18.783 Elliptic Curves
Lecture 24

Andrew Sutherland

December 7, 2023

The modularity theorem

Definition

An elliptic curve $E/\mathbb{Q}$ is modular if it has the same L-function as a modular form.

Theorem (Taylor-Wiles 1995)

Every semistable elliptic curve $E/\mathbb{Q}$ is modular.

Corollary (Wiles 1995)

The equation $x^n + y^n = z^n$ has no integers solutions with $xyz \neq 0$ for $n > 2$.

Theorem (Breuil-Conrad-Diamond-Taylor 2001)

Every elliptic curve $E/\mathbb{Q}$ is modular.

Weak modular forms

Definition

A holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ is a weak modular form of weight k for a congruence subgroup Γ if for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$f(\gamma\tau) = (c\tau + d)^k f(\tau).$$

If $-I \in \Gamma$, for odd k the only weak modular form of weight k is the zero function.

Example

The j-function $j(\tau)$ is a weak modular form of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$, and for $k \geq 3$

$$G_k(\tau) := G_k([1,\tau]) := \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^k},$$

is a weak modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$.

Modular forms

If $\Gamma(N) \subseteq \Gamma$ then $f(\tau + N) = f(\tau)$ for any weak modular form $f\colon \mathcal{H} \to \mathbb{C}$.
It follows that f has a q-expansion (at ∞) of the form

$$f(\tau) = f^*(q^{1/N}) \sum_{n=-\infty}^{\infty} a_n q^{n/N} \qquad (q := e^{2\pi i\tau})$$

Definition

A weak modular form f is holomorphic at ∞ if f^* is holomorphic at 0, and f is holomorphic at the cusps if $f(\gamma\tau)$ is holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

A modular form is a weak modular form that is holomorphic at the cusps.

Example

The j-function is not a modular form, but the Eisenstein series $G_k(\tau)$ is a modular form of weight k for all even $k \geq 4$.

Cusp forms

Definition

A modular form is a cusp form if it vanishes at all the cusps; equivalently its q-expansion has the form $\sum_{n\geq 1} a_n q^n$ (at every cusp).

Example

The Eisenstein series $G_k(\tau)$ are not cusp forms but the discriminant function

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$$

is a cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$.

The set $M_k(\Gamma)$ of modular forms of weight k for Γ is a $\mathbb{C}$-vector space that contains the set of cusp forms $S_k(\Gamma)$ as a subspace. For $k = 2$ we have $\dim S_k(\Gamma) = g(\Gamma)$.

Hecke operators

Definition

For $n \in \mathbb{Z}_{>0}$ the Hecke operator (or Hecke correspondence) T_n is a linear operator on the free abelian group of lattices $L := [\omega_1, \omega_2]$ defined by

$$T_n := \sum_{[L:L']=n} L'.$$

We also define the homethety operator R_λ by $L \mapsto \lambda L$, for all $\lambda \in \mathbb{C}^\times$.

Theorem

The operators T_n and R_λ satisfy the following:

(i) $T_n R_\lambda = R_\lambda T_n$ *and* $R_\lambda R_\mu = R_{\lambda\mu}$.

(ii) $T_{mn} = T_m T_n$ *for all* $m \perp n$.

(iii) $T_{p^{r+1}} = T_{p^r} T_p - p T_{p^{r-1}} R_p$ *for all primes* p *and integers* $r \geq 1$.

The action of Hecke operators on modular forms

Each modular form $f\colon \mathcal{H} \to \mathbb{C}$ of weight k defines a function on lattices $[\omega_1, \omega_2]$ via

$$f([\omega_1,\omega_2]) := f(\omega_1^{-1}[1,\omega_2/\omega_1]) := \omega_1^{-k} f(\omega_2/\omega_1).$$

Definition

For $f \in M_k(\Gamma_0(1))$ we define

$$R_\lambda f(\tau) := f(\lambda[1,\tau]) = \lambda^{-k} f(\tau) \in M_k(\Gamma_0(1)),$$

$$T_n f(\tau) := n^{k-1} \sum_{[[1,\tau]:L]=n} f(L) = n^{k-1} \sum_{ad=n,\, 0\le b<d} d^{-k} f\left(\frac{a\tau+b}{d}\right) \in M_k(\Gamma_0(1)).$$

R_λ and T_n are linear operators on $M_k(\Gamma_0(1))$ that we can restrict to $S_k(\Gamma_0(1))$.
We have $T_{mn} = T_m T_n$ for $m \perp n$, and $T_{p^{r+1}} = T_{p^r} T_p - p^{k-1} T_{p^{r-1}}$ for p prime.

Eigenforms

Theorem

For any $f \in S_k(\Gamma_0(1))$ *and prime* p *we have*

$$a_n(T_p f) = \begin{cases} a_{np}(f) & \text{if } p \nmid n, \\ a_{np}(f) + p^{k-1} a_{n/p}(f) & \text{if } p \mid n. \end{cases}$$

and for all $m \perp n$ *we have* $a_m(T_n f) = a_{mn}(f)$. *In particular* $a_1(T_n(f)) = a_n(f)$.

Definition

An eigenform for $S_k(\Gamma_0(1))$ satisfies $T_n f = \lambda_n f$ for some $\lambda_1, \lambda_2, \ldots \in \mathbb{C}^\times$.
We normalize eigenforms so that $a_1(f) = 1$, and then $\lambda_n = a_n$ for all $n \in \mathbb{Z}_{>0}$.
We then have $a_m a_n = a_{mn}$ for $m \perp n$ and $a_{p^r} = a_p a_{p^{r-1}} - p^{k-1} a_{p^{r-2}}$ for p prime.

A basis of eigenforms

Definition

Let Γ be a congruence subgroup. The Petersson inner product on $S_k(\Gamma)$ is defined by

$$\langle f, g \rangle = \int_{\mathcal{F}} f(\tau)\overline{g(\tau)} y^{k-2} dx dy.$$

It is a positive definite Hermitian form on $S_k(\Gamma)$: it is bilinear and $\langle f, g \rangle = \overline{\langle g, f \rangle}$, with $\langle f, f \rangle = 0$ if and only if $f = 0$. Moreover, we have $\langle f, T_n g \rangle = \langle T_n f, g \rangle$. The Hecke operators are thus Hermitian operators on the space $S_k(\Gamma)$.

Theorem

The space of cusp forms for $S_k(\Gamma_0(1))$ is a direct sum of one-dimensional Hecke eigenspaces, and it has a unique basis of normalized eigenforms $f(\tau) = \sum a_n q^n$ for which a_n is the eigenvalue of T_n on the subspace spanned by f.

The Atkin-Lehner theory of newforms

Definition

A cusp form $f \in S_k(\Gamma_0(N))$ is old if $f \in S_k(\Gamma_0(M))$ for some M properly dividing N. The set of old forms is a subspace $S_k^{\mathrm{old}}(\Gamma_0(N))$ of $S_k(\Gamma_0(N))$. Taking the orthogonal complement with respect to the Petersson inner product yields

$$S_k(\Gamma_0(N)) = S_k^{\mathrm{old}}(\Gamma_0(N)) \oplus S_k^{\mathrm{new}}(\Gamma_0(N)),$$

The level of $f \in S_k(\Gamma_0(N))$ is the least $M|N$ for which $f \in S_k(\Gamma_0(M))$.
Normalized eigenforms $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$ are newforms, and necessarily have level N.

Theorem (Atkin-Lehner)

The space $S_k^{\mathrm{new}}(\Gamma_0(N))$ is a direct sum of one-dimensional Hecke eigenspaces, each generated by a newform $f(\tau) = \sum_n a_n q^n$ for which a_n is the eigenvalue of T_n on $\langle f \rangle$.

Dirichlet series

Definition

A Dirichlet series is a function of the form $L(s) = \sum_{n\geq 1} a_n n^{-s}$ with $a_n \in \mathbb{C}$.
If $|a_n| = O(n^\sigma)$ then $L(s)$ converges locally uniformly in the half plane $\mathrm{re}(s) > 1 + \sigma$.

Example

The Riemann zeta function is the Dirichlet series $\zeta(s) = \sum_{n\geq 1} n^{-s}$.
It converges locally uniformly to a holomorphic function on $\mathrm{re}(s) > 1$,
with a simple pole at $s = 1$ and no other poles. Moreover, the following hold:

- $\zeta(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$;
- $\tilde{\zeta}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ satisfies[1] the functional equation $\hat{\zeta}(s) = \hat{\zeta}(1-s)$;
- we have the Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.

[1] Here $\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt$ is the Euler gamma function.

L-functions of modular forms

Definition

The L-function of a cusp form $f = \sum a_n q^n$ is the Dirichlet series $L(f, s) := \sum a_n n^{-s}$.
If f has weight k then $L(f, s)$ converges locally uniformly on $\mathrm{re}(s) > 1 + k/2$.

Theorem (Hecke)

For $f \in S_k(\Gamma_0(N))$ the L-function $L(f, s)$ has an holomorphic continuation to $\mathbb{C}$ and $\hat{L}(f, s) := N^{s/2}(2\pi)^{-s}\Gamma(s)L(f, s)$ satisfies $\hat{L}(f, s) = \pm\hat{L}(f, k - s)$.

For $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$ the L-function $L(f, s)$ has the Euler product

$$L(f, s) = \sum_{n \geq 1} a_n n^{-s} = \prod_p (1 - a_p p^{-s} + \chi(p) p^{k-1} p^{-2s})^{-1},$$

where the Dirichlet character χ satisfies $\chi(p) = 0$ for $p|N$ and $\chi(p) = 1$ otherwise.

Summary of modular forms for $\Gamma_0(N)$

- A modular form of weight k for $\Gamma_0(N)$ is a holomorphic function $f\colon \mathcal{H}^* \to \mathbb{C}$ satisfying $f(\gamma\tau) = (c\tau+d)^k f(\tau)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.
- A cusp form $f \in S_k(\Gamma_0(N))$ vanishes at the cusps (its q-expansion has $a_0 = 0$).
- The cusp forms $S_k(\Gamma_0(N))$ are a $\mathbb{C}$-vector space with a Petersson inner product.
- The Hecke operators T_n are commuting Hermitian operators on $S_k(\Gamma_0(N))$.
- An eigenform $f = \sum a_n q^n \in S_k(\Gamma_0(N))$ satisfies $T_n f = a_n f$ for all $n \geq 1$.
- A cusp form $f \in S_k\Gamma_0(N))$ is old if $f \in S_k(\Gamma_0(M))$ for some proper divisor $M|N$, and we have $S_k(\Gamma_0(N)) = S_k^{\mathrm{old}}(\Gamma_0(N)) \oplus S_k^{\mathrm{new}}(\Gamma_0(N))$.
- The level of $f \in S_k(\Gamma_0(N))$ is the least $M|N$ for which $f \in S_k^{\mathrm{new}}(\Gamma_0(M))$.
- The newforms of weight k and level N are a canonical basis for $S_k^{\mathrm{new}}(\Gamma_0(N))$.
- The L-function $L(f,s)$ has an analytic continuation, a functional equation satisfied by $\hat{L}(f,s)$, and an Euler product $\prod(1 - a_p p^{-s} + \chi(p)p^{k-1}p^{-2s})^{-1}$.

The L-function of an elliptic curve over $\mathbb{Q}$

Definition

The L-function of an elliptic curve $E/\mathbb{Q}$ is defined by the Euler product

$$L_E(s) = \prod_p L_p(p^{-s})^{-1} = \prod_p \left(1 - a_p p^{-s} + \chi(p) p p^{-2s}\right)^{-1},$$

where $\chi(p)$ is 0 if E has bad reduction at p, and 1 otherwise. For primes of good reduction $a_p := p + 1 - \#\overline{E}(\mathbb{F}_p)$ is the trace of Frobenius, and otherwise

$$L_p(T) = \begin{cases} 1 & \text{if } E \text{ has additive reduction at } p; \\ 1 - T & \text{if } E \text{ has split mulitiplicative reduction at } p; \\ 1 + T & \text{if } E \text{ has non-split multiplicative reduction at } p. \end{cases}$$

This means that $a_p \in \{0, \pm 1\}$ at bad primes.

Primes of bad reduction

Definition

Let K be a number field. An integral model for E/K is a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with $a_1, a_2, a_3, a_4, a_6 \in \mathcal{O}_K$. The minimal discriminant of E/K is the $\mathcal{O}_K$-ideal

$$\Delta_{\min}(E) := \prod_{\mathfrak{p}} \mathfrak{p}^{\min v_{\mathfrak{p}}(\Delta)}$$

where $\mathfrak{p}$ varies over primes of K and Δ over discriminants of integral models for E.
A prime of bad reduction for E is a prime $\mathfrak{p}$ of K that divides the ideal $\Delta_{\min}(E)$.

A global minimal model for E/K is an integral model with discriminant $\Delta_{\min}(E)$.
Such models always exist when K has class number one (and in particular for $K = \mathbb{Q}$).

Why we like (general) Weierstrass equations

Every elliptic curve $E/\mathbb{Q}$ can be defined by an equation of the form $y^2 = x^3 + Ax + B$.

But equations of this form are usually **not** global minimal models, and a prime p that divides the discriminant $-16(4A^3 + 27B^2)$ is not necessarily a prime of bad reduction, even though $y^2 = x^3 + Ax + B$ defines a singular curve over $\mathbb{F}_p$ in this case.

Example

Consider the elliptic curve $y^2 = x^3 - 13392x - 1080432$ over $\mathbb{Q}$.
We have $A = 2^4 \cdot 3^3 \cdot 31$ and $B = 2^4 \cdot 3^3 \cdot 41 \cdot 61$ (so no extraneous powers), and

$$\Delta = -16(4A^3 + 27B^2) = -350572971995136 = -2^{12}3^{12}11^5.$$

But 2 and 3 are not primes of bad reduction!
Indeed, $y^2 + y = x^3 - x^2$ is a global minimal model with discriminant $\Delta_{\min}(E) = -11$.

Types of bad reduction

If p is an odd prime of bad reduction for $E/\mathbb{Q}$ we can find an integral model $y^2 = f(x)$ whose discriminant Δ satisfies $v_p(\Delta) = v_p(\Delta_{\min}) > 0$, and $f(x)$ then has a repeated root r modulo p. Without loss of generality, we assume $r = 0$ (replace x with $x - r$).

Over $\mathbb{F}_p$ we then have the curve $\overline{E}\colon y^2z = x^3 + ax^2z$ with a singular point $(0:0:1)$. Now define $\overline{E}^{\mathrm{ns}}(\mathbb{F}_p) := \overline{E}(\mathbb{F}_p) - \{(0:0:1)\}$ and let $a_p := p - \#\overline{E}^{\mathrm{ns}}(\mathbb{F}_p) \in \mathbb{Z}$.
The set $\overline{E}^{\mathrm{ns}}(\mathbb{F}_p)$ is a finite abelian group (under the usual group law) and we have

$\left(\frac{a}{p}\right)$	$\#\overline{E}^{\mathrm{ns}}(\mathbb{F}_p)$	$\overline{E}^{\mathrm{ns}}(\mathbb{F}_p)$	reduction type
0	p	$\simeq \mathbb{F}_p$	additive
$+1$	$p-1$	$\simeq \mathbb{F}_p^\times$	split multiplicative
-1	$p+1$	$\simeq \{\alpha \in \mathbb{F}_{p^2}^\times : \alpha^{p+1} = 1\}$	non-split multiplicative

Note that $a_p = p - \#\overline{E}^{\mathrm{ns}}(\mathbb{F}_p) = \left(\frac{a}{p}\right)$ in every case. Something similar works for $p = 2$.

The conductor of an elliptic curve

Definition

The conductor of an elliptic curve $E/\mathbb{Q}$ is the integer

$$N_E := \prod_p p^{\varepsilon(p)+\delta(p)}$$

where $\varepsilon(p) = 0, 1, 2$ when E has good, multiplicative, additive reduction at p.

The "wild" exponent $\delta(p)$ is zero unless we have additive reduction at $p = 2, 3$ in which case it can be defined using the ramification of p in the p^n-torsion fields $\mathbb{Q}(E[p^n])$.

We have $N_E|\Delta_{\min}(E)$ with $v_p(N_E) \le 8, 5$ for $p = 2, 3$ and $v_p(N_E) \le 2$ for $p > 3$.

Definition

An elliptic curve $E/\mathbb{Q}$ is semistable if its conductor is squarefree.
Equivalently, E does not have additive reduction at any prime.

Modularity

Definition

For an elliptic curve $E/\mathbb{Q}$ with $L(E,s) = \sum a_n n^{-s}$ we define $f_E \colon \mathcal{H} \to \mathbb{C}$ by

$$f_E(\tau) := \sum_{n\geq 1} a_n q^n \qquad (q := e^{2\pi i \tau})$$

The elliptic curve E is modular if the function f_E is a modular form.
Equivalently, E is modular if and only if $L(E,s)$ is the L-function of a modular form.

If E is modular then f_E must be a cusp form of weight 2 since the Euler factors are

$$1 - a_p p^{-s} + \chi(p) p p^{-2s} = 1 - a_p p^{-s} + \chi(p) p^{k-1} p^{-2s},$$

Theorem (Modularity theorem)

Let $E/\mathbb{Q}$ be an elliptic curve. Then f_E is an eigenform of weight 2 and level N_E.

The functional equation

Corollary

Let $E/\mathbb{Q}$ be an elliptic curve. The L-function $L(E,s)$ has a holomorphic continuation to $\mathbb{C}$ and $\hat{L}(E,s) := N_E^{s/2}(2\pi)^{-s}\Gamma(s)L_E(s)$ satisfies $\hat{L}(E,s) = \pm\hat{L}(E,2-s)$.

Notice that $\hat{L}(E,s) = -\hat{L}(E,2-s)$ is possible only when $\mathrm{ord}_{s=1}L(E,s)$ is odd.

Conjecture (Weak BSD)

We have $E(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus E(\mathbb{Q})_{\mathrm{tors}}$ if and only if $\mathrm{ord}_{s=1}L(E,s) = r$.

Conjecture (Parity conjecture)

If $E(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus E(\mathbb{Q})_{\mathrm{tors}}$ then $\hat{L}(E,s) = (-1)^r\hat{L}(E,2-s)$.

Eichler-Shimura

Definition

Let $f = \sum a_n q^n \in S_2^{\mathrm{new}}(\Gamma_0(N))$ be a newform.
The coefficients a_n are algebraic integers that generate a finite extension $\mathbb{Q}(f)/\mathbb{Q}$.
The dimension of f is $\dim f := [\mathbb{Q}(f) : \mathbb{Q}]$; we call f rational if $\dim f = 1$.

One can associate to any newform in $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ a lattice Λ in $\mathbb{C}^d$ and a corresponding abelian variety $A_f := \mathbb{C}^d/\Lambda$ of dimension $d = \dim f$ defined over $\mathbb{Q}$.
One then has $L(A, s) = \prod_\sigma L(\sigma(f), s)$ where $\sigma(f)$ ranges over the $\mathrm{Aut}(\mathbb{C})$-orbit of f (equivalently, $a_n \in \mathbb{Q}(f)$ and σ varies over embeddings of $\mathbb{Q}(f)$ into $\mathbb{C}$).

Theorem (Eichler-Shimura, Carayol)

For every rational newform $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ there is an elliptic curve $E/\mathbb{Q}$ of conductor N with $f_E = f$ and $L(E, s) = L(f, s)$.

Faltings-Tate

Recall that isogenous elliptic curves over $\mathbb{F}_p$ have the same trace of Frobenius. If E_1 and E_2 are isogenous elliptic curves over $\mathbb{Q}$, then $a_p(E_1) = a_p(E_2)$ for all primes of good reduction, and in fact $a_p(E_1) = a_p(E_2)$ for all primes.

It follows that isogenous elliptic curves over $\mathbb{Q}$ have the same L-function. Remarkably, the converse holds, in fact something even stronger holds.

Theorem (Faltings-Tate)

If two elliptic curves E, E' over $\mathbb{Q}$ satisfy $a_p(E) = a_p(E')$ for all but finitely many primes p then E and E' are isogenous (thus $a_p(E) = a_p(E')$ for all primes p).

Corollary

Elliptic curves over $\mathbb{Q}$ are isogenous if and only if they have the same L-function.

Isogeny classes of elliptic curves and modular forms

Distinct eigenforms $S_2^{\mathrm{new}}(\Gamma_0(N))$ necessarily have distinct L-functions, since their q-expansions $\sum a_n q^n$ must be linearly independent. The modular form f_E given by the modularity theorem thus depends only on the isogeny class of $E/\mathbb{Q}$ and in general there may be non-isomorphic isogenous $E/\mathbb{Q}$ that correspond to the same f_E.

There is thus in general a many-to-one relationship between elliptic curves over $\mathbb{Q}$ and rational eigenforms of weight 2, but a one-to-one relationship between isogeny classes of elliptic curves over $\mathbb{Q}$ and rational eigenforms of weight 2.

You can see this explicitly in the L-functions and Modular Forms Database (LMFDB).

Example

The elliptic curves `11.a1`, `11.a2`, `11.a3` of conductor $N_E = 11$ make up the isogeny class `11.a`, which corresponds to the modular form `11.2.a.a` of weight 2 and level 11. They all have the same L-function 2-11-1.1-c1-0-0, which has $\mathrm{ord}_{s=1} L(s) = 0$.