of the form \((a_v) \otimes x\), where \(a_v \in \mathbb{A}_K\) and \(x \in L\), and there is a natural isomorphism of topological rings
\[
\mathbb{A}_K \otimes_K L \sim \prod_{v \in M_K} (K_v \otimes_K L, \mathcal{O}_v \otimes_\mathcal{O}_K \mathcal{O}_L)
\]
\[
(a_v) \otimes x \mapsto (a_v \otimes x).
\]
Here we are using the general fact that tensor products commute with direct limits (restricted direct products can be viewed as direct limits via Proposition 25.5).\(^6\)

On the LHS we have \( \mathbb{A}_L := \prod_{w \in M_L} (L_w, \mathcal{O}_w) \). But note that \( K_v \otimes_K L \simeq \prod_{w|v} L_w \), by Theorem 11.23 and \( \mathcal{O}_v \otimes_\mathcal{O}_K \mathcal{O}_L \simeq \prod_{w|v} \mathcal{O}_w \), by Corollary 11.26. These isomorphisms preserve both the algebraic and the topological structures of both sides, and it follows that
\[
\mathbb{A}_K \otimes_K L \simeq \prod_{v \in M_K} (K_v \otimes_K L, \mathcal{O}_v \otimes_\mathcal{O}_K \mathcal{O}_L) \simeq \prod_{w \in M_L} (L_w, \mathcal{O}_w) = \mathbb{A}_L
\]
is an isomorphism of topological rings. The image of \( x \in L \) in \( \mathbb{A}_K \otimes_K L \) via the canonical embedding of \( L \) into \( \mathbb{A}_K \otimes_K L \) is \( 1 \otimes x = (1, 1, 1, \ldots) \otimes x \), whose image \((x, x, x, \ldots) \in \mathbb{A}_L\) is equal to the image of \( x \in L \) under the canonical embedding of \( L \) into its adele ring \( \mathbb{A}_L \). \( \square \)

Corollary 25.11. Let \( L \) be a finite separable extension of a global field \( K \) of degree \( n \). There is a natural isomorphism of topological \( K \)-vector spaces (and locally compact groups)
\[
\mathbb{A}_L \simeq \mathbb{A}_K \oplus \cdots \oplus \mathbb{A}_K
\]
that identifies \( \mathbb{A}_K \) with the direct sum of \( n \) copies of \( \mathbb{A}_K \), and this isomorphism restricts to an isomorphism \( L \simeq K \oplus \cdots \oplus K \) of the principal adeles of \( \mathbb{A}_L \) with the \( n \)-fold direct sum of the principal adeles of \( \mathbb{A}_K \).

Theorem 25.12. For each global field \( L \) the principal adeles \( L \subseteq \mathbb{A}_L \) form a discrete cocompact subgroup of the additive group of the adele ring \( \mathbb{A}_L \).

Proof. Let \( K \) be the rational subfield of \( L \) (so \( K = \mathbb{Q} \) or \( K = \mathbb{F}_q(t) \)). It follows from Corollary 25.11 that if the theorem holds for \( K \) then it holds for \( L \), so we will prove the theorem for \( K \). Let us identify \( K \) with its image in \( \mathbb{A}_K \) (the principal adeles).

To show that the topological group \( K \) is discrete in the locally compact group \( \mathbb{A}_K \), it suffices to show that 0 is an isolated point. Consider the open set
\[
U = \{ a \in \mathbb{A}_K : \|a\|_\infty < 1 \text{ and } \|a\|_v \leq 1 \text{ for all } v < \infty \},
\]
where \( \infty \) denotes the unique infinite place of \( K \) (either the real place of \( \mathbb{Q} \) or the place corresponding to the nonarchimedean valuation \( v_\infty(f/g) = \deg g - \deg f \) of \( \mathbb{F}_q(t) \)). The product formula (Theorem 13.21) implies \( \|a\| = 1 \) for all \( a \in K^\times \subseteq \mathbb{A}_K \), so \( U \cap K = \{0\} \).

To prove that the quotient \( \mathbb{A}_K/K \) is compact, we consider the set
\[
W := \{ a \in \mathbb{A}_K : \|a\|_v \leq 1 \text{ for all } v \}.
\]

\(^6\)In general, tensor products do not commute with infinite direct products; there is always a natural map \((\prod_n A_n) \otimes B \to \prod_n (A_n \otimes B)\), but it need not be either a monomorphism or an epimorphism. This is another motivation for using restricted direct products to define the adeles, so that base change works as it should.
If we let \( U_\infty := \{ x \in K_\infty : \| x \|_\infty \leq 1 \} \), then
\[
W = U_\infty \times \prod_{v < \infty} \mathcal{O}_v \subseteq \mathbb{A}_K, \{ \infty \} \subseteq \mathbb{A}_K
\]
is a product of compact sets and therefore compact. We will show that \( W \) contains a complete set of coset representatives for \( K \) in \( \mathbb{A}_K \). This implies that \( \mathbb{A}_K / K \) is the image of the compact set \( W \) under the (continuous) quotient map \( \mathbb{A}_K \rightarrow \mathbb{A}_K / K \), hence compact.

Let \( a = (a_v) \) be any element of \( \mathbb{A}_K \). We wish to show that \( a = b + c \) for some \( b \in W \) and \( c \in K \), which we will do by constructing \( c \in K \) so that \( b = a - c \in W \).

For each \( v < \infty \) define \( x_v \in K \) as follows: put \( x_v := 0 \) if \( \| a_v \|_v \leq 1 \) (almost all \( v \)), and otherwise choose \( x_v \in K \) so that \( \| a_v - x_v \|_v \leq 1 \) and \( \| x_v \|_w \leq 1 \) for \( w \neq v \). To show that such an \( x_v \) exists, let us first suppose \( a_v = r/s \in K \) with \( r, s \in \mathcal{O}_K \) coprime (note that \( \mathcal{O}_K \) is a PID), and let \( p \) be the maximal ideal of \( \mathcal{O}_v \). The ideals \( p^{v(s)} \) and \( p^{-v(s)}(s) \) are coprime, so we can write \( r = r_1 + r_2 \) with \( r_1 \in p^{v(s)} \) and \( r_2 \in p^{-v(s)}(s) \subseteq \mathcal{O}_K \), so that \( a_v = r_1/s + r_2/s \) with \( v(r_1/s) \geq 0 \) and \( w(r_2/s) \geq 0 \) for all \( w \neq v \). If we now put \( x_v := r_2/s \), then \( \| a_v - x_v \|_v = \| r_1/s \|_v \leq 1 \) and \( \| x_v \|_w = \| r_2/s \|_w \leq 1 \) for all \( w \neq v \) as desired. We can approximate any \( a'_v \in K_v \) by such an \( a_v \in K \) with \( \| a'_v - a_v \|_v < \epsilon \) and construct \( x_v \) as above so that \( \| a_v - x_v \|_v \leq 1 \) and \( \| a'_v - x_v \|_v \leq 1 + \epsilon \); but for sufficiently small \( \epsilon \) this implies \( \| a'_v - x_v \|_v \leq 1 \), since the nonarchimedean absolute value \( \| \cdot \|_v \) is discrete.

Finally, let \( x := \sum_{v \leq \infty} x_v \in K \) and choose \( x_\infty \in \mathcal{O}_K \) so that
\[
\| a_\infty - x - x_\infty \|_\infty \leq 1.
\]
For \( a_\infty - x \in Q_\infty \simeq \mathbb{R} \), we can take \( x_\infty \in \mathbb{Z} \) in the real interval \([a_\infty - x - 1, a_\infty - x + 1)\). For \( a_\infty - x \in \mathbb{F}_q(t) \simeq \mathbb{F}_q((t-1)) \) we can take \( x_\infty \in \mathbb{F}_q[t] \) to be the polynomial of least degree for which \( a_\infty - x - x_\infty \in \mathbb{F}_q[[t^{-1}]] \).

Now let \( c := \sum_{v \leq \infty} x_v \in K \subseteq \mathbb{A}_K \), and let \( b := a - c \). Then \( a = b + c \), with \( c \in K \), and we claim that \( b \in W \). For each \( v < \infty \) we have \( x_w \in \mathcal{O}_v \) for all \( w \neq v \) and
\[
\| b \|_v = \| a - c \|_v = \| a_v - \sum_{w \leq \infty} x_w \|_v \leq \max \{ \| a_v - x_v \|_v : w \neq v \} \leq 1,
\]
by the nonarchimedean triangle inequality. For \( v = \infty \) we have \( \| b \|_\infty = \| a_\infty - c \|_\infty \leq 1 \) by our choice of \( x_\infty \), and \( \| b \|_v \leq 1 \) for all \( v \), so \( b \in W \) as claimed and the theorem follows.

**Corollary 25.13.** For any global field \( K \) the quotient \( \mathbb{A}_K / K \) is a compact group.

**Proof.** As explained in Remark 14.4, this follows immediately (in particular, the fact that \( K \) is a discrete subgroup of \( \mathbb{A}_K \) implies that it is closed and therefore \( \mathbb{A}_K / K \) is Hausdorff).

**25.4 Strong approximation**

We are now ready to prove the strong approximation theorem, an important result that has many applications. We begin with an adelic version of the Blichfeldt-Minkowski lemma.

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7Note that while \( \mathbb{F}_q((t^{-1})) \simeq \mathbb{F}_q((t)) \), in order to view \( K = \mathbb{F}_q(t) \) as canonically embedded in its completion with respect to the absolute value \( |f|_\infty = q^{\deg f} \) we need to view \( K_\infty \) as the field of Laurent series in a uniformizer, which we may take to be \( t^{-1} \) (but not \( t \)), and the valuation ring of \( K_\infty \) is \( \mathbb{F}_q([-t]) \) (not \( \mathbb{F}_q([t]) \)).
Lemma 25.14 (Adelic Blichfeldt-Minkowski lemma). Let $K$ be a global field. There is a positive constant $B_K$ such that for any $a \in \mathbb{A}_K$ with $\|a\| > B_K$ there exists a nonzero principal adele $x \in K \subseteq \mathbb{A}_K$ for which $\|x\|_v \leq \|a\|_v$ for all $v \in M_K$.

Proof. Let $b_0 := \text{covol}(K)$ be the measure of a fundamental region for $K$ in $\mathbb{A}_K$ under our normalized Haar measure $\mu$ on $\mathbb{A}_K$ (by Theorem 25.12, $K$ is cocompact, so $b_0$ is finite). Now define
\[ b_1 := \mu \left( \left\{ z \in \mathbb{A}_K : \|z\|_v \leq 1 \text{ for all } v \text{ and } \|z\|_v \leq \frac{1}{4} \text{ if } v \text{ is archimedean} \right\} \right). \]

Then $b_1 \neq 0$, since $K$ has only finitely many archimedean places. Now let $B_K := b_0/b_1$.

Suppose $a \in \mathbb{A}_K$ satisfies $\|a\| > B_K$. We know that $\|a\|_v \leq 1$ for all almost all $v$, so $\|a\|_v \neq 0$ implies that $\|a\|_v = 1$ for almost all $v$. Let us now consider the set
\[ T := \{ t \in \mathbb{A}_K : \|t\|_v \leq \|a\|_v \text{ for all } v \text{ and } \|t\|_v \leq \frac{1}{4}\|a\|_v \text{ if } v \text{ is archimedean} \}. \]

From the definition of $b_1$ we have
\[ \mu(T) = b_1\|a\| > b_1B_K = b_0; \]
this follows from the fact that the Haar measure on $\mathbb{A}_K$ is the product of the normalized Haar measures $\mu_v$ on each of the $K_v$. Since $\mu(T) > b_0$, the set $T$ is not contained in any fundamental region for $K$, so there must be distinct $t_1, t_2 \in T$ with the same image in $\mathbb{A}_K/K$, equivalently, whose difference $x = t_1 - t_2$ is a nonzero element of $K \subseteq \mathbb{A}_K$. We have
\[ \|t_1 - t_2\|_v \leq \begin{cases} \max(\|t_1\|_v, \|t_2\|_v) \leq \|a\|_v & \text{nonarch. } v; \\
\|t_1\|_v + \|t_2\|_v \leq 2 \cdot \frac{1}{2}\|a\|_v \leq \|a\|_v & \text{real } v; \\
\left(\|t_1 - t_2\|_v^{1/2}\right)^2 \leq \left(\|t_1\|_v^{1/2} + \|t_2\|_v^{1/2}\right)^2 \leq (2 \cdot \frac{1}{2}\|a\|_v^{1/2})^2 \leq \|a\|_v & \text{complex } v. \end{cases} \]

Here we have used the fact that the normalized absolute value $\| \|_v$ satisfies the nonarchimedean triangle inequality when $v$ is nonarchimedean, $\| \|_v$ satisfies the archimedean triangle inequality when $v$ is real, and $\| \|_v^{1/2}$ satisfies the archimedean triangle inequality when $v$ is complex. Thus $\|x\|_v = \|t_1 - t_2\|_v \leq \|a\|_v$ for all places $v \in M_K$ as desired. \( \square \)

Remark 25.15. Lemma 25.14 should be viewed as an analog of Minkowski’s lattice point theorem (Theorem 14.12) and a generalization of Proposition 15.9. In Theorem 14.12 we have a discrete cocompact subgroup $\Lambda$ in a real vector space $V \simeq \mathbb{R}^n$ and a sufficiently large symmetric convex set $S$ that must contain a nonzero element of $\Lambda$. In Lemma 25.14 the lattice $\Lambda$ is replaced by $K$, the vector space $V$ is replaced by $\mathbb{A}_K$, the symmetric convex set $S$ is replaced by the set
\[ L(a) := \{ x \in \mathbb{A}_K : \|x\|_v \leq \|a\|_v \text{ for all } v \in M_K \}, \]
and sufficiently large means $\|a\| > B_K$, putting a lower bound on $\mu(L(a))$. Proposition 15.9 is actually equivalent to Lemma 25.14 in the case that $K$ is a number field: use the $M_K$-divisor $c := (\|a\|_v)$ and note that $L(c) = L(a) \cap K$.

Theorem 25.16 (Strong Approximation). Let $M_K = S \sqcup T \sqcup \{w\}$ be a partition of the places of a global field $K$ with $S$ finite. Fix $a_v \in K$ and $\epsilon_v \in \mathbb{R}_{>0}$ for each $v \in S$. There exists an $x \in K$ for which
\[ \|x - a_v\|_v \leq \epsilon_v \text{ for all } v \in S, \]
\[ \|x\|_v \leq 1 \text{ for all } v \in T, \]
(note that there is no constraint on $\|x\|_w$).
Proof. Let \( W = \{ z \in \mathbb{A}_K : \| z\|_v \leq 1 \text{ for all } v \in M_K \} \) as in the proof of Theorem 25.12. Then \( W \) contains a complete set of coset representatives for \( K \subseteq \mathbb{A}_K \), so \( \mathbb{A}_K = K + W \). For any nonzero \( u \in K \subseteq \mathbb{A}_K \) we also have \( \mathbb{A}_K = K + uW \) : given \( c \in \mathbb{A}_K \) write \( u^{-1}c = a + b \) with \( a \in K \) and \( b \in W \) and then \( c = ua + ub \) with \( ua \in K \) and \( ub \in uW \). Now choose \( z \in \mathbb{A}_K \) such that

\[
0 < \| z\|_v \leq \epsilon_v \text{ for } v \in S, \quad 0 < \| z\|_v \leq 1 \text{ for } v \in T, \quad \| z\|_w > B \prod_{v \neq w} \| z\|_v^{-1},
\]

where \( B \) is the constant in the Blichfeldt-Minkowski Lemma 25.14 (this is clearly possible: every \( z = (z_v) \) with \( \| z_v\|_v \leq 1 \) is an element of \( \mathbb{A}_K \)). We have \( \| z\| > B \), so there is a nonzero \( u \in K \subseteq \mathbb{A}_K \) with \( \| u\|_v \leq \| z\|_v \) for all \( v \in M_K \).

Now let \( a = (a_v) \in \mathbb{A}_K \) be the adele with \( a_v \) given by the hypothesis of the theorem for \( v \in S \) and \( a_v = 0 \) for \( v \notin S \). We have \( \mathbb{A}_K = K + uW \), so \( a = x + y \) for some \( x \in K \) and \( y \in uW \). Therefore

\[
\| x - a_v\|_v = \| y\|_v \leq \| u\|_v \leq \| z\|_v \leq \begin{cases} \epsilon_v & \text{for } v \in S, \\ 1 & \text{for } v \in T, \end{cases}
\]

as desired. \( \square \)

**Corollary 25.17.** Let \( K \) be a global field and let \( w \) be any place of \( K \). Then \( K \) is dense in the restricted product \( \prod_{v \neq w} (K_v, \mathcal{O}_v) \).

**Remark 25.18.** Theorem 25.16 and Corollary 25.17 can be generalized to algebraic groups; see [1] for a survey.

**References**