13 Haar measures and the product formula

We now return to our discussion of local and global fields. Our goal in this lecture is to prove a generalization of the product formula that you proved for $\mathbb{Q}$ and $\mathbb{F}_q(t)$ on Problem Set 1 that applies all global fields. The product formula for a global field K is the identity

$$\prod_v \|x\|_v = 1,$$

valid for all $x \in K^\times$. Here $\| \, \|_v$ denotes the *normalized absolute value* associated to v, which ranges over equivalence classes of absolute values on K (also known as *places* of K). We will define $\| \, \|_v$ in terms of the *Haar measure* on the completion of K with respect to v.

13.1 Haar measures

Definition 13.1. Let X be a locally compact Hausdorff space. The σ-algebra Σ of X is the collection of subsets of X generated by the open and closed sets under countable unions and countable intersections. Its elements are called *Borel sets*, or simply *measurable sets*. A *Borel measure* on X is a countably additive function

$$\mu\colon \Sigma \to \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

A *Radon measure* on X is a Borel measure on X that additionally satisfies

1. $\mu(S) < \infty$ if S is compact,
2. $\mu(S) = \inf\{\mu(U) : S \subseteq U,\ U \text{ open}\}$,
3. $\mu(S) = \sup\{\mu(C) : C \subseteq S,\ C \text{ compact}\}$,

for all Borel sets $S \in \Sigma$.[1]

Definition 13.2. A topological group that is both locally compact and Hausdorff is called a *locally compact group*. A (*left*) *Haar measure* μ on a locally compact group is a nonzero Radon measure that is *translation invariant*, meaning that

$$\mu(E) = \mu(x + E)$$

for all $x \in X$ and Borel sets E (we have written the group operation additively because we have in mind the additive group of a local field K) .

One defines a right Haar measure analogously, but in most cases they coincide and in our situation we are working with an abelian group (the additive group of a field), in which case they necessarily do. The key result on Haar measures, is that they exist and are unique up to scaling. For compact groups existence was proved by Haar and uniqueness by von Neumann; the general result for locally compact groups was proved by Weil.

Theorem 13.3 (Weil)**.** *Every locally compact group G has a Haar measure. If μ and μ' are two Haar measure on G, then there is a positive real number λ such that $\mu'(S) = \lambda\mu(S)$ for all measurable sets S.*

[1]Some authors additionally require X to be σ-compact (a countable union of compact sets). Local fields are σ-compact so this distinction will not concern us.

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Proof. See [3, §7.2]. $\square$

Example 13.4. The standard Euclidean measure on $\mathbb{R}^n$ is the unique Haar measure on $\mathbb{R}^n$ for which the unit cube has measure 1.

The additive group of a local field K is a locally compact group (it is a metric space, so it is automatically Hausdorff). For compact groups G, it is standard to normalize the Haar measure so that $\mu(G) = 1$, but local fields are never compact and we will always have $\mu(K) = \infty$. For nonarchimedean local fields the valuation ring $A = B_{\le 1}(0)$ is a compact group, and it is then natural to normalize the Haar measure on K so that $\mu(A) = 1$. But the key point for us is that there is a unique absolute value on K that is compatible with every Haar measure μ on K (regardless of how μ is normalized).

Proposition 13.5. *Let K be a local field with discrete valuation v, residue field k, and absolute value*

$$|\cdot|_v := (\#k)^{-v(\cdot)},$$

and let μ be a Haar measure on K. For every $x \in K$ and measurable set $S \subseteq K$ we have

$$\mu(xS) = |x|_v \mu(S).$$

Moreover, the absolute value $|\ |_v$ is the unique absolute value compatible with the topology on K for which this is true.

Proof. Let A be the valuation ring of K with maximal ideal $\mathfrak{p}$. The proposition clearly holds for $x = 0$, so let $x \neq 0$. The map $\phi_x \colon y \mapsto xy$ is an automorphism of the additive group of K, and it follows that the composition $\mu_x = \mu \circ \phi_x$ is a Haar measure on K, hence a multiple of μ, say $\mu_x = \lambda_x \mu$, for some $\lambda_x \in \mathbb{R}_{>0}$. Define the function $\chi \colon K^\times \to \mathbb{R}_{\ge 0}$ by $\chi(x) := \lambda_x = \mu_x(A)/\mu(A)$. Then $\mu_x = \chi(x)\mu$, and for all $x, y \in K^\times$ we have

$$\chi(xy) = \frac{\mu_{xy}(A)}{\mu(A)} = \frac{\mu_x(yA)}{\mu(A)} = \frac{\chi(x)\mu_y(A)}{\mu(A)} = \frac{\chi(x)\chi(y)\mu(A)}{\mu(A)} = \chi(x)\chi(y).$$

Thus χ is multiplicative, and we claim that in fact $\chi(x) = |x|_v$ for all $x \in K^\times$. Since both χ and $|\cdot|_v$ are multiplicative, it suffices to consider $x \in A - \{0\}$. For any such x, the ideal xA is equal to $\mathfrak{p}^{v(x)}$, since A is a DVR. The residue field $k := A/\mathfrak{p}$ is finite, hence A/xA is also finite; indeed it is a k-vector space of dimension $v(x)$ and has cardinality $[A : xA] = (\#k)^{v(x)}$. Writing A as a finite disjoint union of cosets of xA, we have

$$\mu(A) = [A : xA]\mu(xA) = (\#k)^{v(x)}\chi(x)\mu(A),$$

and therefore $\chi(x) = (\#k)^{-v(x)} = |x|_v$ as claimed. It follows that

$$\mu(xS) = \mu_x(S) = \chi(x)\mu(s) = |x|_v\mu(S),$$

for all $x \in K$ and $S \in \Sigma$. To prove uniqueness, if $|\ |$ is an absolute value on K that induces the same topology as $|\ |_v$ then for some $0 < c < 1$ we have $|x| = |x|_v^c$ for all $x \in K^\times$. Let us fix $x \in K^\times$ with $|x|_v$ (take any x with $v(x) \neq 0$). If $|\ |$ also satisfies $\mu(xS) = |x|\mu(S)$ then

$$\frac{\mu(xA)}{\mu(A)} = |x| = |x|_v^c = \left(\frac{\mu(xA)}{\mu(A)}\right)^c,$$

which implies $c = 1$, meaning that $|\ |$ and $|\ |_v$ are the same absolute value. $\square$

13.2 Places of a global field

Definition 13.6. A *place* of a global field K is an equivalence class of nontrivial absolute values on K. We use M_K to denote the set of places of K. For each place v we may use $|\ |_v$ to denote a representative of its equivalence class, and we use K_v to denote the local field obtained by completing K with respect to $|\ |_v$; note that K_v does not depend on the choice of the representative $|\ |_v$. The place v is called *archimedean* when the absolute value $|\ |_v$ is archimedean, and is *nonarchimedean* otherwise. Every nonarchimedean place v arises from a discrete valuation on K that we may also denote v.

Example 13.7. As proved in Problem Set 1, for $K = \mathbb{Q}$ we have

$$M_K = \{|\ |_p : \text{primes } p \le \infty\},$$

where $|\ |_\infty$ denotes the archimedean absolute value on $\mathbb{Q}$. For $K = \mathbb{F}_q(t)$ we may identify $M_{\mathbb{F}_q(t)}$ with the set of irreducible polynomials in $\mathbb{F}_q[t]$ together with the nonarchimedean absolute value $|r|_\infty = q^{\deg r}$. In both cases the places $p < \infty$ correspond to primes of K (nonzero prime ideals of $\mathcal{O}_K$), while the place $p = \infty$ does not.

Remark 13.8. In contrast with $\mathbb{Q}$, there is nothing special about the absolute value $|\ |_\infty$ on $\mathbb{F}_q(t)$, it is an artifact of our choice of the separating element t, which we could change by applying any automorphism $t \mapsto (at+b)/(ct+d)$ of $\mathbb{F}_q(t)$. If we put $z = 1/t$ and rewrite $\mathbb{F}_q(t)$ as $\mathbb{F}_q(z)$, the absolute value $|\ |_\infty$ on $\mathbb{F}_q(t)$ is the same as the absolute value $|\ |_z$ on $\mathbb{F}_q(z)$ corresponding to the irreducible polynomial $z \in \mathbb{F}_q[z]$.

Definition 13.9. If L/K is an extension of global fields, for every place w of L, any absolute value $|\ |_w$ that represents the equivalence class w restricts to an absolute value on K that represents a place v of K; this v is independent of the choice of $|\ |_w$. We write $w|v$ to indicate this relationship and say that w *extends* v.

A global field L is a finite separable extension of either $K = \mathbb{Q}$ or $K = \mathbb{F}_q(t)$ (for some finite field $\mathbb{F}_q$). Thus every place v of L extends a place $p \le \infty$ of K. When $p < \infty$ we say that v is a *finite place* and write $v \nmid \infty$. In this case v arises from a discrete valuation associated to a prime of L lying above the prime p of K; the finite places of L are in one-to-one correspondence with the primes of L (nonzero prime ideals of its ring of integers). When $p = \infty$ we call v an *infinite place* and write $v|\infty$; infinite places do not correspond to primes of L. If L is a number field the infinite places are precisely the archimedean ones.

Example 13.10. If K is a number field and $v|p$ is a finite place, then K_v is a finite separable extension of $\mathbb{Q}_p$. If we write

$$K \simeq \mathbb{Q}[x]/(f(x)),$$

then

$$K_v \simeq \mathbb{Q}_p[x]/(g(x)),$$

for some irreducible $g \in \mathbb{Q}_p[x]$ appearing in the factorization of f in $\mathbb{Q}_p[x]$. When $v|\infty$ is an infinite place there are only two possibilities: either $K_v = \mathbb{R}$ or $K_v = \mathbb{C}$.

Definition 13.11. Let K be a number field and let $v|\infty$ be an infinite place of K. If $K_v \simeq \mathbb{R}$ then v is a *real place* of K. If $K_v \simeq \mathbb{C}$ then v is a *complex place* of K.

Theorem 13.12. *Let L/K be a finite separable extension of global fields and let v be a place of K. Then there is an isomorphism of finite étale K_v-algebras*

$$L \otimes_K K_v \xrightarrow{\sim} \prod_{w|v} L_w$$

defined by $\ell \otimes x \mapsto (\ell x, \ldots, \ell x)$.

For nonarchimedean places this follows from part (v) of Theorem 11.20, but here we give a more general proof that works for any place of K.

Proof. By Proposition 4.35, $L \otimes_K K_v$ is finite étale K_v-algebra and therefore isomorphic to a finite product $\prod_{i \in I} L_i$ of finite separable extensions L_i/K_v. We need to show that each L_i is the completion L_w of L at a place $w|v$ and every L_w appears exactly once in $\prod_i L_i$.

Each L_i is a local field, since it is a finite extension of K_v, and it has a unique absolute value $|\ |_w$ that extends the absolute value $|\ |_v$ on K_v (for any choice of $|\ |_v$ representing the place v); this follows from Theorem 10.7 when v is nonarchimedean and is obviously the case if $K_v \simeq \mathbb{R}, \mathbb{C}$ is archimedean, since then either $L_w = K_v$ or $L_w \simeq \mathbb{C}$ and $K_v \simeq \mathbb{R}$.[2] The map $L \hookrightarrow L \otimes_K K_v \simeq \prod_i L_i \twoheadrightarrow L_i$ allows us to view L as a subfield of each L_i, so the absolute value $|\ |_w$ on L_i restricts to an absolute value on L that uniquely determines a place $w|v$. This defines a map $\{i \in I\} \to \{w|v\}$; we need to show that it is a bijection and that the induced map $\phi\colon \{L_i : i \in I\} \to \{L_w : w|v\}$ sends each L_i to an isomorphic L_w.

We may view $L \otimes_K K_v \simeq \prod_i L_i$ as an isomorphism of topological rings: on the LHS the étale K_v-algebra $L \otimes_K K_v$ is a finite dimensional K_v-vector space with a canonical topology induced by the sup norm, and on the RHS we have the product topology; these topologies coincide because the absolute value on each L_i restricts to the absolute value on K_v, allowing us to also view the RHS as a normed K_v-vector space, and all norms on a finite dimensional vector space over a complete field induce the same topology (Proposition 10.6).

The image of the canonical embedding $L \hookrightarrow L \otimes_K K_v$ defined by $\ell \mapsto \ell \otimes 1$ is dense because $K \subseteq L$ is dense in K_v: for any nonzero $\ell \otimes x$ in $L \otimes_K K_v$ we can approximate it arbitrarily closely by $\ell/y \otimes y = \ell \otimes 1$ for some nonzero $y \in K$ (and similarly for sums of pure tensors). The image of L is therefore dense in $\prod_i L_i$, and in the projection to any L_i, or any $L_i \times L_j$ $(i \neq j)$. If ϕ maps L_i to L_w then we necessarily have $L_i \simeq L_w$ by the universal property of completions (Proposition 8.3): L_i is complete, L is dense in L_i, and L_w is the completion of L with respect to the restriction of the absolute value on L_i to L.

If ϕ is not injective then some L_w appears as two distinct L_i and L_j in $L \otimes_K K_v \simeq \prod L_i$, but this is impossible because the image of the diagonal embedding $L \to L_w \times L_w$ is not dense but the image of L is dense in $L_i \times L_j$.

For each $w|v$ we may define a continuous homomorphism of finite étale K_v-algebras:

$$\begin{aligned} \varphi\colon L \otimes_K K_v &\to L_w \\ \ell \otimes x &\mapsto \ell x. \end{aligned}$$

The map φ is surjective because its image contains L and is complete, and L_w is the completion of L. It then follows from Corollary 4.31 that L_w is isomorphic one of the

[2] The isomorphisms $K_v \simeq \mathbb{R}$ and $L_w \simeq \mathbb{C}$ are isomorphism of topological fields whose archimedean topology is induced by an absolute value; we always view $\mathbb{R}$ and $\mathbb{C}$ as local fields whose topology is induced by the standard Euclidean metric. There are plenty of nonarchimedean topologies on $\mathbb{R}$ and $\mathbb{C}$ (for each prime p the field isomorphism $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ lets us put an extension of the p-adic absolute value on $\mathbb{C}$ which we can restrict to $\mathbb{R}$), but none correspond to local fields because they are not locally compact.

factors L_i in $L \otimes_K K_v \simeq \prod L_i$; the absolute value on L_i must correspond to the place w, thus $\phi(L_i) = L_w$ and ϕ is surjective. $\square$

Corollary 13.13. *Let K be a number field and $p \leq \infty$ a prime of $\mathbb{Q}$. There is a one-to-one-correspondence*

$$\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}}_p)/\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longleftrightarrow \{v \in M_K : v|p\},$$

between $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-orbits of $\mathbb{Q}$-embeddings of K into $\overline{\mathbb{Q}}_p$ and the places $v|p$ of K.

Before proving the corollary, lets make sure we understand the set of Galois orbits on the LHS. Each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on a $\mathbb{Q}$-embedding $\tau\colon K \to \overline{\mathbb{Q}}_p$ by composition: $\sigma \circ \tau$ is also a $\mathbb{Q}$-embedding of K into $\overline{\mathbb{Q}}_p$.

Proof. Theorem 13.12 gives us an isomorphism $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{v|p} K_v$ of finite étale $\mathbb{Q}_p$-algebras. Each $\varphi \in \mathrm{Hom}_{\mathbb{Q}_p}(K \otimes_{\mathbb{Q}} \mathbb{Q}_p, \overline{\mathbb{Q}}_p)$ can therefore be written as $\varphi = \phi \circ \pi$, where $\pi\colon K \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \prod_{v|p} K_v \to K_v$ is a projection to one of the K_v and $\phi \in \mathrm{Hom}_{\mathbb{Q}_p}(K_v, \overline{\mathbb{Q}}_p)$. The key point is that the image of φ is a field (it is an étale $\mathbb{Q}_p$-algebra that lies in $\overline{\mathbb{Q}}_p$), and therefore must be isomorphic to one of the factors in $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{v|p} K_v$ by Proposition 4.31.

It follows that we can identify the set $\mathrm{Hom}_{\mathbb{Q}_p}(K \otimes_{\mathbb{Q}} \mathbb{Q}_p, \overline{\mathbb{Q}}_p)$ with the disjoint union of sets $\bigsqcup_{v|p} \mathrm{Hom}_{\mathbb{Q}_p}(K_v, \overline{\mathbb{Q}}_p)$. We then have bijections of finite sets

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}}_p) &\longleftrightarrow \mathrm{Hom}_{\mathbb{Q}_p}(K \otimes_{\mathbb{Q}} \mathbb{Q}_p, \overline{\mathbb{Q}}_p)\\ &\longleftrightarrow \bigsqcup_{v|p} \mathrm{Hom}_{\mathbb{Q}_p}(K_v, \overline{\mathbb{Q}}_p).\end{aligned}$$

Each $\mathrm{Hom}_{\mathbb{Q}_p}(K_v, \overline{\mathbb{Q}}_p)$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-orbit in $\mathrm{Hom}_{\mathbb{Q}_p}(K \otimes_{\mathbb{Q}} \mathbb{Q}_p, \overline{\mathbb{Q}}_p)$: if we write K_v as $\mathbb{Q}_p(\alpha)$ where $\alpha \in K_v$ has minimal polynomial $f \in \mathbb{Q}_p[x]$, we have a bijection between $\mathbb{Q}_p$-embeddings $K_v \to \overline{\mathbb{Q}}_p$ and roots of f in $\overline{\mathbb{Q}}_p$, and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts transitively on both. $\square$

The corollary implies that $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is in bijection with the set $\{v|\infty\}$ of archimedean places of K; note that $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is just a group of order 2 whose non-trivial element is complex conjugation. We can partition $\{v|\infty\}$ into *real* and *complex* places, based on whether $K_v \simeq \mathbb{R}$ or $K_v \simeq \mathbb{C}$. Each real place corresponds to an element of $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{R})$; these are fixed by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and thus correspond to trivial $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbits of $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$ (orbits of size one). Each complex place corresponds to a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbit of size two in $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$; these are conjugate pairs of embeddings $K \to \mathbb{C}$ whose image does not lie in $\mathbb{R}$.

Definition 13.14. Let K be a number field. Elements of $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{R})$ are *real embeddings* and elements of $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$ whose image does not lie in $\mathbb{R}$ are *complex embeddings*.

There is a one-to-one correspondence between real embeddings and real places, but complex embeddings come in conjugate pairs; and each pair corresponds to a single complex place.

Corollary 13.15. *Let K be a number field with r real places and s complex places. Then*

$$[K : \mathbb{Q}] = r + 2s.$$

Proof. Recall that $[K:\mathbb{Q}] = \#\operatorname{Hom}_{\mathbb{Q}}(K,\mathbb{C})$ (write $K = \mathbb{Q}[x]/(f(x))$ and note that the elements of $\operatorname{Hom}_{\mathbb{Q}}(K,\mathbb{C})$ are determined by choosing a root of f in $\mathbb{C}$ to be the image of x). The action of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ on $\operatorname{Hom}_{\mathbb{Q}}(K,\mathbb{C})$ has r orbits of size 1, and s orbits of size 2. ☐

Example 13.16. Let $K = \mathbb{Q}[x]/(x^3-2)$. There are three embeddings $K \hookrightarrow \mathbb{C}$, one for each root of x^3-2; explicitly:

$$(1)\ x \mapsto \sqrt[3]{2}, \qquad (2)\ x \mapsto e^{2\pi i/3}\cdot\sqrt[3]{2}, \qquad (3)\ x \mapsto e^{4\pi i/3}\cdot\sqrt[3]{2}.$$

The first embedding is real, while the second two are complex and conjugate to each other. Thus K has $r=1$ real place and $s=1$ complex place, and we have $[K:\mathbb{Q}] = 1\cdot 1 + 2\cdot 1 = 3$.

We conclude this section with a result originally due to Brill [2], which relates the parity of the number of complex places to the sign of the absolute discriminant of a number field.

Proposition 13.17. *Let K be a number field with s complex places, and let $\alpha_1,\ldots,\alpha_n$ be a $\mathbb{Z}$-basis for $\mathcal{O}_K$. The sign of $D_K \coloneqq \operatorname{disc}(\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}$ is $(-1)^s$.*

Proof. Let $\operatorname{Hom}_{\mathbb{Q}}(K,\mathbb{C}) = \{\sigma_1,\ldots,\sigma_n\}$ and consider the matrix $A \coloneqq [\sigma_i(a_j)]_{ij}$ with determinant $\det A =: x + yi \in \mathbb{C}$; recall that $D_K = (\det A)^2$, by Proposition 12.6. Each real embedding σ_i corresponds to a row of A fixed by complex conjugation, while each pair of complex conjugate embeddings $\sigma_i, \overline{\sigma_i}$ corresponds to a pair of rows of A that are interchanged by complex conjugation. Swapping two rows negates the determinant, thus

$$x + yi = \det A = (-1)^s \det \bar{A} = (-1)^s(x - yi).$$

Either $(-1)^s = 1$, in which case $y = 0$ and $D_K = x^2$ has sign $+1 = (-1)^s$, or $(-1)^s = -1$, in which case $x = 0$ and $D_K = -y^2$ has sign $-1 = (-1)^s$. ☐

13.3 The product formula for global fields

Definition 13.18. Let K be a global field. For each place v of K the *normalized absolute value* $\|\ \|_v \colon K_v \to \mathbb{R}_{\geq 0}$ on the completion of K at v is defined by

$$\|x\|_v \coloneqq \frac{\mu(xS)}{\mu(S)},$$

where μ is a Haar measure on K_v and S is any measurable set with $\mu(S) \neq 0$ (we can always take $S = A_v \coloneqq \{x \in K_v : |x|_v \leq 1\}$ of K_v).

This definition is independent of the choice of μ and S (by Theorem 13.3). If v is nonarchimedean then the normalized absolute value $\|\ \|_v$ is precisely the absolute value $|\ |_v$ defined in Proposition 13.5. If v is a real place then the normalized absolute value $\|\ \|_v$ is just the usual Euclidean absolute value $|\ |_{\mathbb{R}}$ on $\mathbb{R}$, since for the Euclidean Haar measure $\mu_{\mathbb{R}}$ on $\mathbb{R}$ we have $\mu_{\mathbb{R}}(xS) = |x|_{\mathbb{R}}\mu_{\mathbb{R}}(S)$ for every measurable set S. But when v is a complex place the normalized absolute value $\|\ \|_v$ is the **square** of the Euclidean absolute value $|\ |_{\mathbb{C}}$ on $\mathbb{C}$, since in $\mathbb{C}$ we have $\mu_{\mathbb{C}}(xS) = |x|_{\mathbb{C}}^2\mu_{\mathbb{C}}(S)$.

Remark 13.19. When v is a complex place the normalized absolute value $\|\ \|_v$ is **not** an absolute value, because it does not satisfy the triangle inequality. For example, if $K = \mathbb{Q}(i)$ and $v|\infty$ is the complex place of K then $\|1\|_v = |1|_{\mathbb{C}}^2 = 1$ but

$$\|1+1\|_v = \|2\|_v = |2|_{\mathbb{C}}^2 = 4 > 2 = \|1\|_v + \|1\|_v.$$

Nevertheless, the normalized absolute value $\| \ \|_v$ is always multiplicative and compatible with the topology on K_v in the sense that the open balls $B_{<r}(x) := \{y \in K_v : \|y - x\|_v < r\}$ are a basis for the topology on K_v; these are the properties that we care about for the product formula (and for the topology on the ring of adéles $\mathbb{A}_K$ that we will see later).

Lemma 13.20. *Let L/K be a finite separable extension of global fields, let v be a place of K and let $w|v$ be a place of L. Then*

$$\|x\|_w = \|\mathrm{N}_{L_w/K_v}(x)\|_v.$$

Proof. The lemma is trivially true if $[L_w : K_v] = 1$ so assume $[L_w : K_v] > 1$. If v is archimedean then $L_w \simeq \mathbb{C}$ and $K_v \simeq \mathbb{R}$, in which case for any $x \in L_w$ we have

$$\|x\|_w = \mu(xS)/\mu(S) = |x|_{\mathbb{C}}^2 = |x\overline{x}|_{\mathbb{R}} = |\mathrm{N}_{\mathbb{C}/\mathbb{R}}(x)|_{\mathbb{R}} = \|\mathrm{N}_{L_w/K_v}(x)\|_v,$$

where $| \ |_{\mathbb{R}}$ and $| \ |_{\mathbb{C}}$ are the Euclidean absolute values on $\mathbb{R}$ and $\mathbb{C}$.

We now assume v is nonarchimedean. Let π_w and π_v be uniformizers for the local fields K_w and L_v, respectively, and let f be the degree of the corresponding residue field extension k_w/k_v. Without loss of generality, we may assume $x = \pi_w^{w(x)}$, since $\|x\|_v = |x|_v$ depends only on $w(x)$. Theorem 6.9 and Proposition 13.5 imply

$$\|\mathrm{N}_{L_w/K_v}(\pi_w)\|_v = \|\pi_v^f\|_v = (\#k_v)^{-f},$$

so $\|\mathrm{N}_{L_w/K_v}(x)\|_v = (\#k_v)^{-fw(x)}$. Proposition 13.5 then implies

$$\|x\|_w = (\#k_w)^{-w(x)} = (\#k_v)^{-fw(x)} = \|\mathrm{N}_{L_w/K_v}(x)\|_v. \qquad \square$$

Remark 13.21. Note that if v is a nonarchimedean place of K extended by a place $w|v$ of L/K, the absolute value $\| \ \|_w$ is **not** the unique absolute value on L_w that extends the absolute value on $\| \ \|_v$ on K_v given by Theorem 10.7, it differs by a power of $n = [L_w : K_v]$, but it is equivalent to it. It might seem strange to use a normalization here that does not agree with the one we used when considering extensions of local fields in Lecture 9. The difference is that here we are thinking about a single global field K that has many different completions K_v, and we want the normalized absolute values on the various K_v to be compatible (so that the product formula will hold). By contrast, in Lecture 9 we considered various extensions L_w of a single local field K_v and wanted to normalize the absolute values on the L_w compatibly so that we could work in K_v and any of its extensions (all the way up to $\overline{K}_v$) using the same absolute value. These two objectives cannot be met simultaneously and it is better to use the "right" normalization in each setting.

Theorem 13.22 (Product Formula)**.** *Let L be a global field. For all $x \in L^\times$ we have*

$$\prod_{v \in M_L} \|x\|_v = 1,$$

where $\| \ \|_v$ denotes the normalized absolute value for each place $v \in M_L$.

Proof. The global field L is a finite separable extension of $K = \mathbb{Q}$ or $K = \mathbb{F}_q(t)$.[3] Let p be a place of K. By Theorem 13.12, any basis for L as a K-vector space is also a basis for

$$L \otimes_K K_p \simeq \prod_{v|p} L_v$$

[3]Here we are using the fact that if $\mathbb{F}_q$ is the field of constants of L (the largest finite field in L), then L is a finite extension of $\mathbb{F}_q(z)$ and we can choose some $t \in \mathbb{F}_q(z) - \mathbb{F}_q$ so that $\mathbb{F}_q(z) \simeq \mathbb{F}_q(t)$ and $L/\mathbb{F}_q(t)$ is separable (such a t is called a *separating element*).

as a K_v-vector space. Thus

$$\mathrm{N}_{L/K}(x) = \mathrm{N}_{(L\otimes_K K_p)/K_p}(x) = \prod_{v|p} \mathrm{N}_{L_v/K_p}(x).$$

Taking normalized absolute values on both sides yields

$$\left\|\mathrm{N}_{L/K}(x)\right\|_p = \prod_{v|p} \|\mathrm{N}_{L_v/K_p}(x)\|_p = \prod_{v|p} \|x\|_v.$$

We now take the product of both sides over all places $p \in M_K$ to obtain

$$\prod_{p\in M_K} \|\mathrm{N}_{L/K}(x)\|_p = \prod_{p\in M_K} \prod_{v|p} \|x\|_v = \prod_{v\in M_L} \|x\|_v.$$

The LHS is equal to 1, by the product formula for K proved on Problem Set 1. □

With the product formula in hand, we can now give an axiomatic definition of a global field, which up to now we have simply defined as a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$, due to Emil Artin and George Whaples [1].

Definition 13.23. A *global field* is a field K whose completion at each of its places $v \in M_K$ is a local field, and which has a product formula of the form

$$\prod_{v\in M_K} \|x\|_v = 1,$$

where each normalized absolute value $\| \ \|_v \colon K_v \to \mathbb{R}_{\ge 0}$ satisfies $\| \ \|_v = | \ |_v^{m_v}$ for some absolute value $| \ |_v$ representing v and some fixed $m_v \in \mathbb{R}_{>0}$.

Theorem 13.24 (Artin-Whaples)**.** *Every global field is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$.*

Proof. See Problem Set 7. □

References

[1] Emil Artin and George Whaples, *Axiomatic characterization of fields by the product formula for valuations*, Bull. Amer. Math. Soc. **51** (1945), 469–492.

[2] Alexander von Brill, *Ueber die Discriminante*, Math. Ann. **12** (1877), 87–89.

[3] Joe Diestel and Angela Spalsbury, *The Joys of Haar Measure*, American Mathematical Society, 2014.