18.783 Elliptic Curves
Lecture 4

Andrew Sutherland

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The function field of a curve

Definition

Let C/k be a plane projective curve $f(x,y,z)=0$ with $f\in k[x,y,z]$ nonconstant, homogeneous, and irreducible in $\bar{k}[x,y,z]$. The function field $k(C)$ is the set of equivalence classes of rational functions g/h such that:

- **(i)** g and h are homogeneous polynomials in $k[x,y,z]$ of the same degree;
- **(ii)** h is not divisible by f, equivalently, h is not an element of the ideal (f);
- **(iii)** g_1/h_1 and g_2/h_2 are considered equivalent whenever $g_1h_2-g_2h_1\in(f)$.

Addition: $\frac{g_1}{h_1}+\frac{g_2}{h_2}=\frac{g_1h_2+g_2h_1}{h_1h_2}$, Multiplication $\frac{g_1}{h_1}\cdot\frac{g_2}{h_2}=\frac{g_1g_2}{h_1h_2}$, Inverse: $\left(\frac{g}{h}\right)^{-1}=\frac{h}{g}$.

If $g\in(f)$ then $g/h=0$ in $k(C)$, so we don't define $(g/h)^{-1}$ in this case.

The field $k(C)$ is a transcendental extension of k (of transcendence degree 1).

😎 Pro tips: • Don't confuse $k(C)$ and $C(k)$. • Don't assume $k[x,y,z]/(f)$ is a UFD.

Evaluating functions in $k(C)$ at a point in $C(\bar{k})$

For $g/h \in k(C)$ with $\deg g = \deg h = d$ and any $\lambda \in k^\times$ we have

$$\frac{g(\lambda x, \lambda y, \lambda z)}{h(\lambda x, \lambda y, \lambda z)} = \frac{\lambda^d g(x,y,z)}{\lambda^d h(x,y,z)} = \frac{g(x,y,z)}{h(x,y,z)}$$ ✓

For any $P \in C(\bar{k})$ we have $f(P) = 0$, so if $g_1/h_1 = g_2/h_2$ with $h_1(P), h_2(P) \neq 0$, then $g_1(P)h_2(P) - g_2(P)h_1(P) = f(P) = 0$, so $(g_1/h_1)(P) = (g_2/h_2)(P)$. ✓

To evaluate $\alpha \in k(C)$ at $P \in C(\bar{k})$ we need to choose $\alpha = g/h$ with $h(P) \neq 0$.

Example

$f(x,y,z) = zy^2 - x^3 - z^2x$, $P = (0:0:1)$, $\alpha = 3xz/y^2$. We have

$$\alpha(P) = \frac{3xz}{y^2}(0:0:1) = \frac{3xz^2}{x^3 + z^2x}(0:0:1) = \frac{3z^2}{x^2+z^2}(0:0:1) = 3$$

Rational maps

Definition

We say that $\alpha \in k(C)$ is defined at $P \in C(\bar{k})$ if $\alpha = g/h$ with $h(P) \neq 0$.

Definition

Let C_1/k and C_2/k be projective plane curves. A rational map $\phi\colon C_1 \to C_2$ is a triple $(\phi_x : \phi_y : \phi_z) \in \mathbb{P}^2(k(C_1))$ such that for any $P \in C_1(\bar{k})$ where ϕ_x, ϕ_y, ϕ_z are defined and not all zero we have $(\phi_x(P) : \phi_y(P) : \phi_z(P)) \in C_2(\bar{k})$.

The rational map ϕ is defined at P if there exists $\lambda \in k(C_1)^\times$ such that $\lambda\phi_x, \lambda\phi_y, \lambda\phi_z$ are defined and not all zero at P.

Rational maps (alternative approach)

Let $C_1 : f_1(x,y,z) = 0$ and $C_2 : f_2(x,y,z) = 0$ be projective curves over k.
If $\psi_x, \psi_z, \psi_z \in k[x,y,z]$ are homogeneous of the same degree, not all in (f_1), and $f_2(\psi_x, \psi_y, \psi_z) \in (f_1)$, then at least one and possibly all of

$$(\psi_x/\psi_z : \psi_y/\psi_z : 1), \qquad (\psi_x/\psi_y : 1 : \psi_z/\psi_y), \qquad (1 : \psi_y/\psi_x : \psi_z/\psi_x)$$

is a rational map $\psi \colon C_1 \to C_2$. Call two such triples $(\psi_x : \psi_y : \psi_z]$ and $(\psi'_x : \psi'_y : \psi'_z)$ equivalent if $\psi'_x\psi_y - \psi_x\psi'_y$ and $\psi'_x\psi_z - \psi_x\psi'_z$ and $\psi'_y\psi_z - \psi_y\psi'_z$ all lie in (f_1).

This holds in particular when $\psi'_* = \lambda\psi_*$ for some nonzero homogeneous $\lambda \in k[x,y,z]$, so we can always remove any common factor of ψ_x, ψ_y, ψ_z.

Equivalent triples define the same rational map, and every rational map can be defined this way: if $\phi = (\frac{g_x}{h_x} : \frac{g_y}{h_y} : \frac{g_z}{h_z})$ then take $\psi_x := g_x h_y h_z$, $\psi_y := g_x h_x h_z$, $\psi_z := g_x h_x h_y$.

The rational map given by $[\psi_x, \psi_y, \psi_z]$ is defined at $P \in C_1(\bar{k})$ whenever any of $\psi_x(P), \psi_y(P), \psi_z(P)$ is nonzero, in which case $(\psi_x(P) : \psi_y(P) : \psi_z(P)) \in C_2(\bar{k})$.

Morphisms

Definition

A morphism is a rational map $\phi\colon C_1 \to C_2$ that is defined at every $P \in C_1(\bar{k})$.

Theorem

If C_1 is a smooth projective curve then every rational map $\phi\colon C_1 \to C_2$ is a morphism.

(Because when C_1 is smooth its coordinate ring $k[C_1]$ is a Dedekind domain.)

Theorem

A morphism of projective curves is either surjective or constant.

(Because projective varieties are complete/proper.)

Projective curves are isomorphic if there is an invertible morphism $\phi : C_1 \to C_2$.
We then have a bijection $C_1(\bar{k}) \to C_2(\bar{k})$, but this necessary condition is not sufficient!

An equivalence of categories

Every surjective morphism of projective of curves $\phi\colon C_1 \to C_2$ induces an injective morphism $\phi^*\colon k(C_2) \to k(C_1)$ of function fields defined by $\alpha \mapsto \alpha \circ \phi$.

Theorem

The categories of smooth projective curves over k with surjective morphisms and function fields of transcendence degree one over k are contravariantly equivalent via the functor $C \mapsto k(C)$ and $\phi \mapsto \phi^*$.

Every curve C, even singular affine curves, has a function field (for plane curves $f(x,y) = 0$, $k(C)$ is the fraction field of $k[C] := k[x,y]/(f)$). The function field $k(C)$ is categorically equivalent to a smooth projective curve $\tilde{C}$, the desingularization of C.

One can construct $\tilde{C}$ from C geometrically (using blow ups), but its existence is categorical, and in many applications the function field is all that matters.

Isogenies

Let E_1, E_2 be elliptic curves over k, with distinguished points O_1, O_2.

Definition

An isogeny $\phi\colon E_1 \to E_2$ is a surjective morphism that is also a group homomorphism.

Definition (apparently weaker but actually equivalent)

An isogeny $\phi\colon E_1 \to E_2$ is a non-constant rational map with $\phi(O_1) = O_2$.

E_1 and E_2 are isomorphic if there are isogenies $\phi_1\colon E_1 \to E_2$ and $\phi_2\colon E_2 \to E_1$ whose composition is the identity (the isogenies ϕ_1 and ϕ_2 are then called isomorphisms).

Morphisms $\phi\colon E_1 \to E_1$ with $\phi(O_1) = O_1$ are endomorphisms.
Note that $E_1 \to O_1$ is an endormophism, but it is **not an isogeny** (for us at least).

Endomorphisms that are isomorphisms are called automorphisms.

Examples of isogenies and endomorphisms

- The negation map $[-1]\colon P \mapsto -P$ defined by $(x : y : z) \mapsto (x : -y : z)$ is an isogeny, an endomorphism, an isomorphism, and an automorphism.

- For any integer n the multiplication by n map $[n]\colon P \mapsto nP$ is an endomorphism. It is an isogeny for $n \neq 0$ and an automorphism for $n = \pm 1$.

- For $E/\mathbb{F}_q$ we have the Frobenius endomorphism $\pi_E\colon (x : y : z) \mapsto (x^q : y^q : z^q)$. It induces a group isomorphism $E(\overline{\mathbb{F}}_q) \to E(\overline{\mathbb{F}}_q)$, but it is **not an isomorphism**.

- For $E/\mathbb{F}_q$ of characteristic p the map $\pi\colon (x : y : z) \mapsto (x^p : y^p : z^p)$ is an isogeny, but typically not an endomorphism. For $E\colon y^2 = x^3 + Ax + B$ the image of π is the elliptic curve $E^{(p)}\colon y^2 = x^3 + A^p x + B^p$, which need not be isomorphic to E.

The multiplication-by-2 map

Let E/k be defined by $y^2 = x^3 + Ax + B$ and let ϕ be the endomorphism $P \mapsto 2P$. The doubling formula for affine $P = (x : y : 1) \in E(\bar{k})$ is given by

$$\phi_x(x,y) = m(x,y)^2 - 2x = \frac{(3x^2+A)^2 - 8xy^2}{4y^2},$$

$$\phi_y(x,y) = m(x,y)(x - \phi_x(x,y)) - y = \frac{12xy^2(3x^2+A) - (3x^2+A)^3 - 8y^4}{8y^3},$$

with $m(x,y) := (3x^2+A)/(2y)$. We then have $\phi := (\psi_x/\psi_z : \psi_y/\psi_z : 1)$ with

$$\psi_x(x,y,z) = 2yz((3x^2+Az^2)^2 - 8xy^2z),$$

$$\psi_y(x,y,z) = 12xy^2z(3x^2+Az^2) - (3x^2+Az^2)^3 - 8y^4z^2,$$

$$\psi_z(x,y,z) = 8y^3z^3.$$

How do we evaluate this morphism at the point $O := (0 : 1 : 0)$? 🤷

The multiplication-by-2 map

How do we evaluate this morphism at the point $O := (0:1:0)$?

We can add any multiple of $f(x,y,z) = y^2z - x^3 - Axz^2 - Bz^3$ to any of ψ_x, ψ_y, ψ_z; this won't change the morphism ϕ.

Replacing ψ_x by $\psi_x + 18xyzf$ and ψ_y by $\psi_y + (27f - 18y^2z)f$, and simplifying yields

$$\begin{aligned}
\psi_x(x,y,z) &= 2y\big(xy^2 - 9Bxz^2 + A^2z^3 - 3Ax^2z\big),\\
\psi_y(x,y,z) &= y^4 - 12y^2z(2Ax + 3Bz) - A^3z^4 + 27Bz(2x^3 + 2Axz^2 + Bz^3) + 9Ax^2(3x^2 + 2Az^2),\\
\psi_z(x,y,z) &= 8y^3z.
\end{aligned}$$

Now $\phi(O) = (\psi_x(0,1,0) : \psi_y(0,1,0) : \psi_z(0,1,0)) = (0:1:0) = O$, as expected.

That wasn't particularly fun. 😠 But there is a way to completely avoid this! 😀

A standard form for isogenies

Lemma

Let $E_1\colon y^2 = f_1(x)$ and $E_2\colon y^2 = f_2(x)$ be elliptic curves over k and let $\alpha\colon E_1 \to E_2$ be an isogeny. Then α can be put in the affine standard form

$$\alpha(x,y) = \left(\frac{u(x)}{v(x)}, \frac{s(x)}{t(x)}y\right),$$

where $u, v, s, t \in k[x]$ are polynomials with $u \perp v$ and $s \perp t$.

Corollary

When $\alpha\colon E_1 \to E_2$ is defined as above we necessarily have $v^3 | t^2$ and $t^2 | v^3 f_1$.

It follows that $v(x)$ and $t(x)$ have the same set of roots in $\bar{k}$, and these roots are precisely the x-coordinates of the affine points in $E(\bar{k})$ that lie in the kernel of α. In particular, $\ker\alpha$ is a finite subgroup of $E(\bar{k})$.

Degree and separability

Definition

Let $\alpha(x,y) = \left(\frac{u(x)}{v(x)}, \frac{s(x)}{t(x)}y\right)$ be an isogeny in standard form.

The degree of α is $\deg \alpha := \max(\deg u, \deg v)$.
We say that α is separable if $(u/v)'$ is nonzero, otherwise α is inseparable.

Definition (equivalent)

Let $\alpha\colon E_1 \to E_2$ be an isogeny, let $\alpha^*\colon k(E_2) \to k(E_1)$ be the corresponding embedding of function fields, and consider the field extension $k(E_1)/\alpha^*(k(E_2))$.

The degree of α the degree of the field extension $k(E_1)/\alpha^*(k(E_2))$.
We say that α is separable if $k(E_1)/\alpha^*(k(E_2))$ is separable, otherwise α is inseparable.

Examples

- The standard form of the negation map $[-1]$ is $[-1](x, y) = (x, -y)$.
  It is separable and has degree 1.

- The standard form of the multiplication-by-2 map $[2]$ is

  $$[2](x, y) = \left(\frac{x^4 - 2Ax^2 - 8Bx + A^2}{4(x^3 + Ax + B)}, \frac{x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - A^3 - 8B^2}{8(x^3 + Ax + B)^2} y \right).$$

  It is separable and has degree 4.

- The standard form of the Frobenius endomorphism of $E/\mathbb{F}_q$ is

  $$\pi_E(x, y) = \left(x^q, (x^3 + Ax + B)^{(q-1)/2} y \right).$$

  Note that we have used the curve equation to transform y^q (here q is odd).
  It is inseparable, because $(x^q)' = qx^{q-1} = 0$, and it has degree q.