**18.783 Elliptic Curves** **Spring 2022**
**Problem Set #10** **Due: 04/22/2022**

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**Description**: These problems are related to the material covered in Lectures 17–19.

**Instructions**: Pick any combination of problems to solve that sums to 100 points. Your solutions are to be written up in latex and submitted as a pdf-file to Gradescope.

Collaboration is permitted/encouraged, but you must identify your collaborators or your group on pset partners, as well any references you consulted that are not listed in the syllabus or lecture notes. If there are none write "**Sources consulted: none**" at the top of your solutions. Note that each student is expected to write their own solutions; it is fine to discuss the problems with others, but your writing must be your own.

The first person to spot each non-trivial typo/error in the problem sets or lecture notes will receive 1-5 points of extra credit.

In cases where your solution involves writing code, please either include your code in your write up (as part of the pdf), or the name of a notebook in your 18.783 CoCalc project containing you code (please use a separate notebook for each problem).

## Problem 1. Congruence subgroups (98 points)

Let $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ denote the modular group and $\mathcal{H}^* := \{\tau : \operatorname{im}\tau > 0\} \cup \mathbb{Q} \cup \{\infty\}$ the extended upper half plane. The diagram on the next page depicts a fundamental region $\mathcal{F}$ for $\mathcal{H}^*/\Gamma(1)$ in $\mathcal{H}^*$, along with nine of its translates. Each translate $\gamma\mathcal{F}$ is labeled by $\gamma$, where $\gamma$ is expressed in terms of the generators $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ for $\Gamma(1)$.

The colored labels $\rho, i, \infty$ within the region labeled by $\gamma$ indicate the points $\gamma\rho$, $\gamma i$, and $\gamma\infty$, respectively (where $\rho = e^{2\pi i/3}$). Note that the red, black, and blue colored $\infty$ along the top of the diagram are all the same point, but there are three distinct translates of $\infty$ on the real axis (at $-1, 0, 1$), each of which lies in two translates of $\mathcal{F}$ (this illustrates a key point: translates of $\mathcal{F}$ may overlap at points whose stabilizers act non-trivially, namely, the points $i, \rho, \infty$). The region $\mathcal{F}$ includes the arc from $i$ to $\rho$ along the unit circle and the line from $\rho$ to $\infty$ along the imaginary axis, but no other points on its boundary other than $\infty$; translates of these have been colored and oriented.

**Remark.** The lines/arcs connecting $i, \rho, \infty$ in each translate of $\mathcal{F}$ are *geodesics*, meaning they are shortest paths between two points, under the *hyperbolic metric* in which the length of a continuously differentiable path $\gamma\colon [a,b] \to \mathbb{H}$ is given by $\int_a^b \frac{|\gamma'(t)|}{\operatorname{Im}(\gamma(t))}dt$. Such paths are necessarily either vertical lines or arcs on a semicircle that intersects the real axis in right angles. For more background on hyperbolic geometry see [2, Ch. 33].

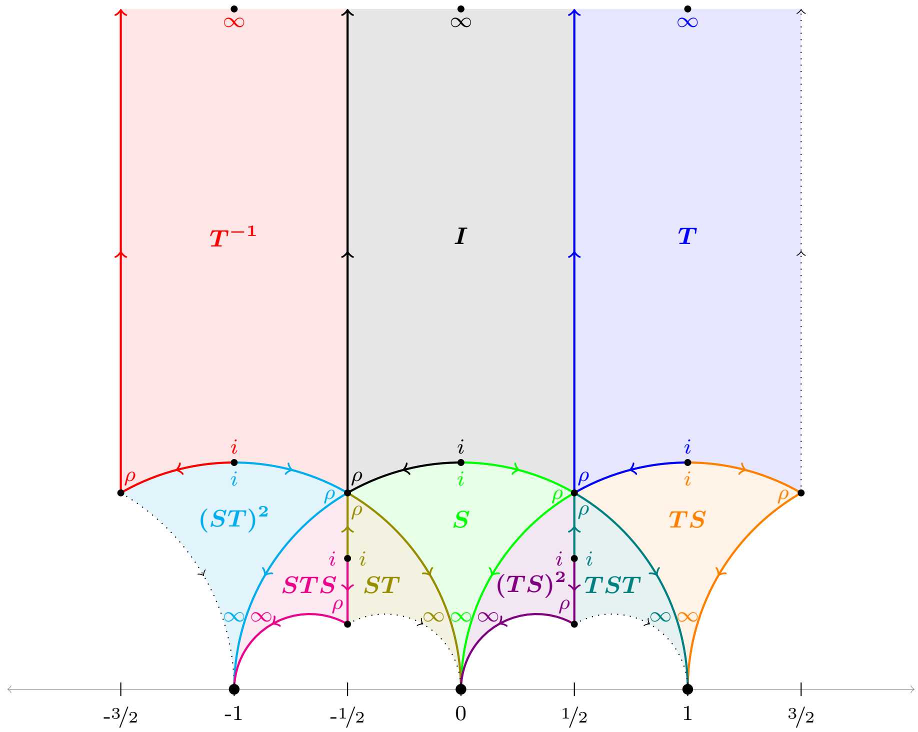
We recall the following subgroups of $\mathrm{SL}_2(\mathbb{Z})$, defined for each integer $N \geq 1$:

$$\Gamma(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) \bmod N\},$$
$$\Gamma_1(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right) \bmod N\},$$
$$\Gamma_0(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix}* & *\\ 0 & *\end{smallmatrix}\right) \bmod N\},$$

and corresponding modular curves

$$X(N) := \mathcal{H}^*/\Gamma(N), \qquad X_1(N) := \mathcal{H}^*/\Gamma_1(N), \qquad X_0(N) := \mathcal{H}^*/\Gamma_0(N).$$

For any congruence subgroup $\Gamma$ we call the $\Gamma(1)$-translates of $\infty$ in $\mathcal{H}^*$ or $\mathcal{H}^*/\Gamma$ *cusps*.




**(a)** Determine the index of $\Gamma(2)$ in $\Gamma(1)$, and the number of $\Gamma(2)$ cusp orbits. Then give a connected fundamental region for $\mathcal{H}^*/\Gamma(2)$ by listing a subset of the translates of $\mathcal{F}$ in the diagram above and identify the cusps that lie in your region. Compute the genus of $X(2)$ by triangulating your fundamental region and applying Euler's formula $V - E + F = 2 - 2g$. Be careful to count vertices and edges correctly — initially specify vertices and edges as $\mathcal{H}^*$-points in the diagram (e.g. $ST\rho$), then determine which vertices and edges are $\Gamma(2)$-equivalent (note that edges whose end points are equivalent need not be equivalent). Do the same for $\Gamma_0(2)$ and $X_0(2)$.

**(b)** For each of the following congruence subgroups, determine its index in $\Gamma(1)$, the number of cusp orbits, and a set of cusp representatives: $\Gamma_0(3)$, $\Gamma_1(3)$, $\Gamma(3)$.

**(c)** Prove that for each integer $N \geq 1$ we have an exact sequence

$$1 \longrightarrow \Gamma(N) \longrightarrow \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow 1.$$

Show that in general one cannot replace $\mathrm{SL}_2$ with $\mathrm{GL}_2$ in the sequence above (so your proof for $\mathrm{SL}_2$ needs to use more than the fact that $\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ is surjective).

**(d)** Derive formulas for the index $[\Gamma(1) : \Gamma]$ for $\Gamma = \Gamma(N), \Gamma_1(N), \Gamma_0(N)$ and any $N \geq 1$. Use the Euler function $\phi(N) := \#(\mathbb{Z}/N\mathbb{Z})^\times$ where appropriate.

For any congruence subgroup $\Gamma$, let $\nu_2(\Gamma)$ and $\nu_3(\Gamma)$ count the number of $\mathrm{SL}_2(\mathbb{Z})$ translates of $i$ and $\rho$, respectively, that lie in a fundamental region of $\mathcal{H}^*$ for $\Gamma$ and are fixed by some $\gamma \in \Gamma$ other than $\pm I$. Let $\nu_\infty(\Gamma)$ be the number of cusp-orbits for $\Gamma$.

**(e)** For $\Gamma = \Gamma(p), \Gamma_1(p), \Gamma_0(p)$ derive formulas for $\nu_2(\Gamma), \nu_3(\Gamma), \nu_\infty(\Gamma)$, where $p$ is a prime (hint: show that for any $\delta \in \mathrm{SL}_2(\mathbb{Z})$, if $\gamma \in \mathrm{SL}_2(\mathbb{Z}) - \{\pm I\}$ stabilizes $\delta i$ then it has trace 0, and if it stabilizes $\delta\rho$ then it has trace $\pm 1$).

Let $\overline{\Gamma}(N), \overline{\Gamma}_1(N), \overline{\Gamma}_0(N)$ denote the images of the groups $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$ in $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(Z)/\{\pm I\}$ respectively, and for any congruence subgroup $\Gamma$ with image $\overline{\Gamma}$ in $\mathrm{PSL}_2(\mathbb{Z})$ define

$$\mu(\Gamma) := [\overline{\Gamma}(1) : \overline{\Gamma}] = \begin{cases} [\Gamma(1) : \Gamma] & \text{if } -I \in \Gamma \\ [\Gamma(1) : \Gamma]/2 & \text{if } -I \notin \Gamma \end{cases}$$

Using the Riemann-Hurwitz genus formula one can prove that for any congruence subgroup $\Gamma$ the genus of the modular curve $X_\Gamma := \mathcal{H}^*/\Gamma$ is given by the formula

$$g(X_\Gamma) = 1 + \frac{\mu(\Gamma)}{12} - \frac{\nu_2(\Gamma)}{4} - \frac{\nu_3(\Gamma)}{3} - \frac{\nu_\infty(\Gamma)}{2}.$$

For convenience we may write $g(\Gamma)$ for $g(X_\Gamma)$.

**(f)** Use your answers to (d) and (e) to give asymptotic approximations for $g(\Gamma)$ for $\Gamma = \Gamma(p), \Gamma_1(p), \Gamma_0(p)$ and increasing primes $p$ that have an exact leading term (so of the form $f(p) + O(g(p))$ for some functions $f$ and $g$ with $g = o(f)$). Conclude that the set of primes $p$ for which $g(\Gamma)$ takes any fixed value is finite.

Modular curves of genus 0 and 1 are of particular interest because we can use these curves to obtain infinite families of elliptic curves over $\mathbb{Q}$ (or a number field) that have particular properties, for example, a torsion point of order $p$. By Faltings' Theorem, over a number field a curve of genus $g \geq 2$ has only a finite number of rational points.

**(g)** For $\Gamma = \Gamma(p), \Gamma_1(p), \Gamma_0(p)$ determine the primes $p$ for which $g(\Gamma) = 0$, and the primes $p$ for which $g(\Gamma) = 1$.

You may use Sage to check your answers (and gain intuition), but your proofs must stand on their own. To create the congruence subgroups $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$ in Sage use `Gamma(N)`, `Gamma1(N)`, `Gamma0(N)`, respectively. The returned objects support `index()`, `nu2()`, `nu3()`, `cusps()`, and `genus()` methods that you may find useful.

## Problem 2. Non-congruence subgroups of finite index (98 points)

Recall that a *congruence subgroup* is a subgroup of $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ that contains $\Gamma(N)$ for some $N \geq 1$. Every congruence subgroup is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. In this problem you will prove that the converse does not hold; there exist finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ that are not congruence subgroups.

Let $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$, let $\alpha$ be the image of $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathrm{PSL}_2(\mathbb{Z})$, and let $\beta$ be the image of $ST = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ in $\mathrm{PSL}_2(\mathbb{Z})$.

**(a)** Let $\mathbf{Z}_{2,3}$ be the finitely presented group with generators $x, y$ satisfying the relations $x^2 = y^3 = 1$ (and no others). Prove that the map $\mathbf{Z}_{2,3} \to \mathrm{PSL}_2(\mathbb{Z})$ defined by $x \mapsto \alpha$ and $y \mapsto \beta$ is an isomorphism. You may find the diagram from Problem 1 helpful.

Part (a) implies that for any finite group $H = \langle a, b \rangle$ with $|a| = 2$ and $|b| = 3$ we have a surjective group homomorphism

$$\mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{Z}) \twoheadrightarrow H,$$

where the first map is quotient map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$ and the second is the composition of the isomorphism $\mathrm{PSL}_2(\mathbb{Z}) \xrightarrow{\sim} \mathbf{Z}_{2,3}$ and the surjective homomorphism $\mathbf{Z}_{2,3} \to H$ defined by $x \mapsto a$, $y \mapsto b$. The kernel $\Gamma_H$ of such a homomorphism is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Our strategy is to show that for many finite groups $H = \langle a, b\rangle$, this kernel cannot contain $\Gamma(N)$ for any integer $N$, and is therefore not a congruence subgroup. To simplify matters, we will focus on cases where $H$ is a *simple* group, meaning that $H$ is a non-trivial group that contains no normal subgroups other than the trivial group and itself. Every non-trivial finite group $G$ has a *composition series*

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = G$$

in which each $G_i$ is a normal subgroup of $G_{i+1}$ and each quotient $G_{i+1}/G_i$ is simple. The quotients $G_{i+1}/G_i$ are called the *simple factors* of $G$ (analogous to prime factors of an integer). This composition series is not unique, but the Jordan-Hölder theorem states that the simple factors $G_{i+1}/G_i$ that appear in any composition series for $G$ are unique up to isomorphism (and occur with the same multiplicity).

**(b)** Prove that if a finite simple group $S$ is a quotient of $G$ (meaning $S = G/K$ for some $K \triangleleft G$) then $S$ is a simple factor of $G$ but that the converse does not hold in general.

**(c)** Prove that if a finite group $G$ is the direct product of non-trivial groups $H_1, \ldots, H_n$ then the simple factors of $G$ are precisely the simple factors of the $H_i$ (counted with multiplicity). Conclude that if $S$ is a simple quotient of $G$ then it is a quotient of one of the $H_i$.

**(d)** Prove part (c) of Problem 1.

**(e)** Let $N = p_1^{e_1} \cdots p_r^{e_r} > 1$ by with $p_1, \ldots, p_r$ distinct primes. Prove that every simple quotient of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is a simple quotient of $\mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$ for some $i$.

**(f)** Using the fact that $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$ is a non-abelian simple group for primes $p \geq 5$, show that the simple factors of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ are: a cyclic group of order 2, $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$, and $3e-3$ cyclic groups of order $p$, and that in particular, $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$ is the unique non-abelian simple factor of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$, for all primes $p \geq 5$.

**(g)** Using the fact that the alternating group $A_n$ is a non-abelian simple group for all $n \geq 5$, prove that $A_n$ is not a quotient of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for any $N$ and any $n > 5$.

**(h)** Using Sage, find elements $a$ of order 2 and $b$ of order 3 that generate $A_9$ and list them in cycle notation. You don't need to write down a proof that they generate $A_9$ but you should verify this in Sage. To create $A_9$ use `A9=AlternatingGroup(9)`, and to check whether $a$ and $b$ generate $A_9$ use `A9.subgroup([a,b]) == A9`.

In fact, $A_n$ is generated by an element of order 2 and an element of order 3 for all $n \geq 9$ (see [1]), but you are not asked to prove this. It follows from the discussion after (a) that there is a surjective homomorphism $\mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow A_9$ that sends $\pm\alpha$ to $a$ and $\pm\beta$ to $b$. The kernel $\Gamma$ of this homomorphism is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

**(i)** Prove that $\Gamma$ is not a congruence subgroup.

We now want to construct a short list of generators for $\Gamma$. The first step is to convert the representation of $A_9$ with generators $a$ and $b$ of orders 2 and 3 into a finitely presented group that is a finite quotient of $\mathbf{Z}_{2,3}$ specified by relations. To do this use the Sage command:

```
H=A9.subgroup([a,b]).as_finitely_presented_group().simplified()
```

This may take a few seconds. The second step is to plug $S$ and $ST$ into all the relations in the finite presentation of $H$ you created above. **Important**: Sage may swap the roles of $a$ and $b$ when it constructs the finite presentation – check the relations to see if this happened (if you see $a^3$ and $b^2$ in the list of relations rather than $a^2$ and $b^3$ then you know they were swapped). Assuming $a^2$ and $b^3$ are the first two relations, you can use

```
G=SL(2,Integers()); S=G([0,-1,1,0]); T=G([1,1,0,1])
for i in range(2,len(H.relations())):
        print(H.relations()[i].subs(a=S,b=S*T))
```

to get a list of matrices in $\mathrm{SL}_2(\mathbb{Z})$ that, together with $S$ and $T$ generate $\Gamma$. Note that the length of the list you get will depend on your choice of $a$ and $b$, but shouldn't be more than 10 or 20 matrices (in fact one can do it with 4).

**(j)** Record the number of matrices in your list above (not including $S$ and $T$), and a smallest and largest matrix in your list according to the $L^\infty$-norm (maximum of absolute values of matrix entries).

## Problem 3. Polycyclic presentations (98 points)

Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be a sequence of generators for a finite abelian group $G$, and let $G_i = \langle \alpha_1, \ldots, \alpha_i \rangle$ be the subgroup generated by $\alpha_1, \ldots, \alpha_i$. The subnormal series

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = G,$$

is a *polycyclic series*: each $G_{i-1}$ is a normal subgroup of $G_i$ and each of the quotients $G_i/G_{i-1} = \langle \alpha_i G_{i-1} \rangle$ is a cyclic group (which we don't require to have prime order, but one can always further decompose the series so that they are). Every finite solvable group admits a polycyclic series, but we restrict ourselves here to abelian groups (written multiplicatively).

When $G$ is the internal direct product of the cyclic groups $\langle \alpha_i \rangle$, we have $G_i/G_{i-1} \cong \langle \alpha_i \rangle$ and call $\vec{\alpha}$ a *basis* for $G$, but this is a special case. For abelian groups, $G_i/G_{i-1}$ is isomorphic to a subgroup of $\langle \alpha_i \rangle$, but it may be a proper subgroup, even when $G$ is cyclic.

The sequence $r(\vec{\alpha}) = (r_1, \ldots, r_k)$ of *relative orders* for $\vec{\alpha}$ is defined by

$$r_i = |G_i : G_{i-1}|,$$

and satisfies $r_i = \min\{r : \alpha_i^r \in G_{i-1}\}$. We necessarily have $r_i \leq |\alpha_i|$, but equality typically does not hold ($\vec{\alpha}$ is a basis precisely when $r_i = |\alpha_i|$ for all $i$). In any case, we always have $\prod_i r_i = |G|$, thus computing the $r_i$ determines the order of $G$.

**(a)** Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be a sequence of generators for a finite abelian group $G$, with relative orders $r(\vec{\alpha}) = (r_1, \ldots, r_k)$. Prove that every $\beta \in G$ can be uniquely represented in the form

$$\beta = \vec{x} \cdot \vec{\alpha} = \alpha_1^{x_1} \cdots \alpha_k^{x_k},$$

where each $x_i \in \mathbb{Z}$ satisfies $0 \leq x_i < r_i$. Show that if $\beta = \alpha_i^{r_i}$, then $x_j = 0$ for $j \geq i$.

By analogy with the case $r = 1$, we call $\vec{x}$ the *discrete logarithm* of $\beta$ with respect to $\vec{\alpha}$ (but note that the discrete logarithm of the identity element is now the zero vector). The vector $\vec{x}$ can be conveniently encoded as an integer $x$ in the interval $[0, |G| - 1]$ via

$$x = \sum_{1 \le i \le k} x_i N_i, \qquad\qquad N_i = \prod_{1 \le j < i} r_j,$$

and we may simply write $x = \log_{\vec{\alpha}} \beta$ to indicate that $x$ is the integer encoding the vector $\vec{x} = \log_{\vec{\alpha}} \beta$. Note that $x_i = \lfloor x/N_i \rfloor \bmod r_i$, so it is easy to recover $\vec{x}$ from its encoding $x$.

**(b)** Design a generic group algorithm that given generators $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$ for a finite abelian group $G$, constructs a table $T$ with entries $T[0], ..., T[|G| - 1]$ with the property that if $T[n] = \beta$, then $n = \log_{\vec{\alpha}} \beta$. Your algorithm should also output the relative orders $r_i$, and the integers $s_i$ for which $T[s_i] = \alpha_i^{r_i}$.

This allows us to compute a *polycyclic presentation* for $G$, which consists of the sequence $\vec{\alpha}$, the relative orders $r(\vec{\alpha}) = (r_1, \ldots, r_k)$, and the vector of integers $s(\vec{\alpha}) = (s_1, \ldots, s_k)$. With this presentation in hand, we can effectively simulate any computation in $G$ without actually performing any group operations (i.e. calls to the black box). This can be very useful when the group operation is expensive.

**(c)** Let $\vec{\alpha}$, $r(\vec{\alpha})$, and $s(\vec{\alpha})$ be a polycyclic presentation for a finite abelian group $G$. Given integers $x = \log_{\vec{\alpha}} \beta$ and $y = \log_{\vec{\alpha}} \gamma$, explain how to compute the integer $z = \log_{\vec{\alpha}} \beta\gamma$ using $r(\vec{\alpha})$ and $s(\vec{\alpha})$, without performing any group operations. Also explain how to compute the integer $w = \log_{\vec{\alpha}} \beta^{-1}$.

As a side benefit, the algorithm you designed in part (b) gives a more efficient way to enumerate the class group $\mathrm{cl}(D)$ than we used in Problem Set 9, since the class number $h(D)$ is asymptotically on the order of $\sqrt{|D|}$ (this is a theorem of Siegel).

But first we need to figure out how to construct a set of generators for $G$. We will do this using *prime forms*. These are forms $f = (a, b, c)$ for which $a$ is prime and $-a < b \le a$ (but we do not require $a \le c$, so prime forms need not be reduced). Prime forms correspond to prime ideals whose norm is prime (degree-1 primes). Recall that imaginary quadratic orders $\mathcal{O}$ are determined by their discriminant $D$, which can always be written in the form $D = u^2 D_K$, where $D_K$ is the discriminant of the maximal order $\mathcal{O}_K$ and $u = [\mathcal{O}_K : \mathcal{O}]$ is the conductor of $\mathcal{O}$.

**(d)** Let $a$ be a prime. Prove that if $a$ divides the conductor then there are no prime forms of norm $a$, and that otherwise there are exactly $1 + (\frac{D}{a})$ prime forms of norm $a$, where $(\frac{D}{a})$ is the Kronecker symbol.[1] Write a program that, given a prime $a$, either outputs a prime form $(a, b, c)$ with $b \ge 0$ or determines that none exist.

When $D$ is fundamental, we can generate $\mathrm{cl}(D)$ using prime forms of norm at most $\sqrt{|D|/3}$; this follows from the bound proved in Problem Set 9 and the fact that the maximal order $\mathcal{O}_K$ is a Dedekind domain (so ideals can be uniquely factored into prime ideals). We can still generate $\mathrm{cl}(D)$ with prime forms when $D$ is non-fundamental, but bounding the primes involved is slightly more complicated, so we will restrict ourselves to fundamental discriminants for now.

[1] Thus $(\frac{D}{2})$ is 0 if $D$ is even, 1 if $D \equiv 1 \bmod 8$, and $-1$ if $D \equiv 5 \bmod 8$. Note that we refer to $a$ as the "norm" of the form $(a, b, c)$, since the corresponding ideal has norm $a$.

**(e)** Implement the algorithm you designed in part (b), using the program from part (d) to enumerate the prime forms of norm $a \le \sqrt{|D|/3}$ in increasing order by $a$. Use the prime forms as generators, but use a table lookup to discard prime forms that are already present in your table so that your $\alpha_i$ all have relative orders $r_i > 1$ (**warning**: prime forms need not be reduced: be sure to reduce them before making any comparisons). For the group operation, you can create binary quadratic forms in Sage using `BinaryQF([a,b,c])`, and then compose forms $f$ and $g$ using `h=f*g`. Use `h.reduced_form()` to get the reduced form. You will only be using this code on small examples, so don't worry about the efficiency of your implementation.

**(f)** Run your algorithm on $D = -5291$, and then run it on the first fundamental discriminant $D < -N$, where $N$ is the first five digits of your student ID. Don't list all the elements of $\mathrm{cl}(D)$, just give the reduced forms for the elements of $\vec{\alpha}$ and the integer vectors $r(\vec{\alpha})$ and $s(\vec{\alpha})$. Sanity check your results by verifying that you at least get the right class number for $D$ (you can check this in Sage using `NumberField(x**2-D,'t').class_number()`).

**(g)** Recall that every finite abelian group is isomorphic to a unique product of non-trivial cyclic groups $\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}$ for which $n_1|n_2|\cdots|n_r$. The sequence of integers $(n_1, \ldots, n_r)$ are the *invariant factors* of $G$ and uniquely identify its isomorphism class. Design an algorithm that takes a polycyclic presentation for a finite abelian group $G$ as input and outputs its invariant factors along with a corresponding basis $\alpha_1, \ldots, \alpha_r$ for $G$ with $|\alpha_i| = n_i$.

**(h)** Use your algorithm from part (g) to compute the invariant factors of the two class groups you computed in part (f), along with corresponding generators. Express each generator as a reduced form and give its discrete logarithm with respect to the generators for the polycyclic presentations you computed in part (f).

## Problem 4. Survey (2 points)

Complete the following survey by rating each of the problems you attempted on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

| | Interest | Difficulty | Time Spent |
|---|---|---|---|
| Problem 1 | | | |
| Problem 2 | | | |
| Problem 3 | | | |

Also, please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material (1="old hat", 10="all new").

| Date | Lecture Topic | Material | Presentation | Pace | Novelty |
|---|---|---|---|---|---|
| 4/20 | The Hilbert class polynomial | | | | |

Please feel free to record any additional comments you have on the problem sets or lectures, in particular, ways in which they might be improved.

## References

[1] I.M.S. Dey and J. Wiegold, *Generators for alternating and symmetric groups*, J. Australian Mathematical Society **12** (1971), 63–68.

[2] J. Voight, *Quaternion algebras*, Springer, 2021.