17 The functional equation

In the previous lecture we proved that the Riemann zeta function $\zeta(s)$ has an Euler product and an analytic continuation to the right half-plane $\text{Re}(s) > 0$. In this lecture we complete the picture by deriving a functional equation that relates the values of $\zeta(s)$ to those of $\zeta(1 - s)$. This will then also allow us to extend $\zeta(s)$ to a meromorphic function on $\mathbb{C}$ that is holomorphic except for a simple pole at $s = 1$.

17.1 Fourier transforms and Poisson summation

A key tool we will use to derive the functional equation is the Poisson summation formula, a result from harmonic analysis that we now recall.

**Definition 17.1.** A Schwartz function on $\mathbb{R}$ is a complex-valued $C^\infty$ function $f: \mathbb{R} \to \mathbb{C}$ that decays rapidly to zero: for all $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$\sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)| < \infty,$$

where $f^{(n)}$ denotes the $n$th derivative of $f$. The Schwartz space $\mathcal{S}(\mathbb{R})$ of all Schwartz functions on $\mathbb{R}$ is a (non-unital) $\mathbb{C}$-algebra of infinite dimension.

**Example 17.2.** All compactly supported $C^\infty$ functions are Schwartz functions, as is the Gaussian function $g(x) := e^{-\pi x^2}$. Non-examples include functions that do not tend to zero as $x \to \pm \infty$ (such as polynomials), and functions like $(1 + x^{2n})^{-1}$ and $e^{-x^2} \sin(e^x^2)$ that either do not tend to zero quickly enough, or have derivatives that do not tend to zero as $x \to \pm \infty$.

**Remark 17.3.** For any $p \in \mathbb{R}_{\geq 1}$, the Schwartz space $\mathcal{S}(\mathbb{R})$ is contained in the space $L^p(\mathbb{R})$ of functions on $f: \mathbb{R} \to \mathbb{C}$ for which the Lebesgue integral $\int_{\mathbb{R}} |f(x)|^p dx$ exists. The space $L^p(\mathbb{R})$ is a complete normed $\mathbb{C}$-vector space under the $L^p$-norm $\|f\|_p := (\int_{\mathbb{R}} |f(x)|^p dx)^{1/p}$, and is thus a Banach space. The Schwartz space $\mathcal{S}(\mathbb{R})$ is not complete under the $L^p$-norm, but it is dense in $L^p(\mathbb{R})$ (in the subspace topology). One can equip the Schwartz space with a translation-invariant metric of its own under which it is a complete metric space (and thus a Fréchet space, since it is also locally convex), but the topology of $\mathcal{S}(\mathbb{R})$ will not concern us here. Similar comments apply to $\mathcal{S}(\mathbb{R}^n)$.

It follows immediately from the definition and standard properties of the derivative that the Schwartz space $\mathcal{S}(\mathbb{R})$ is closed under differentiation, multiplication by polynomials, and linear change of variable. It is also closed under convolution: for any $f, g \in \mathcal{S}(\mathbb{R})$ the function

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x - y)dy$$

is also an element of $\mathcal{S}(\mathbb{R})$. Convolution is commutative, associative, and bilinear.

**Definition 17.4.** The Fourier transform of a Schwartz function $f \in \mathcal{S}(\mathbb{R})$ is the function

$$\hat{f}(y) := \int_{\mathbb{R}} f(x)e^{-2\pi ixy}dx,$$
which is also a Schwartz function [1, Thm. 5.1.3]. We can recover \( f(x) \) from \( \hat{f}(y) \) via the inverse transform

\[
f(x) = \int_{\mathbb{R}} \hat{f}(y)e^{2\pi i xy}dy;
\]

see [1, Thm. 5.1.9] for a proof of this fact. The maps \( f \mapsto \hat{f} \) and \( \hat{f} \mapsto f \) are thus inverse linear operators on \( S(\mathbb{R}) \) (they are also continuous in the metric topology of \( S(\mathbb{R}) \) and thus homeomorphisms).

**Remark 17.5.** The invertibility of the Fourier transform on the Schwartz space \( S(\mathbb{R}) \) is a key motivation for its definition. For functions in \( L^1(\mathbb{R}) \) (the largest space of functions for which our definition of the Fourier transform makes sense), the Fourier transform of a smooth function decays rapidly to zero, and the Fourier transform of a function that decays rapidly to zero is smooth; this leads one to consider the subspace \( S(\mathbb{R}) \) of smooth functions that decay rapidly to zero. One can show that \( S(\mathbb{R}) \) is the largest subspace of \( L^1(\mathbb{R}) \) closed under multiplication by polynomials on which the Fourier transform is invertible.\(^1\)

The Fourier transform changes convolutions into products, and vice versa. We have

\[
\hat{f} \ast g = \hat{f} \hat{g} \quad \text{and} \quad \hat{f} \ast \hat{g} = f \ast g,
\]

for all \( f, g \in S(\mathbb{R}) \) (see Problem Set 8). One can thus view the Fourier transform as an isomorphism of (non-unital) \( \mathbb{C} \)-algebras that sends \((S(\mathbb{R}), +, \times)\) to \((S(\mathbb{R}), +, \ast)\).

**Lemma 17.6.** For all \( a \in \mathbb{R}_{>0} \) and \( f \in S(\mathbb{R}) \), we have \( \hat{f}(ax)(y) = \frac{1}{a} \hat{f}(\frac{y}{a}) \).

**Proof.** Applying the substitution \( t = ax \) yields

\[
\hat{f}(ax)(y) = \int_{\mathbb{R}} f(ax)e^{-2\pi i y/dx}dx = \frac{1}{a} \int_{\mathbb{R}} f(t)e^{-2\pi i y/da}dt = \frac{1}{a} \hat{f}\left(\frac{y}{a}\right).
\]

\( \square \)

**Lemma 17.7.** For \( f \in S(\mathbb{R}) \) we have \( \frac{d}{dy} \hat{f}(y) = -2\pi i xf(x)(y) \) and \( \frac{d}{dx}f(x)(y) = 2\pi iy\hat{f}(y) \).

**Proof.** Noting that \( xf \in S(\mathbb{R}) \), the first identity follows from

\[
\frac{d}{dy} \hat{f}(y) = \frac{d}{dy} \left( \int_{\mathbb{R}} f(x)e^{-2\pi i y/dx}dx \right) = \int_{\mathbb{R}} f(x)(-2\pi i)xe^{-2\pi i y/dx}dx = -2\pi i x\hat{f}(x)(y),
\]

since we may differentiate under the integral via dominated convergence. For the second, we note that \( \lim_{x \to \pm \infty} f(x) = 0 \), so integration by parts yields

\[
\frac{d}{dx}f(x)(y) = \int_{\mathbb{R}} f'(x)e^{-2\pi i y/dx}dx = 0 - \int_{\mathbb{R}} f(x)(-2\pi iy)e^{-2\pi i y/dx}dx = 2\pi iy\hat{f}(y).
\]

\( \square \)

The Fourier transform is compatible with the inner product \( \langle f, g \rangle := \int_{\mathbb{R}} f(x)g(x)dx \) on \( L^2(\mathbb{R}) \) (which contains \( S(\mathbb{R}) \)). Indeed, we can easily derive **Parseval’s identity:**

\[
\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x)dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(y)\overline{g(x)}e^{2\pi i xy}dxdy = \int_{\mathbb{R}} \hat{f}(y)\overline{g(y)}dy = \langle \hat{f}, \hat{g} \rangle,
\]

which when applied to \( g = f \) yields **Plancherel’s identity:**

\[
\|f\|_2^2 = \langle f, f \rangle = \langle \hat{f}, \hat{f} \rangle = \|\hat{f}\|_2^2,
\]

where \( \|f\|_2 = (\int_{\mathbb{R}} |f(x)|^2dx)^{1/2} \) is the \( L^2 \)-norm. For number-theoretic applications there is an analogous result due to Poisson.

---

\(^1\)I thank Keith Conrad and Terry Tao for clarifying this point.
Theorem 17.8 (Poisson Summation Formula). For all $f \in S(\mathbb{R})$ we have the identity

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

Proof. We first note that both sums are well defined; the rapid decay property of Schwartz functions guarantees absolute convergence. Let $F(x) := \sum_{n \in \mathbb{Z}} f(x + n)$. Then $F$ is a periodic $C^\infty$-function, so it has a Fourier series expansion

$$F(x) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i nx},$$

with Fourier coefficients

$$c_n = \int_0^1 F(t)e^{-2\pi i nt}dt = \int_0^1 \sum_{m \in \mathbb{Z}} f(t + m)e^{-2\pi i nt}dt = \int_{\mathbb{R}} f(t)e^{-2\pi i nt}dt = \hat{f}(n).$$

We then note that

$$\sum_{n \in \mathbb{Z}} f(n) = F(0) = \sum_{n \in \mathbb{Z}} c_n = \sum_{n \in \mathbb{Z}} \hat{f}(n). \quad \square$$

Finally, we note that the Gaussian function $e^{-\pi x^2}$ is its own Fourier transform.

Lemma 17.9. Let $g(x) := e^{-\pi x^2}$. Then $\hat{g}(y) = g(y)$.

Proof. The function $g(x)$ satisfies the first order ordinary differential equation

$$g' + 2\pi x g = 0, \quad (1)$$

with initial value $g(0) = 1$. Multiplying both sides by $-i$ and taking Fourier transforms yields

$$-i(\hat{g'} + 2\pi x \hat{g}) = -i(2\pi i x \hat{g} + ig') = \hat{g'} + 2\pi x \hat{g} = 0,$$

via Lemma 17.7. So $\hat{g}$ also satisfies (1), and $\hat{g}(0) = \int_{\mathbb{R}} e^{-\pi x^2}dx = 1$, so $\hat{g} = g. \quad \square$

17.1.1 Jacobi’s theta function

We now define the theta function\(^2\)

$$\Theta(\tau) := \sum_{n \in \mathbb{Z}} e^{\pi in^2 \tau}.$$ 

The sum is absolutely convergent for $\text{im} \tau > 0$ and thus defines a holomorphic function on the upper half plane. It is easy to see that $\Theta(\tau)$ is periodic modulo 2, that is,

$$\Theta(\tau + 2) = \Theta(\tau),$$

but it it also satisfies another functional equation.

Lemma 17.10. For all $a \in \mathbb{R}_{>0}$ we have $\Theta(ia) = \Theta(i/a)/\sqrt{a}$.

Proof. Put $g(x) := e^{-\pi x^2}$ and $h(x) := g(\sqrt{a}x) = e^{-\pi x^2 a}$. Lemmas 17.6 and 17.9 imply

$$\hat{h}(y) = g(\sqrt{a}x)(y) = \hat{g}(y/\sqrt{a})/\sqrt{a} = g(y/\sqrt{a})/\sqrt{a}.$$ 

Plugging $\tau = ia$ into $\Theta(\tau)$ and applying Poisson summation (Theorem 17.8) yields

$$\Theta(ia) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 a} = \sum_{n \in \mathbb{Z}} \hat{h}(n) = \sum_{n \in \mathbb{Z}} g(n/\sqrt{a})/\sqrt{a} = \Theta(i/a)/\sqrt{a}. \quad \square$$

\(^2\)The function $\Theta(\tau)$ we define here is a special case of one of four parameterized families of theta functions $\Theta_i(z : \tau)$ originally defined by Jacobi for $i = 0, 1, 2, 3$, which play an important role in the theory of elliptic functions and modular forms; in terms of Jacobi’s notation, $\Theta(i) = \Theta_3(0; \tau)$. 

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17.1.2 Euler’s gamma function

You are probably familiar with the gamma function $\Gamma(s)$, which plays a key role in the functional equation of not only the Riemann zeta function but many of the more general zeta functions and $L$-series we wish to consider. Here we recall some of its analytic properties. We begin with the definition of $\Gamma(s)$ as a Mellin transform.

**Definition 17.11.** The Mellin transform of a function $f : \mathbb{R}_{>0} \rightarrow \mathbb{C}$ is the complex function defined by

\[ \mathcal{M}(f)(s) := \int_0^{\infty} f(t)t^{s-1}dt, \]

whenever this integral converges. It is holomorphic on $\Re s \in (a,b)$ for any interval $(a,b)$ in which the integral $\int_0^{\infty} |f(t)|t^{\sigma-1}dt$ converges for all $\sigma \in (a,b)$.

**Definition 17.12.** The Gamma function $\Gamma(s) := \mathcal{M}(e^{-t})(s) = \int_0^{\infty} e^{-t}t^{s-1}dt,$ is the Mellin transform of $e^{-t}$. Since $\int_0^{\infty} |e^{-t}|t^{\sigma-1}dt$ converges for all $\sigma > 0$, the integral defines a holomorphic function on $\Re(s) > 0$.

Integration by parts yields

\[ \Gamma(s) = \frac{t^se^{-t}}{s}\bigg|_{0}^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-t}t^sdt = \frac{\Gamma(s+1)}{s}, \]

thus $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1 (since $\Gamma(1) = \int_0^{\infty} e^{-t}dt = 1$), and

\[ \Gamma(s+1) = s\Gamma(s) \quad (2) \]

for $\Re(s) > 0$. Equation (2) allows us to extend $\Gamma(s)$ to a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$, and no other poles.

An immediate consequence of (2) is that for integers $n > 0$ we have

\[ \Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\cdots2\cdot1\cdot\Gamma(1) = n!, \]

thus the gamma function can be viewed as an extension of the factorial function. The gamma function satisfies many useful identities in addition to (2), including the following.

**Theorem 17.13 (Euler’s Reflection Formula).** We have

\[ \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \]

as meromorphic functions on $\mathbb{C}$ with simple poles at each integer $s \in \mathbb{Z}$.

**Proof.** Let $f(s) := \Gamma(s)\Gamma(1-s)\sin(\pi s)$. The function $\Gamma(s)\Gamma(1-s)$ has a simple pole at each $s \in \mathbb{Z}$ and no other poles, while $\sin(\pi s)$ has a zero at each $s \in \mathbb{Z}$ and no poles, so $f(s)$ is holomorphic on $\mathbb{C}$. We now note that

\[ f(s+1) = \Gamma(s+1)\Gamma(-s)\sin(\pi s + \pi) = -s\Gamma(s)\Gamma(-s)\sin(\pi s) = \Gamma(s)\Gamma(1-s)\sin(\pi s) = f(s), \]

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so \( f \) is periodic (with period 1). Using the substitution \( u = e^t \) we obtain

\[
|\Gamma(s)| \leq \int_0^\infty |e^{-t} t^{s-1}|dt = \int_{-\infty}^\infty |e^{-u} e^{u(s-1)}| e^u du = \int_{-\infty}^\infty e^{u Re(s) - e^u} du.
\]

This implies \( |\Gamma(s)| \) is bounded on \( \Re(s) \in [1, 2] \), hence on \( \Re(s) \in [0, 1] \cap \Im(s) \geq 1 \), via (2). It follows that in the strip \( \Re(s) \in [0, 1] \) we have

\[
|f(s)| = |\Gamma(s)||\Gamma(1 - s)||\sin(\pi s)| = O(e^{\Im(s)}),
\]

as \( \Im(s) \to \infty \), since \( |\sin(\pi s)| = \frac{1}{2}|e^{is} - e^{-is}| = O(e^{\Im(s)}) \). By Lemma 17.14 below, \( f(s) \) is constant. To determine the constant, as \( s \to 0 \) we have \( \Gamma(s) \sim \frac{1}{s} \) and \( \sin(\pi s) \sim \pi s \), thus

\[
f(0) = \lim_{s \to 0} f(s) = \lim_{s \to 0} \Gamma(s) \Gamma(1 - s) \sin(\pi s) = \lim_{s \to 0} \frac{1}{s} \cdot 1 \cdot \pi s = \pi,
\]

and the theorem follows.

**Lemma 17.14.** Let \( f(s) \) be a holomorphic function on \( \mathbb{C} \) such that \( f(s + 1) = f(s) \) and \( |f(s)| = O(e^{\Im(s)}) \) as \( \Im(s) \to \infty \) in the vertical strip \( \Re(s) \in [0, 1] \). Then \( f \) is constant.

**Proof.** The function

\[
g(s) = \frac{f(s) - f(a)}{\sin(\pi(s - a))}
\]

is holomorphic on \( \mathbb{C} \), since \( f(s) - f(a) \) is holomorphic and vanishes at the zeros \( a + \mathbb{Z} \) of \( \sin(\pi(s - a)) \) (all of which are simple). We also have \( g(s + 1) = g(s) \), and \( |g(s)| \) is bounded on \( \Re(s) \in [0, 1] \), since as \( \Im(s) \to \infty \) we have \( |f(s) - f(a)| = O(e^{\Im(s)}) \) and \( \Im(\pi(s - a)) \sim e^{\pi \Im(s)} \). It follows that \( g(s) \) is bounded on \( \mathbb{C} \), hence constant, by Liouville’s theorem. We must have \( g = 0 \), since \( |g(s)| = O(e^{(1 - \pi)\Im(s)}) = o(1) \) as \( \Im(s) \to \infty \), and this implies \( f(s) = f(a) \) for all \( s \in \mathbb{C} \).

**Example 17.15.** Putting \( s = \frac{1}{2} \) in the reflection formula yields \( \Gamma(\frac{1}{2})^2 = \pi \), so \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \).

**Corollary 17.16.** The function \( \Gamma(s) \) has no zeros on \( \mathbb{C} \).

**Proof.** Suppose \( \Gamma(s_0) = 0 \). The RHS of the reflection formula \( \Gamma(s) \Gamma(1 - s) = \pi/\sin(\pi s) \) is never zero, since \( \sin(\pi s) \) has no poles, so \( \Gamma(1 - s) \) must have a pole at \( s_0 \). Therefore \( 1 - s_0 \in \mathbb{Z} \), equivalently, \( s_0 \in \mathbb{Z} \), but for \( s_0 \in \mathbb{Z}_{>0} \) we have \( \Gamma(s_0) = (s_0 - 1)! \neq 0 \), and for \( s_0 \in \mathbb{Z}_{\leq0} \) we cannot have \( \Gamma(s_0) = 0 \) because \( \Gamma(s) \) has a pole at all non-positive integers.

### 17.1.3 Completing the zeta function

Let us now consider the function

\[
F(s) := \pi^{-s} \Gamma(s) \zeta(2s),
\]

which is holomorphic on \( \Re(s) > 1/2 \). In the region \( \Re(s) > 1/2 \) we have an absolutely convergent sum

\[
F(s) = \pi^{-s} \Gamma(s) \sum_{n \geq 1} n^{-2s} = \sum_{n \geq 1} (\pi n^2)^{-s} \Gamma(s) = \sum_{n \geq 1} \int_0^\infty (\pi n^2)^{-s} t^{s-1} e^{-t} dt,
\]
and the substitution \( t = \pi n^2 y \) with \( dt = \pi n^2 dy \) yields
\[
F(s) = \sum_{n \geq 1} \int_0^\infty (\pi n^2)^{-s} (\pi n^2 y)^{s-1} e^{-\pi n^2 y} \pi n^2 dy = \sum_{n \geq 1} \int_0^\infty y^{s-1} e^{-\pi n^2 y} dy.
\]
By the Fubini-Tonelli theorem, we can swap the sum and the integral to obtain
\[
F(s) = \int_0^\infty y^{s-1} \sum_{n \geq 1} e^{-\pi n^2 y} dy.
\]
We have \( \Theta(iy) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 y} = 1 + 2 \sum_{n \geq 1} e^{-\pi n^2 y} \), thus
\[
F(s) = \frac{1}{2} \int_0^1 y^{s-1} (\Theta(iy) - 1) dy
= \frac{1}{2} \left( \int_0^1 y^{s-1} \Theta(iy) dy - \frac{1}{s} + \int_1^\infty y^{s-1} (\Theta(iy) - 1) dy \right)
\]
We now focus on the first integral on the RHS. The change of variable \( t = \frac{1}{s} y \) yields
\[
\int_0^1 y^{s-1} \Theta(iy) dy = \int_1^\infty t^{1-s} \Theta(i/t)(-t^{-2}) dt = \int_1^\infty t^{-s-1} \Theta(i/t) dt.
\]
By Lemma 17.10, \( \Theta(i/t) = \sqrt{t} \Theta(it) \), and adding \(-\int_1^\infty t^{-s-1/2} dt + \int_1^\infty t^{-s-1/2} dt = 0\) yields
\[
\int_1^\infty t^{-s-1/2} (\Theta(it) - 1) dt + \int_1^\infty t^{-s-1/2} dt
= \int_1^\infty t^{-s-1/2} (\Theta(it) - 1) dt - \frac{1}{1/2 - s}.
\]
Plugging this back into our equation for \( F(s) \) we obtain the identity
\[
F(s) = \frac{1}{2} \int_1^\infty (y^{s-1} + y^{-s-1/2}) (\Theta(iy) - 1) dy - \frac{1}{2s} - \frac{1}{1 - 2s},
\]
valid on \( \text{Re}(s) > 1/2 \). We now observe that \( F(s) = F(1/2 - s) \) for \( s \neq 0, 1/2 \), which allows us to analytically extend \( F(s) \) to a meromorphic function on \( \mathbb{C} \) with poles only at \( s = 0, 1/2 \). Replacing \( s \) with \( s/2 \) leads us to define the completed zeta function
\[
Z(s) := \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s),
\]
which is meromorphic on \( \mathbb{C} \) and satisfies the functional equation
\[
Z(s) = Z(1-s).
\]
It has simple poles at 0 and 1 (and no other poles). The only zeros of \( Z(s) \) on \( \text{Re}(s) > 0 \) are the zeros of \( \zeta(s) \), since by Corollary 17.16, the gamma function \( \Gamma(s) \) has no zeros (and neither does \( \pi^{-s/2} \)). Thus the zeros of \( Z(s) \) on \( \mathbb{C} \) all lie in the critical strip \( 0 < \text{Re}(s) < 1 \).

The functional equation also allows us to analytically extend \( \zeta(s) \) to a meromorphic function on \( \mathbb{C} \) whose only pole is a simple pole at \( s = 1 \); the pole of \( Z(s) \) at \( s = 0 \) comes from the pole of \( \Gamma(s/2) \) at \( s = 0 \). The function \( \Gamma(s/2) \) also has poles at \(-2, -4, \ldots \) where \( Z(s) \) does not, so our extended \( \zeta(s) \) must have zeros at \(-2, -4, \ldots \). These are trivial zeros;
all the interesting zeros of \( \zeta(s) \) lie in the critical strip and are conjectured to lie only on the critical line \( \text{Re}(s) = 1/2 \) (this is the Riemann hypothesis).

We can compute \( \zeta(0) \) using the functional equation. From (3) and (4) we have

\[
\zeta(s) = \frac{Z(s)}{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)} = \frac{Z(1-s)}{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)} = \frac{\pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)}{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)} \zeta(1-s) = \frac{\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s). \tag{5}
\]

We know that \( \zeta(s) \) has a simple pole with residue 1 at \( s = 1 \), so

\[
1 = \lim_{s \to 1^+} (s-1)\zeta(s) = \lim_{s \to 1^+} \frac{(s-1)\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s).
\]

When \( s = 1 \), the denominator on the RHS is \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \), which cancels \( \pi^{1-1/2} = \sqrt{\pi} \) in the numerator. Using \( \Gamma(z) = \frac{1}{2}\Gamma(z+1) \) to shift \( \Gamma\left(\frac{1-s}{2}\right) \) in the numerator yields

\[
1 = \lim_{s \to 1^+} (s-1)\frac{2}{1-s}\Gamma\left(\frac{3-s}{2}\right) \zeta(1-s) = -2\Gamma(1)\zeta(0) = -2\zeta(0).
\]

Thus \( \zeta(0) = -1/2. \)

Using the reflection formula to replace \( \Gamma\left(\frac{s}{2}\right) = \pi/\left(\Gamma\left(\frac{2-s}{2}\right)\sin\left(\frac{\pi s}{2}\right)\right) \) in (5), we have

\[
\zeta(s) = \pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{2-s}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s).
\]

Applying the duplication formula \( \Gamma(2z) = \pi^{-1/2}2^{-2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) \) with \( z = \frac{1-s}{2} \) then yields

\[
\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s), \tag{6}
\]

which is how one often sees the functional equation for \( \zeta(s) \) written.

### 17.2 Gamma factors and a holomorphic zeta function

If we write out the Euler product for the completed zeta function, we have

\[
Z(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \cdot \prod_p (1-p^{-s})^{-1}.
\]

One should think of this as a product over the places of the field \( \mathbb{Q} \); the leading factor

\[
\Gamma_\mathbb{R}(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)
\]

that distinguishes the completed zeta function \( Z(s) \) from \( \zeta(s) \) corresponds to the real archimedean place of \( \mathbb{Q} \). When we discuss Dedekind zeta functions in a later lecture we will see that there are gamma factors \( \Gamma_\mathbb{R} \) and \( \Gamma_\mathbb{C} \) associated to each of the real and complex places of a number field.

If we insert an additional factor of \( \left(\frac{s}{2}\right) := \frac{s(s-1)}{2} \) in \( Z(s) \) we can remove the poles at 0 and 1, yielding a function \( \xi(s) \) holomorphic on \( \mathbb{C} \). This yields Riemann’s seminal result.

**Theorem 17.17 (Analytic Continuation II).** The function

\[
\xi(s) := \left(\frac{s}{2}\right)\Gamma_\mathbb{R}(s)\zeta(s)
\]

is holomorphic on \( \mathbb{C} \) and satisfies the functional equation

\[
\xi(s) = \xi(1-s).
\]

The zeros of \( \xi(s) \) all lie in the critical strip \( 0 < \text{Re}(s) < 1 \).

**Remark 17.18.** We will usually work with \( Z(s) \) and deal with the poles rather than making it holomorphic by introducing additional factors; some authors use \( \xi(s) \) to denote our \( Z(s) \).
17.3 Zeros in the critical strip

The zeros of $\xi(s)$ in the critical strip are precisely the zeros of $\zeta(s)$ in the critical strip. Let $N(T)$ denote the number of zeros of $\xi(s)$ in the critical strip that satisfy $0 < \text{im} \ s < T$. If we fix $\epsilon > 0$ and let $R$ be the rectangle $\{-\epsilon \leq \text{Re}(s) \leq 1 + \epsilon, \ 0 \leq \text{im} \ s \leq T\}$, we can compute $N(T)$ using Cauchy’s argument principle via

$$N(T) = \frac{1}{2\pi i} \int_{\partial R} \frac{\xi'(s)}{\xi(s)} ds,$$

provided that there are no zeros on the lines $\text{im} \ s = 0$ and $\text{im} \ s = T$. From this formula and the functional equation one derive the asymptotic formula

$$N(T) \sim \frac{1}{2\pi} T \log \left( \frac{T}{2\pi e} \right),$$

along with an explicit error term that allows one to compute the integer $N(T)$ exactly. Note that this formula implies that there are infinitely many zeros in the critical strip. The Riemann hypothesis states that all of these zeros lie on the critical line $\text{im} \ s = 1/2$.

One can count zeros on the critical line by counting zeros of the Hardy $Z$-function

$$e^{i\theta(t)} \zeta(1/2 + it)$$

in a region $0 \leq t \leq T$, where $\theta(t)$ is the Riemann-Siegel function

$$\theta(t) := \arg \left( \Gamma \left( \frac{2it + 1}{4} \right) \right) - \frac{\log \pi}{2} t.$$

There are asymptotic expansions of the Hardy $Z$-function that allow one to do this efficiently (one counts sign changes and checks for multiple roots). By comparing the result to $N(T)$ one can determine whether all the zeros in the critical strip with $0 < \text{im} \ s < T$ lie on the critical line or not. This has been done for values of $T$ exceeding $10^{13}$; more precisely, it has been verified that when ordered by their imaginary parts, the first $10^{13}$ zeros above the real axis all lie on the critical line; see [2] for details.

References
