22 The ring of adeles, strong approximation

22.1 Introduction to adelic rings

Recall that we have a canonical injection

$$\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z} \simeq \prod_p \mathbb{Z}_p,$$

that embeds $\mathbb{Z}$ into the product of its nonarchimedean completions. Each of the rings $\mathbb{Z}_p$ is compact, hence $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is compact (by Tychonoff's theorem). But notice that if we consider the analogous product $\prod_p \mathbb{Q}_p$ of the completions of $\mathbb{Q}$, each of the local fields $\mathbb{Q}_p$ is locally compact (including the archimedean field $\mathbb{Q}_\infty = \mathbb{R}$), but the product $\prod_p \mathbb{Q}_p$ is **not locally compact**. Local compactness is important to us, because it gives us a Haar measure (recall that any locally compact group has a translation-invariant measure that is unique up to scaling), a tool we would very much like to have at our disposal.

To see where the problem arises, recall that for any family of topological spaces $(X_i)_{i\in I}$ (here the index set I may be any set), the product topology on the set $X := \prod X_i$ is, by definition, the weakest topology that makes the projection maps $\pi_i\colon X \to X_i$ continuous; this means it is generated by open sets of the form $\pi_i^{-1}(U_i)$ with $U_i \subseteq X_i$ open, and therefore every open set in X is a (possibly empty) union of open sets of the form

$$\prod_{i\in S} U_i \times \prod_{i\in I-S} X_i,$$

with $S \subseteq I$ finite and each $U_i \subseteq X_i$ open (these sets form a *basis* for the topology on X). In particular, every open set $U \subseteq X$ will have $\pi_i(U) = X_i$ for all but finitely many $i \in I$, so unless all but finitely many of the X_i are compact, the space X cannot possibly be locally compact for the simple reason that no compact set C in X contains a nonempty open set (if it did then we would have $\pi_i(C) = X_i$ compact for all but finitely many $i \in I$). Recall that for X to be locally compact means that every $x \in X$ we have $x \in U \subseteq C$ for some open set U and compact set C (so C is a compact neighborhood of x).

To solve this problem we want to take the product of the fields $\mathbb{Q}_p$ (or more generally, the completions of any global field) in a different way that yields a locally compact topological ring. This leads us to the *restricted product* which is a purely topological construction, but one that was invented essentially for the purpose of solving this number-theoretic problem.

22.2 Restricted products

This section is purely about the topology of restricted products; readers familiar with restricted products should feel free to skip to the next section.

Definition 22.1. Let (X_i) be a family of topological spaces indexed by $i \in I$, and let (U_i) be a family of open sets $U_i \subseteq X_i$. The *restricted product* $\prod(X_i, U_i)$ is the topological space

$$\prod(X_i, U_i) := \left\{(x_i) \in \prod X_i : x_i \in U_i \text{ for almost all } i \in I\right\}$$

with the basis of open sets

$$\mathcal{B} := \left\{\prod V_i : V_i \subseteq X_i \text{ is open for all } i \in I \text{ and } V_i = U_i \text{ for almost all } i \in I\right\},$$

where *almost all* means all but finitely many.

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For each $i \in I$ we have a projection map $\pi_i \colon \prod(X_i, U_i) \to X_i$ defined by $(x_i) \mapsto x_i$; each π_i is continuous, since if U_i is an open subset of X_i, then $\pi_i^{-1}(U_i)$ is the union of all $V = \prod V_i \in \mathcal{B}$ with $V_i = U_i$, which is open.

As sets, we always have

$$\prod U_i \subseteq \prod(X_i, U_i) \subseteq \prod X_i,$$

but in general the restricted product topology on $\prod(X_i, U_i)$ is not the same as the subspace topology it inherits as a subset of $\prod X_i$; it has more open sets. For example, $\prod U_i$ is open in $\prod(X_i, U_i)$ but not in $\prod X_i$, unless $U_i = X_i$ for almost all i, in which case $\prod(X_i, U_i) = \prod X_i$ (both as sets and as topological spaces). Thus the restricted product is a generalization of the direct product and the two coincide if and only if $U_i = X_i$ for almost all i; note that this is automatically true when I is finite, so only infinite restricted products are interesting.

Remark 22.2. The restricted product does not depend on any particular U_i. Indeed,

$$\prod(X_i, U_i) = \prod(X_i, U_i')$$

whenever $U_i' = U_i$ for almost all i; note that the two restricted products are not merely isomorphic, they are identical, both as sets and as topological spaces. It is thus enough to specify the U_i for all but finitely many $i \in I$.

Each $x \in X := \prod(X_i, U_i)$ distinguishes a finite subset $S = S(x) \subseteq I$, namely, the set of indices i for which $x_i \notin U_i$ (this may be the empty set). It is thus natural to consider

$$X_S := \{x \in X : S(x) = S\} = \prod_{i \in S} X_i \times \prod_{i \notin S} U_i.$$

Notice that $X_S \in \mathcal{B}$ is an open set, and we can view it as a topological space in two ways: as a subspace of X or as a direct product of certain X_i and U_i. But notice that restricting the basis $\mathcal{B}$ for X to a basis for the subspace X_S yields

$$\mathcal{B}_S := \left\{\prod V_i : V_i \subseteq \pi_i(X_S) \text{ is open and } V_i = U_i = \pi_i(X_S) \text{ for almost all } i \in I\right\},$$

which is just the standard basis for the product topology on X_S, so the two coincide.

We have $X_S \subseteq X_T$ if and only if $S \subseteq T$, thus if we partially order the finite subsets $S \subseteq I$ by inclusion, the X_S and the inclusion maps $i_{ST} \colon X_S \hookrightarrow X_T$ form a *direct system*, and we can consider the corresponding *direct limit*

$$\varinjlim_S X_S,$$

which is the quotient of the coproduct space[1] $\coprod X_S$ by the equivalence relation $x \sim i_{ST}(x)$ for all $x \in S \subseteq T$. This direct limit is canonically isomorphic to the restricted product X, which gives us another way to define the restricted product; before proving this list us recall the general definition of a direct limit of topological spaces.

Definition 22.3. A *direct system* (or *inductive system*) in a category is a family of objects $\{X_i : i \in I\}$ indexed by a directed set I (see Definition 8.15) and a family of morphisms $\{f_{ij} \colon X_i \to X_j : i \leq j\}$ such that each f_{ii} is the identity and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

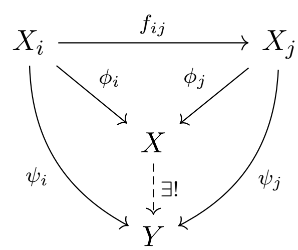
[1]The topology on the coproduct $\coprod X_S$ is the weakest topology that makes all the injections $X_S \hookrightarrow \coprod X_S$ continuous; its open sets are simply unions of open sets the X_S.

Definition 22.4. Let (X_i, f_{ij}) be a direct system of topological spaces. The *direct limit* (or *inductive limit*) of (X_i, f_{ij}) is the quotient space

$$X = \varinjlim X_i := \coprod_{i\in I} X_i / \sim,$$

where $x_i \sim f_{ij}(x_i)$ for all $i \le j$. The pullbacks $\phi_i \colon X_i \to X$ of the quotient map $\coprod X_i \to X$ satisfy $\phi_i = \phi_j \circ f_{ij}$ for $i \le j$.

The topological space $X = \varinjlim X_i$ has the universal property that if Y is another topological space with continuous maps $\psi_i \colon X_i \to Y$ that satisfy $\psi_i = \psi_j \circ f_{ij}$ for $i \le j$, then there is a unique continuous map $X \to Y$ for which all of the diagrams



commute (this universal property defines the direct limit in any category with coproducts).

We now prove that that $\prod(X_i, U_i) \simeq \varinjlim X_S$ as claimed above.

Proposition 22.5. *Let (X_i) be a family of topological spaces indexed by $i \in I$, let (U_i) be a family of open sets $U_i \subseteq X_i$, and let $X := \prod(X_i, U_i)$ be the corresponding restricted product. For each finite $S \subseteq I$ define*

$$X_S := \prod_{i\in S} X_i \times \prod_{i\notin S} U_i \subseteq X,$$

and inclusion maps $i_{ST} \colon X_S \hookrightarrow X_T$, and let $\varinjlim X_S$ be the corresponding direct limit.

There is a canonical homeomorphism of topological spaces

$$\varphi \colon X \xrightarrow{\sim} \varinjlim X_S$$

that sends $x \in X$ to the equivalence class of $x \in X_{S(x)} \subseteq \coprod X_S$ in $\varinjlim X_S := \coprod X_S / \sim$, where $S(x) := \{i \in I : x_i \notin U_i\}$.

Proof. To prove that the map $\varphi \colon X \to \varinjlim X_S$ is a homeomorphism, we need to show that it is (1) a bijection, (2) continuous, and (3) an open map.

(1) For each equivalence class $\mathcal{C} \in \varinjlim X_S := \coprod X_S / \sim$, let $S(\mathcal{C})$ be the intersection of all the sets S for which $\mathcal{C}$ contains an element of $\coprod X_S$ in X_S. Then $S(x) = S(\mathcal{C})$ for all $x \in \mathcal{C}$, and $\mathcal{C}$ contains a unique element for which $x \in X_{S(x)} \subseteq \coprod X_S$. Thus φ is a bijection.

(2) Let U be an open set in $\varinjlim X_S = \coprod X_S / \sim$. The inverse image V of U in $\coprod X_S$ is open, as are the inverse images V_S of V under the canonical injections $\iota \colon X_S \hookrightarrow \coprod X_S$. The union of the V_S in X is equal to $\varphi^{-1}(U)$ and is an open set in X; thus φ is continuous.

(3) Let U be an open set in X. Since the X_S form an open cover of X, we can cover U with open sets $U_S = U \cap X_S$, and then $\coprod U_S$ is an open set in $\coprod X_S$. Moreover, for each $x \in \coprod U_S$, if $y \sim x$ for some $y \in \coprod X_S$ then y and x must correspond to the same element in U; in particular, $y \in \coprod U_S$, so $\coprod U_S$ is a union of equivalence classes in $\coprod X_S$. It follows that its image in $\varinjlim X_S = \coprod X_S / \sim$ is open. $\square$

Proposition 22.5 gives us another way to construct the restricted product $\prod(X_i, U_i)$: rather than defining it as a subset of $\prod X_i$ with a modified topology, we can instead construct it as a limit of direct products that are subspaces of $\prod X_i$.

We now specialize to the case of interest, where we are forming a restricted product using a family $(X_i)_{i\in I}$ of locally compact spaces and a family of open subsets (U_i) that are almost all compact. Under these conditions the restricted product $\prod(X_i, U_i)$ is locally compact, even though the product $\prod X_i$ is not unless the index set I is finite.

Proposition 22.6. *Let $(X_i)_{i\in I}$ be a family of locally compact topological spaces and let $(U_i)_{i\in I}$ be a corresponding family of open subsets $U_i \subseteq X_i$ almost all of which are compact. Then the restricted product $X := \prod(X_i, U_i)$ is locally compact.*

Proof. We first note that for each finite set $S \subseteq I$ the topological space

$$X_S := \prod_{i\in S} X_i \times \prod_{i\notin S} U_i$$

can be viewed as a finite product of locally compact spaces, since all but finitely many of the U_i are compact and the product of these is compact (by Tychonoff's theorem), hence locally compact. A finite product of locally compact spaces is always locally compact, since we can construct compact neighborhoods as products of compact neighborhoods in each factor (the key point is that in a finite product, products of open sets are open); thus the X_S are all locally compact, and the X_S cover X (since each $x \in X$ lies in $X_{S(x)}$). It follows that X is locally compact, since each $x \in X_S$ has a compact neighborhood $x \in U \subseteq C \subseteq X_S$ that is also a compact neighborhood in X (every open cover of C in X restricts to an open cover of C in X_S that must have a finite subcover, so C is compact in X, and U is open in X because X_S is open). $\square$

22.3 The ring of adeles

Recall that for a global field K (finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$), we use M_K to denote the set of places of K (equivalence classes of absolute values), and for any $v \in M_K$ we use K_v to denote the corresponding local field (the completion of K with respect to v), and define $\mathcal{O}_v := K_v$ when v is nonarchimedean.[2]

Definition 22.7. Let K be a global field. The *adele ring*[3] of K is the restricted product

$$\mathbb{A}_K := \prod(K_v, \mathcal{O}_v)_{v\in M_K},$$

which we may view as a subset (but not a subspace!) of $\prod_v K_v$; indeed

$$\mathbb{A}_K = \left\{(a_v) \in \prod K_v : a_v \in \mathcal{O}_v \text{ for almost all } v\right\},$$

and for each $a \in \mathbb{A}_K$ we use a_v to denote its projection in K_v; we make $\mathbb{A}_K$ a ring by defining addition and multiplication component-wise (closure is clear).

For each finite set of places S we have the subring of *S-adeles*

$$\mathbb{A}_{K,S} := \prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v,$$

[2]Per Remark 22.2, as far as the topology goes it doesn't matter how we define $\mathcal{O}_v$ at the archimedean places, but we would like every $\mathcal{O}_v$ to be a topological ring, which motivates this choice.

[3]In French one writes *adèle*, but it is common practice to omit the accent when writing in English.

which is a direct product of topological rings. By Proposition 22.5, $\mathbb{A}_K \simeq \varinjlim \mathbb{A}_{K,S}$ is the direct limit of the S-adele rings, which makes it clear that $\mathbb{A}_K$ is also a topological ring.

The canonical embeddings $K \hookrightarrow K_v$ induce the canonical embedding

$$\begin{aligned} K &\hookrightarrow \mathbb{A}_K \\ x &\mapsto (x, x, x, \ldots) \end{aligned}$$

since for each $x \in K$ we have $x \in \mathcal{O}_v$ for all but finitely many v. The image of K in $\mathbb{A}_K$ forms the subring of *principal adeles* (which of course is also a field).

We extend the normalized absolute value $\| \ \|_v$ of K_v (see Definition 12.28) to $\mathbb{A}_K$ via

$$\|a\|_v := \|a_v\|_v,$$

and define the *adelic absolute value* (or *adelic norm*)

$$\|a\| := \prod_{v \in M_K} \|a\|_v \in \mathbb{R}_{\geq 0}$$

which we note converges because $\|a\|_v \leq 1$ for almost all v. For $\|a\| \neq 0$ this is equal to the size of the M_K*-divisor* $(\|a\|_v)$ we defined in Lecture 14 (see Definition 14.1). For any nonzero principal adele a we necessarily have $\|a\| = 1$, by the product formula (Theorem 12.32).

Example 22.8. For $K = \mathbb{Q}$ the adele ring $\mathbb{A}_{\mathbb{Q}}$ is the union of the rings

$$\mathbb{A}_{\mathbb{Q},S} = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p.$$

Taking $S = \emptyset$ yields the ring $\mathbb{A}_{\mathbb{Z}} := \mathbb{R} \times \prod_{p<\infty} \mathbb{Z}_p \simeq \mathbb{R} \times \hat{\mathbb{Z}}$ of *integral adeles*. We can also write $\mathbb{A}_{\mathbb{Q}}$ as

$$\mathbb{A}_{\mathbb{Q}} = \left\{ a \in \prod_{p \leq \infty} \mathbb{Q}_p : \|a\|_p \leq 1 \text{ for almost all } p \right\}.$$

Proposition 22.9. *The adele ring $\mathbb{A}_K$ of a global field K is locally compact and Hausdorff.*

Proof. Local compactness follows from Proposition 22.6, since the local fields K_v are all locally compact and all but finitely many $\mathcal{O}_v$ are valuation rings of a nonarchimedean local field, hence compact ($\mathcal{O}_v = \{x \in K_v : \|x\|_v \leq 1\}$ is a closed ball in a metric space). If $x, y \in \mathbb{A}_K$ are distinct then $x_v \neq y_v$ for some $v \in M_K$, and since K_v is Hausdorff we can separate x_v and y_v by open sets whose inverse images under the projection map $\pi_v \colon \mathbb{A}_K \to K_v$ are open sets separating x and y; thus $\mathbb{A}_K$ is Hausdorff. □

Proposition 22.9 implies that the additive group of $\mathbb{A}_K$ (which is sometimes denoted $\mathbb{A}_K^+$ to emphasize that we are viewing it as a group rather than a ring) is a locally compact group, and therefore has a Haar measure that is unique up to scaling. Each of the completions K_v is a local field with a Haar measure μ_v that we normalize as follows:

- $\mu_v(\mathcal{O}_v) = 1$ for all nonarchimedean v;
- $\mu_v(S) = \mu_{\mathbb{R}}(S)$ for $K_v \simeq \mathbb{R}$, where $\mu_{\mathbb{R}}(S)$ is the standard Euclidean measure on $\mathbb{R}$;
- $\mu_v(S) = 2\mu_{\mathbb{C}}(S)$ for $K_v \simeq \mathbb{C}$, where $\mu_{\mathbb{C}}(S)$ is the standard Euclidean measure on $\mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$.

Note that the normalization of μ_v at the archimedean places is consistent with the canonical measure μ on $K_\mathbb{R} \simeq \mathbb{R}^r \times \mathbb{C}^s \simeq \mathbb{R}^n$ induced by the canonical inner product on $K_\mathbb{R} \subseteq K_\mathbb{C}$ that we defined in Lecture 13 (see §13.2).

We now define a measure μ on $\mathbb{A}_K$ as follows. We take as a basis for the σ-algebra of measurable sets all sets of the form $\prod_v B_v$ with $\mu_v(B_v) < \infty$ for all $v \in M_K$ and $B_v = \mathcal{O}_v$ for almost all v. We then define

$$\mu\left(\prod_v B_v\right) := \prod_v \mu_v(B_v).$$

It is easy to verify that μ is a Radon measure, and it is clearly translation invariant since each of the Haar measures μ_v is translation invariant and addition is defined componentwise; note that for any $x \in \mathbb{A}_K$ and measurable set $B = \prod_v B_v$ the set $x+B = \prod_v (x_v + B_v)$ is also measurable, since $x_v + B_v = \mathcal{O}_v$ whenever $x_v \in \mathcal{O}_v$ and $B_v = \mathcal{O}_v$, and this applies to almost all v. It follows from uniqueness of the Haar measure (up to scaling) that μ is a Haar measure on $\mathbb{A}_K$ which we henceforth adopt as our normalized Haar measure on $\mathbb{A}_K$.

We now want to understand the behavior of the adele ring $\mathbb{A}_K$ under base change. Note that the canonical embedding $K \hookrightarrow \mathbb{A}_K$ makes $\mathbb{A}_K$ a K-vector space, and if L/K is any finite separable extension of K (also a K-vector space), we may consider the tensor product

$$\mathbb{A}_K \otimes L,$$

which is also an L-vector space. As a topological K-vector space, the topology on $\mathbb{A}_K \otimes L$ is just the product topology on $[L:K]$ copies of of $\mathbb{A}_K$ (this applies whenever we take a tensor product of topological vector spaces, one of which has finite dimension).

Proposition 22.10. *Let L be a finite separable extension of a global field K. There is a canonical isomorphism of topological rings*

$$\mathbb{A}_K \otimes_K L \simeq \mathbb{A}_L$$

in which the canonical embeddings of $L \simeq K \otimes_K L$ into $\mathbb{A}_K \otimes_K L$ and L into $\mathbb{A}_L$ agree.

Proof. The LHS $\mathbb{A}_K \otimes_K L$ is isomorphic to the restricted product

$$\prod\nolimits_v (K_v \otimes_K L, \mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L).$$

Explicitly, each element of $\mathbb{A}_K \otimes_K L$ is a finite sum of elements of the form $(a_v) \otimes x$, where $(a_v) \in \mathbb{A}_K$ and $x \in L$, and there is a natural isomorphism

$$\begin{aligned} \mathbb{A}_K \otimes_K L &\xrightarrow{\sim} \prod\nolimits_{v \in M_K} (K_v \otimes_K L, \mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L) \\ (a_v) \otimes x &\mapsto (a_v \otimes x) \end{aligned}$$

that is both a ring isomorphism and a homeomorphism of topological spaces.

On the RHS we have $\mathbb{A}_L := \prod_{w \in M_L} (L_w, \mathcal{O}_w)$. But note that $K_v \otimes_K L \simeq \prod_{w|v} L_w$, by Theorem 11.4 and $\mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L \simeq \prod_{w|v} \mathcal{O}_w$, by Corollary 11.7. These isomorphisms preserve both the algebraic and the topological structures of both sides, and it follows that

$$\mathbb{A}_K \otimes_K L \simeq \prod\nolimits_{v \in M_K} (K_v \otimes_K L, \mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L) \simeq \prod\nolimits_{w \in M_L} (L_w, \mathcal{O}_w) = \mathbb{A}_L$$

is an isomorphism of topological rings. The image of $x \in L$ in $\mathbb{A}_K \otimes_K L$ via the canonical embedding of L into $\mathbb{A}_K \otimes_K L$ is $1 \otimes x = (1,1,1,\ldots) \otimes x$, whose image $(x,x,x,\ldots) \in \mathbb{A}_L$ is equal to the image of $x \in L$ under the canonical embedding of L into its adele ring $\mathbb{A}_L$. □

Corollary 22.11. *Let L be a finite separable extension of a global field K of degree n. There is a canonical isomorphism of topological K-vector spaces (and locally compact groups)*

$$\mathbb{A}_L \simeq \mathbb{A}_K \oplus \cdots \oplus \mathbb{A}_K$$

that identifies $\mathbb{A}_K$ with the direct sum of n copies of $\mathbb{A}_K$, and this isomorphism restricts to an isomorphism $L \simeq K \oplus \cdots \oplus K$ of the principal adeles of $\mathbb{A}_L$ with the n-fold direct sum of the principal adeles of $\mathbb{A}_K$.

Theorem 22.12. *Let L be a global field. The principal adeles $L \hookrightarrow \mathbb{A}_L$ are a discrete subgroup of the additive group of $\mathbb{A}_L$ and the quotient $\mathbb{A}_L/L$ of topological groups is compact.*

Proof. Let K be the rational subfield of L (so $K = \mathbb{Q}$ or $K = \mathbb{F}_q(t)$). It follows from the previous corollary, that if the theorem holds for K then it holds for L, so we will prove the theorem for K. Let us identify $K \hookrightarrow \mathbb{A}_K$ with its image in $\mathbb{A}_K$ (the principal adeles).

To show that the topological group K is discrete in $\mathbb{A}_K$, it suffices to show that 0 is an isolated point. Consider the open set

$$U = \{a \in \mathbb{A}_K : \|a\|_\infty < 1 \text{ and } \|a\|_v \le 1 \text{ for all } v < \infty\},$$

where the place $v = \infty$ is the archimedean place when $K = \mathbb{Q}$ and the nonarchimedean place corresponding to the degree valuation $v_\infty(f/g) = \deg f - \deg g$ when $K = \mathbb{F}_q(t)$). The product formula (Theorem 12.32) implies $\|a\| = 1$ for all nonzero principal adeles $a \in \mathbb{A}_K$, so the only principal adele in U is 0. Thus K is a discrete subgroup of $\mathbb{A}_K$.

To prove that the quotient $\mathbb{A}_K/K$ is compact, we consider the set

$$W := \{a \in \mathbb{A}_K : \|a\|_v \le 1 \text{ for all } v\}.$$

Let $S = \{\infty\} \subseteq M_K$ and put $U_\infty = \{x \in K_\infty : \|x\|_\infty \le 1\}$. Then

$$W = U_\infty \times \prod_{v \notin S} \mathcal{O}_v \subseteq \mathbb{A}_{K,S}$$

is a product of compact sets and therefore compact as a subspace of $\mathbb{A}_{K,S} \subseteq \mathbb{A}_K$.

We now show that W contains a complete set of coset representatives for K in $\mathbb{A}_K$. Let $a = (a_v)$ be any element of $\mathbb{A}_K$. We claim $a = b + c$ for some $b \in W$ and $c \in K$.

For $v < \infty$, let $x_v = 0$ if $\|a_v\| \le 1$ (true for almost all v), and otherwise choose $x_v \in K$ so that $\|a_v - x_v\|_v \le 1$ and $\|x_v\|_w \le 1$ for $w \ne v$; such a x_v exists by the "pretty strong" approximation theorem (Theorem 3.29). Now let $c' = \sum_{v<\infty} x_v \in K \subseteq \mathbb{A}_K$ (this is a finite sum because almost all the x_v are zero), and choose $x_\infty \in \mathcal{O}_K$ so that

$$\|a_\infty - c'_\infty - x_\infty\|_\infty \le 1.$$

When $K = \mathbb{Q}$ we can take $x_\infty \in \mathbb{Z}$ to be the nearest integer to the rational number $a_\infty - c'_\infty$. When $K = \mathbb{F}_q(t)$, if $a_\infty - c'_\infty = f/g$ with $f, g \in \mathbb{F}_q[t]$ relatively prime, we can write $f = hg + f'$ for some $h, f' \in \mathbb{F}_q[t]$ with $\deg f' < \deg g$ and let $x_\infty = -h$.

Now let $c := \sum_{v\le\infty} x_v \in K \subseteq \mathbb{A}_K$, and let $b := a - c$. Then $a = b + c$, with $c \in K$, and we claim that $b \in W$. For each $v < \infty$ we have $x_w \in \mathcal{O}_v$ for all $w \ne v$ and

$$\|b\|_v = \|a - c\|_v = \left\| a_v - \sum_{w\le\infty} x_w \right\|_v \le \max\left(\|a_v - x_v\|_v, \max(\{\|x_w\|_v : w \ne v\})\right) \le 1,$$

by the nonarchimedean triangle inequality. For $v = \infty$ we have $\|b\|_\infty = \|a_\infty - c_\infty\| = \|a_\infty - c'_\infty - x_\infty\| \le 1$ by our choice of x_∞, and therefore $b \in W$ as claimed.

Thus W surjects onto A_K/K under the quotient map $\mathbb{A}_K \twoheadrightarrow \mathbb{A}_K/K$. The quotient map is continuous, so the image $\mathbb{A}_K/K$ of the compact set W must be compact. □

22.4 Strong approximation

We are now ready to prove the strong approximation theorem (Theorem 3.27) that we recorded back in Lecture 3 but have so far not used.[4] In order to prove it we first prove an adelic version of the Blichfeldt-Minkowski lemma.

Lemma 22.13 (Blichfeldt-Minkowski lemma)**.** *Let K be a global field. There is a positive constant C such that for any $x \in \mathbb{A}_K$ with $\|x\| > C$ there exists a nonzero principal adele $y \in K \subseteq \mathbb{A}_K$ for which $\|y\|_v \le \|x\|_v$ for all $v \in M_K$.*

Proof. Let $c_0 := \operatorname{covol}(K)$ be the measure of a fundamental region for K in $\mathbb{A}_K$ under our normalized Haar measure μ on $\mathbb{A}_K$ (by Theorem 22.12, K is cocompact so c_0 is finite). Now define

$$c_1 := \mu\left(\left\{z \in \mathbb{A}_K : \|z\|_v \le 1 \text{ and } \|z\|_v \le \tfrac{1}{4} \text{ if } v \text{ is archimedean}\right\}\right).$$

Then $c_1 \ne 0$, since K has only finitely many archimedean places, and we put $C := c_0/c_1$.[5]

Suppose $x \in \mathbb{A}_K$ satisfies $\|x\| > C$. We know that $\|x\|_v \le 1$ for all almost all v, so $\|x\| > C$ implies that $\|x\|_v = 1$ for almost all v. Let us now consider the set

$$T := \left\{t \in \mathbb{A}_K : \|t\|_v \le \|x\|_v \text{ and } \|t\|_v \le \tfrac{1}{4}\|x\|_v \text{ if } v \text{ is archimedean}\right\}.$$

From the definition of c_1 we have

$$\mu(T) = c_1\|x\| > c_1 C = c_0;$$

this follows from the fact that the Haar measure on $\mathbb{A}_K$ is the product of the normalized Haar measures μ_v on each of the K_v. Since $\mu(T) > c_0$, the set T cannot lie in a fundamental region for K, so there must be distinct $t_1, t_2 \in T$ with the same image in $\mathbb{A}_K/K$, equivalently, whose difference $y = t_1 - t_2$ is a nonzero element of $K \subseteq \mathbb{A}_K$. We then have

$$\|t_1 - t_2\|_v \le \begin{cases} \max(\|t_1\|_v, \|t_2\|_v) \le \|x\|_v & \text{nonarch. } v; \\ \|t_1\|_v + \|t_2\|_v \le 2 \cdot \frac{1}{4}\|x\|_v \le \frac{1}{2}\|x\|_v & \text{real } v; \\ (\|t_1 - t_2\|_v^{1/2})^2 \le (\|t_1\|_v^{1/2} + \|t_2\|_v^{1/2})^2 \le (2 \cdot \frac{1}{2}\|x\|_v^{1/2})^2 \le \|x\|_v & \text{complex } v. \end{cases}$$

Here we have used the fact that the normalized absolute value $\| \ \|_v$ satisfies the nonarchimedean triangle inequality when v is nonarchimedean, $\| \ \|_v$ satisfies the archimedean triangle inequality when v is real, and $\| \ \|_v^{1/2}$ satisfies the archimedean triangle inequality when v is complex. Thus $\|y\|_v = \|t_1 - t_2\| \le \|x\|_v$ for all places $v \in M_K$ as desired. □

Theorem 22.14 (Strong Approximation)**.** *Let K be a global field and let $M_K = S \sqcup T \sqcup \{w\}$ be a partition of the places of K with S finite. For each $v \in S$, let a_v be an element of K and let $\epsilon_v \in \mathbb{R}_{>0}$. Then there exists an $x \in K$ for which*

$$\begin{aligned} \|x - a_v\|_v \le \epsilon_v &\text{ for all } v \in S, \\ \|x\|_v \le 1 &\text{ for all } v \in T, \end{aligned}$$

(with no constraint on $\|x\|_w$).

[4] We have made do with the pretty strong approximation theorem (Theorem 3.29).

[5] With our canonical normalization of μ we will actually get the same C for all K, but we don't need this. With a little more care one can show that in fact $C = 1$ works.

Proof. Let $W = \{z \in \mathbb{A}_K : \|z\|_v \le 1 \text{ for all } v \in M_K\}$ as in the proof of Theorem 22.12. Then W contains a complete set of coset representatives for $K \subseteq \mathbb{A}_K$, so $\mathbb{A}_K = K + W$. For any nonzero $u \in K \subseteq \mathbb{A}_K$ we also have $\mathbb{A}_K = K + uW$: given $c \in \mathbb{A}_K$ write $u^{-1}c \in \mathbb{A}_K$ as $u^{-1}c = a + b$ with $a \in K$ and $b \in W$ and then $c = ua + ub$ with $ua \in K$ and $ub \in uW$. Now choose $z \in \mathbb{A}_K$ such that

$$0 < \|z\|_v \le \epsilon_v \text{ for } v \in S, \quad 0 < \|z\|_v \le 1 \text{ for } v \in T, \quad \|z\|_w > C \prod_{v \ne w} \|z\|_v^{-1},$$

where C is the constant in Lemma 22.13 (this is clearly possible). We then have $\|z\| > C$, and by Lemma 22.13 there is a nonzero $u \in K \subseteq \mathbb{A}_K$ with $\|u\|_v \le \|z\|_v$ for all $v \in M_K$.

Now consider the adele $a = (a_v) \in \mathbb{A}_K$ with $a_v = 0$ for $v \notin S$ (for $v \in S$ the value of a_v is given by the hypothesis of the theorem). We have $\mathbb{A}_K = K + uW$, so $a = x + y$ for some $x \in K$ and $y \in uW$. Therefore

$$\|x - a_v\| = \|y\|_v \le \|u\|_v \le \|z\|_v \le \begin{cases} \epsilon_v & \text{for } v \in S, \\ 1 & \text{for } v \in T, \end{cases}$$

as desired. □

Remark 22.15. Theorem 22.14 can be generalized to algebraic groups (the global field K can be viewed as the algebraic group $\mathrm{GL}_1(K)$, an affine line); see [1] for a survey.

Corollary 22.16. *Let K be a global field and let w be any place of K. Then K is dense in the restricted product $\prod_{v \ne w}(K_v, \mathcal{O}_v)$.*

References

[1] Andrei S. Rapinchuk, *Strong approximation for algebraic groups*, Thin groups and superstrong approximation, MSRI Publications **61**, 2013.