

## 15 Elliptic curves over $\mathbb{C}$ (part 2)

Last time we showed that every lattice  $L \subseteq \mathbb{C}$  gives rise to an elliptic curve

$$E_L: y^2 = 4x^3 - g_2(L)x - g_3(L),$$

where

$$g_2(L) = 60G_4(L) := 60 \sum_{L^*} \frac{1}{\omega^4}, \quad g_3(L) = 140G_6(L) = 140 \sum_{L^*} \frac{1}{\omega^6},$$

with  $L^* := L - \{0\}$ , and we defined a map

$$\Phi: \mathbb{C}/L \rightarrow E_L(\mathbb{C})$$

$$z \mapsto \begin{cases} (\wp(z), \wp'(z)) & z \notin L \\ 0 & z \in L \end{cases}$$

where

$$\wp(z) = \wp(z; L) = \frac{1}{z^2} + \sum_{\omega \in L^*} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right)$$

is the Weierstrass  $\wp$ -function for the lattice  $L$ , and

$$\wp'(z) = -2 \sum_{\omega \in L} \frac{1}{(z - \omega)^3}.$$

In this lecture we will prove two theorems. First we will prove that  $\Phi$  is an isomorphism of additive groups; it is also an isomorphism of complex manifolds [3, Cor. 5.1.1], and of complex Lie groups, but we won't prove this right now.<sup>1</sup> Second, we will prove that every elliptic curve  $E/\mathbb{C}$  is isomorphic to  $E_L$  for some lattice  $L$ ; this is the *Uniformization Theorem*.

### 15.1 The isomorphism from a torus to the corresponding elliptic curve

**Theorem 15.1.** *Let  $L \subseteq \mathbb{C}$  be a lattice and let  $E_L: y^2 = 4x^3 - g_2(L)x - g_3(L)$  be the corresponding elliptic curve. The map  $\Phi: \mathbb{C}/L \rightarrow E_L(\mathbb{C})$  is a group isomorphism.*

*Proof.* We first note that  $\Phi(0) = 0$ , so  $\Phi$  preserves the identity, and for all  $z \notin L$  we have

$$\Phi(-z) = (\wp(-z), \wp'(-z)) = (\wp(z), -\wp'(z)) = -\Phi(z),$$

since  $\wp$  is even and  $\wp'$  is odd, so  $\Phi$  is compatible with taking inverses.

Let  $L = [\omega_1, \omega_2]$ . There are three points of order 2 in  $\mathbb{C}/L$ ; if  $L = [\omega_1, \omega_2]$  these are  $\omega_1/2, \omega_2/2$ , and  $(\omega_1 + \omega_2)/2$ . By Lemma 14.31,  $\wp'$  vanishes at these points, hence  $\Phi$  maps points of order 2 in  $\mathbb{C}/L$  to points of order 2 in  $E_L(\mathbb{C})$ , since the latter are the points with  $y$ -coordinate zero. Moreover,  $\Phi$  is injective on points of order 2, since  $\wp(z)$  maps each point of order 2 in  $\mathbb{C}/L$  to a distinct root of  $4\wp(z)^3 - g_2(L)\wp(z) - g_3(L)$ , as shown in the proof of Lemma 14.33. The restriction of  $\Phi$  to  $(\mathbb{C}/L)[2]$  defines a bijection of  $(\mathbb{C}/L)[2] \xrightarrow{\sim} E_L[2] \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  with  $\Phi(0) = 0$ , which must be a group isomorphism.

<sup>1</sup>This is not difficult to show, but it would distract us from our immediate goal. We will see an explicit isomorphism of complex manifolds in a few lectures when we study modular curves, and in that case we will take the time to define precisely what this means and to prove it.

To show that  $\Phi$  is surjective, let  $(x_0, y_0) \in E_L(\mathbb{C})$ . The elliptic function  $f(z) = \wp(z) - x_0$  has order 2, hence it has two zeros in the fundamental parallelogram  $\mathcal{F}_0$ , by Theorem 14.18. Neither of these zeros occurs at  $z = 0$ , since  $f$  has a pole at 0. So let  $z_0 \neq 0$  be a zero of  $f(z)$  in  $\mathcal{F}_0$ . Then  $\wp(z_0) = x_0$ , which implies  $\Phi(z_0) = (x_0, \pm y_0)$  and therefore  $(x_0, y_0) = \Phi(\pm z_0)$ ; thus  $\Phi$  is surjective.

We now show that  $\Phi$  is injective. Let  $z_1, z_2 \in \mathcal{F}_0$  and suppose that  $\Phi(z_1) = \Phi(z_2)$ . If  $2z_1 \in L$  then  $z_1$  is a 2-torsion element and we have already shown that  $\Phi$  restricts to a bijection on  $(\mathbb{C}/L)[2]$ , so we must have  $z_1 = z_2$ . We now assume  $2z_1 \notin L$ , which implies  $\wp'(z_1) \neq 0$ . As argued above, the roots of  $f(z) = \wp(z) - \wp(z_1)$  in  $\mathcal{F}_0$  are  $\pm z_1$ , thus  $z_2 \equiv \pm z_1 \pmod{L}$ . We also have  $\wp'(z_1) = \wp'(z_2)$ , and this forces  $z_2 \equiv z_1 \pmod{L}$ , since  $\wp'(-z_1) = -\wp'(z_1) \neq \wp'(z_1)$  because  $\wp'(z_1) \neq 0$ .

It remains only to show that  $\Phi(z_1 + z_2) = \Phi(z_1) + \Phi(z_2)$ . So let  $z_1, z_2 \in \mathcal{F}_0$ ; we may assume that  $z_1, z_2, z_1 + z_2 \notin L$  since the case where either  $z_1$  or  $z_2$  lies in  $L$  is immediate, and if  $z_1 + z_2 \in L$  then  $z_1$  and  $z_2$  are inverses modulo  $L$ , a case treated above.

The points  $P_1 = \Phi(z_1)$  and  $P_2 = \Phi(z_2)$  are affine points in  $E_L(\mathbb{C})$ , and the line  $\ell$  between them cannot be vertical because  $P_1$  and  $P_2$  are not inverses (since  $z_1$  and  $z_2$  are not). So let  $y = mx + b$  be an equation for this line, and let  $P_3$  be the third point where the line intersects the curve  $E_L$ . Then  $P_1 + P_2 + P_3 = 0$ , by the definition of the group law on  $E_L(\mathbb{C})$ .

Now consider the function  $\ell(z) = -\wp'(z) + m\wp(z) + b$ . It is an elliptic function of order 3 with a triple pole at 0, so it has three zeros in the fundamental region  $\mathcal{F}_0$ , two of which are  $z_1$  and  $z_2$ . Let  $z_3$  be the third zero in  $\mathcal{F}_0$ . The point  $\Phi(z_3)$  lies on both the line  $\ell$  and the elliptic curve  $E_L(\mathbb{C})$ , hence it must lie in  $\{P_1, P_2, P_3\}$ ; moreover, we have a bijection from  $\{z_1, z_2, z_3\}$  to  $\{\Phi(z_1), \Phi(z_2), \Phi(z_3)\} = \{P_1, P_2, P_3\}$ , and this bijection must send  $z_3$  to  $P_3$  if  $P_3$  is distinct from  $P_1$  and  $P_2$ . If  $P_3$  coincides with exactly one of  $P_1$  or  $P_2$ , say  $P_1$ , then  $\ell(z)$  has a double zero at  $z_1$  and we must have  $z_3 = z_1$ ; and if  $P_1 = P_2 = P_3$  then clearly  $z_1 = z_2 = z_3$ . Thus in every case we must have  $P_3 = \Phi(z_3)$ .

We have  $P_1 + P_2 + P_3 = 0$ , so it suffices to show  $z_1 + z_2 + z_3 \in L$ , since this implies

$$\Phi(z_1 + z_2) = \Phi(-z_3) = -\Phi(z_3) = -P_3 = P_1 + P_2 = \Phi(z_1) + \Phi(z_2).$$

Let  $\mathcal{F}_\alpha$  be a fundamental region for  $L$  whose boundary does not contain any zeros or poles of  $\ell(z)$  and replace  $z_1, z_2, z_3$  by equivalent points in  $\mathcal{F}_\alpha$  if necessary.

Applying Theorem 14.17 to  $g(z) = z$  and  $f(z) = \ell(z)$  yields

$$\frac{1}{2\pi i} \int_{\partial \mathcal{F}_\alpha} z \frac{\ell'(z)}{\ell(z)} dz = \sum_{w \in \mathcal{F}_\alpha} \text{ord}_w(\ell) w = z_1 + z_2 + z_3 - 3 \cdot 0 = z_1 + z_2 + z_3, \quad (1)$$

where the boundary  $\partial \mathcal{F}_\alpha$  of  $\mathcal{F}_\alpha$  is oriented counter-clockwise.

Let us now evaluate the integral in (1); to ease the notation, define  $f(z) := \ell'(z)/\ell(z)$ , which we note is an elliptic function (hence periodic with respect to  $L$ ). We then have

$$\begin{aligned} \int_{\partial \mathcal{F}_\alpha} z f(z) dz &= \int_{\alpha}^{\alpha+\omega_1} z f(z) dz + \int_{\alpha+\omega_1}^{\alpha+\omega_1+\omega_2} z f(z) dz + \int_{\alpha+\omega_1+\omega_2}^{\alpha+\omega_2} z f(z) dz + \int_{\alpha+\omega_2}^{\alpha} z f(z) dz \\ &= \int_{\alpha}^{\alpha+\omega_1} z f(z) dz + \int_{\alpha}^{\alpha+\omega_2} (z + \omega_1) f(z) dz + \int_{\alpha+\omega_1}^{\alpha} (z + \omega_2) f(z) dz + \int_{\alpha+\omega_2}^{\alpha} z f(z) dz \\ &= \omega_1 \int_{\alpha}^{\alpha+\omega_2} f(z) dz + \omega_2 \int_{\alpha+\omega_1}^{\alpha} f(z) dz. \end{aligned} \quad (2)$$

Note that we have used the periodicity of  $f(z)$  to replace  $f(z + \omega_i)$  by  $f(z)$ , and to cancel integrals in opposite directions along lines that are equivalent modulo  $L$ .

For any closed (not necessarily simple) curve  $C$  and a point  $z_0 \notin C$ , the quantity

$$\frac{1}{2\pi i} \int_C \frac{dz}{z - z_0}$$

is the *winding number* of  $C$  about  $z_0$ , and it is an integer (it counts the number of times the curve  $C$  “winds around” the point  $z_0$ ); see [1, Lem. 4.2.1] or [4, Lem. B.1.3].

The function  $\ell(\alpha + t\omega_2)$  parametrizes a closed curve  $C_1$  from  $\ell(\alpha)$  to  $\ell(\alpha + \omega_2) = \ell(\alpha)$ , as  $t$  ranges from 0 to 1. The winding number of  $C_1$  about the point 0 is the integer

$$c_1 := \frac{1}{2\pi i} \int_{C_1} \frac{dz}{z - 0} = \frac{1}{2\pi i} \int_0^1 \frac{\ell'(\alpha + t\omega_2)}{\ell(\alpha + t\omega_2)} \omega_2 dt = \frac{1}{2\pi i} \int_\alpha^{\alpha+\omega_2} \frac{\ell'(z)}{\ell(z)} dz = \frac{1}{2\pi i} \int_\alpha^{\alpha+\omega_2} f(z) dz. \quad (3)$$

Similarly, the function  $\ell(\alpha + t\omega_1)$  parameterizes a closed curve  $C_2$  from  $\ell(\alpha)$  to  $\ell(\alpha + \omega_1)$ , and we obtain the integer

$$c_2 := \frac{1}{2\pi i} \int_{C_2} \frac{dz}{z - 0} = \frac{1}{2\pi i} \int_0^1 \frac{\ell'(\alpha + t\omega_1)}{\ell(\alpha + t\omega_1)} \omega_1 dt = \frac{1}{2\pi i} \int_\alpha^{\alpha+\omega_1} \frac{\ell'(z)}{\ell(z)} dz = \frac{1}{2\pi i} \int_\alpha^{\alpha+\omega_1} f(z) dz. \quad (4)$$

Plugging (3), and (4) into (2), and applying (1), we see that

$$z_1 + z_2 + z_3 = c_1\omega_1 - c_2\omega_2 \in L,$$

as desired. □

## 15.2 The $j$ -invariant of a lattice

**Definition 15.2.** The  $j$ -invariant of a lattice  $L$  is defined by

$$j(L) = 1728 \frac{g_2(L)^3}{\Delta(L)} = 1728 \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2}.$$

Recall that  $\Delta(L) \neq 0$ , by Lemma 14.33, so  $j(L)$  is always defined.

The elliptic curve  $E_L: y^2 = 4x^3 - g_2(L)x - g_3(L)$  is isomorphic to the elliptic curve  $y^2 = x^3 + Ax + B$ , where  $g_2(L) = -4A$  and  $g_3(L) = -4B$ . Thus

$$j(L) = 1728 \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2} = 1728 \frac{(-4A)^3}{(-4A)^3 - 27(-4B)^2} = 1728 \frac{4A^3}{4A^3 + 27B^2} = j(E_L).$$

Thus the  $j$ -invariant of a lattice  $L$  is the same as the  $j$ -invariant of the corresponding elliptic curve  $E_L$ . We now define the discriminant of an elliptic curve so that it agrees with the discriminant of the corresponding lattice.

**Definition 15.3.** The *discriminant* of an elliptic curve  $E: y^2 = x^3 + Ax + B$  is

$$\Delta(E) = -16(4A^3 + 27B^2).$$

This definition applies to any elliptic curve  $E/k$  defined by a short Weierstrass equation, whether  $k = \mathbb{C}$  or not, but for the moment we continue to focus on elliptic curves over  $\mathbb{C}$ .

Recall from Theorem 13.14 that elliptic curves  $E/k$  and  $E'/k$  are isomorphic over  $\bar{k}$  if and only if  $j(E) = j(E')$ . Thus over an algebraically closed field like  $\mathbb{C}$ , the  $j$ -invariant characterizes elliptic curves up to isomorphism. We now define an analogous notion of isomorphism for lattices.

**Definition 15.4.** Lattices  $L$  and  $L'$  are said to be *homothetic* if  $L' = \lambda L$  for some  $\lambda \in \mathbb{C}^\times$ .

**Theorem 15.5.** *Two lattices  $L$  and  $L'$  are homothetic if and only if  $j(L) = j(L')$ .*

*Proof.* Suppose  $L$  and  $L'$  are homothetic, with  $L' = \lambda L$ . Then

$$g_2(L') = 60 \sum_{\omega \in L'^*} \frac{1}{\omega^4} = 60 \sum_{\omega \in L^*} \frac{1}{(\lambda\omega)^4} = \lambda^{-4} g_2(L).$$

Similarly,  $g_3(L') = \lambda^{-6} g_3(L)$ , and we have

$$j(L') = 1728 \frac{(\lambda^{-4} g_2(L))^3}{(\lambda^{-4} g_2(L))^3 - 27(\lambda^{-6} g_3(L))^2} = 1728 \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2} = j(L).$$

To show the converse, let us now assume  $j(L) = j(L')$ . Let  $E_L$  and  $E_{L'}$  be the corresponding elliptic curves. Then  $j(E_L) = j(E_{L'})$ . We may write

$$E_L: y^2 = x^3 + Ax + B,$$

with  $-4A = g_2(L)$  and  $-4B = g_3(L)$ , and similarly for  $E_{L'}$ , with  $-4A' = g_2(L')$  and  $-4B' = g_3(L')$ . By Theorem 13.13, there is a  $\mu \in \mathbb{C}^\times$  such that  $A' = \mu^4 A$  and  $B' = \mu^6 B$ , and if we let  $\lambda = 1/\mu$ , then  $g_2(L') = \lambda^{-4} g_2(L) = g_2(\lambda L)$  and  $g_3(L') = \lambda^{-6} g_3(L) = g_3(\lambda L)$ , as above. We now show that this implies  $L' = \lambda L$ .

Recall from Theorem 14.29 that the Weierstrass  $\wp$ -function satisfies

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Differentiating both sides yields

$$\begin{aligned} 2\wp'(z)\wp''(z) &= 12\wp(z)^2\wp'(z) - g_2\wp'(z) \\ \wp''(z) &= 6\wp(z)^2 - \frac{g_2}{2}. \end{aligned} \tag{5}$$

By Theorem 14.28, the Laurent series for  $\wp(z; L)$  at  $z = 0$  is

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)G_{2n+2}z^{2n} = \frac{1}{z^2} + \sum_{n=1}^{\infty} a_n z^{2n},$$

where  $a_1 = g_2/20$  and  $a_2 = g_3/28$ .

Comparing coefficients for the  $z^{2n}$  term in (5), we find that for  $n \geq 2$  we have

$$(2n+2)(2n+1)a_{n+1} = 6 \left( \sum_{k=1}^{n-1} a_k a_{n-k} + 2a_{n+1} \right),$$

and therefore

$$a_{n+1} = \frac{6}{(2n+2)(2n+1) - 12} \sum_{k=1}^{n-1} a_k a_{n-k}.$$

This allows us to compute  $a_{n+1}$  from  $a_1, \dots, a_{n-1}$ , for all  $n \geq 2$ . It follows that  $g_2(L)$  and  $g_3(L)$  uniquely determine the function  $\wp(z) = \wp(z; L)$  (and therefore the lattice  $L$  where  $\wp(z)$  has poles), since  $\wp(z)$  is uniquely determined by its Laurent series expansion about 0.

Now consider  $L'$  and  $\lambda L$ , where we have  $g_2(L') = g_2(\lambda L)$  and  $g_3(L') = g_3(\lambda L)$ . It follows that  $\wp(z; L') = \wp(z; \lambda L)$  and  $L' = \lambda L$ , as desired.  $\square$

**Corollary 15.6.** *Two lattices  $L$  and  $L'$  are homothetic if and only if the corresponding elliptic curves  $E_L$  and  $E_{L'}$  are isomorphic.*

Thus homothety classes of lattices correspond to isomorphism classes of elliptic curves over  $\mathbb{C}$ , and both are classified by the  $j$ -invariant. Recall from Theorem 13.12 that every complex number is the  $j$ -invariant of an elliptic curve  $E/\mathbb{C}$ . To prove the Uniformization Theorem we just need to show that the same is true of lattices.

### 15.3 The $j$ -function

Every lattice  $[\omega_1, \omega_2]$  is homothetic to a lattice of the form  $[1, \tau]$ , with  $\tau$  in the upper half plane  $\mathbb{H} = \{z \in \mathbb{C} : \text{im } z > 0\}$ ; we may take  $\tau = \pm\omega_2/\omega_1$  with the sign chosen so that  $\text{im } \tau > 0$ . This leads to the following definition of the  $j$ -function.

**Definition 15.7.** The  $j$ -function  $j: \mathbb{H} \rightarrow \mathbb{C}$  is defined by  $j(\tau) = j([1, \tau])$ . We similarly define  $g_2(\tau) = g_2([1, \tau])$ ,  $g_3(\tau) = g_3([1, \tau])$ , and  $\Delta(\tau) = \Delta([1, \tau])$ .

Note that for any  $\tau \in \mathbb{H}$ , both  $-1/\tau$  and  $\tau + 1$  lie in  $\mathbb{H}$  (the maps  $\tau \mapsto 1/\tau$  and  $\tau \mapsto -\tau$  both swap the upper and lower half-planes; their composition preserves them).

**Theorem 15.8.** *The  $j$ -function is holomorphic on  $\mathbb{H}$ , and satisfies  $j(-1/\tau) = j(\tau)$  and  $j(\tau + 1) = j(\tau)$ .*

*Proof.* From the definition of  $j(\tau) = j([1, \tau])$  we have

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)} = 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}.$$

The series defining

$$g_2(\tau) = 60 \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(m + n\tau)^4} \quad \text{and} \quad g_3(\tau) = 140 \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(m + n\tau)^6}$$

converge absolutely for any fixed  $\tau \in \mathbb{H}$ , by Lemma 14.22, and they converge uniformly over  $\tau$  in any compact subset of  $\mathbb{H}$ . The proof of this last fact is straight-forward but slightly technical; see [2, Thm. 1.15] for the details. It follows that  $g_2(\tau)$  and  $g_3(\tau)$  are holomorphic on  $\mathbb{H}$ , and therefore  $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$  is also holomorphic on  $\mathbb{H}$ . Since  $\Delta(\tau)$  is nonzero for all  $\tau \in \mathbb{H}$ , by Lemma 14.33, the  $j$ -function  $j(\tau)$  is holomorphic on  $\mathbb{H}$  as well.

The lattices  $[1, \tau]$  and  $[1, -1/\tau] = -1/\tau[1, \tau]$  are homothetic, and the lattices  $[1, \tau + 1]$  and  $[1, \tau]$  are equal; thus  $j(-1/\tau) = j(\tau)$  and  $j(\tau + 1) = j(\tau)$ , by Theorem 15.5.  $\square$

### 15.4 The modular group

We now consider the *modular group*

$$\Gamma = \text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

As proved in Problem Set 8, the group  $\Gamma$  acts on  $\mathbb{H}$  via linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d},$$

and it is generated by the matrices  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . This implies that the  $j$ -function is invariant under the action of the modular group; in fact, more is true.

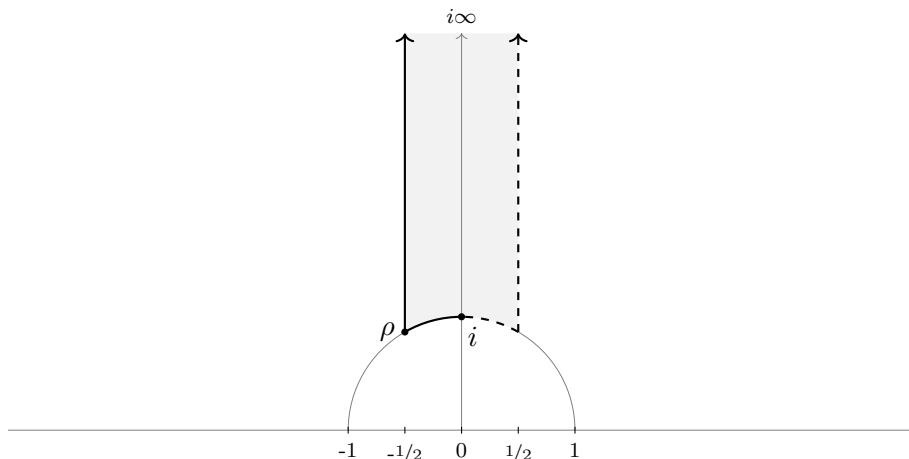


Figure 1: Fundamental domain  $\mathcal{F}$  for  $\mathbb{H}/\Gamma$ , with  $i = e^{\pi/2}$  and  $\rho = e^{2\pi i/3}$ .

**Lemma 15.9.** *We have  $j(\tau) = j(\tau')$  if and only if  $\tau' = \gamma\tau$  for some  $\gamma \in \Gamma$ .*

*Proof.* We have  $j(S\tau) = j(-1/\tau) = j(\tau)$  and  $j(T\tau) = j(\tau + 1) = j(\tau)$ , by Theorem 15.8, It follows that if  $\tau' = \gamma\tau$  then  $j(\tau') = j(\tau)$ , since  $S$  and  $T$  generate  $\Gamma$ .

To prove the converse, let us suppose that  $j(\tau) = j(\tau')$ . Then by Theorem 15.5, the lattices  $[1, \tau]$  and  $[1, \tau']$  are homothetic. So  $[1, \tau'] = \lambda[1, \tau]$ , for some  $\lambda \in \mathbb{C}^\times$ . There thus exist integers  $a, b, c$ , and  $d$  such that

$$\begin{aligned}\tau' &= a\lambda\tau + b\lambda \\ 1 &= c\lambda\tau + d\lambda\end{aligned}$$

From the second equation, we see that  $\lambda = \frac{1}{c\tau + d}$ . Substituting this into the first, we have

$$\tau' = \frac{a\tau + b}{c\tau + d} = \gamma\tau, \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2 \times 2}.$$

Similarly, using  $[1, \tau] = \lambda^{-1}[1, \tau']$ , we can write  $\tau = \gamma'\tau'$  for some integer matrix  $\gamma'$ . The fact that  $\tau' = \gamma\gamma'\tau'$  implies that  $\det \gamma = \pm 1$  (since  $\gamma$  and  $\gamma'$  are integer matrices). But  $\tau$  and  $\tau'$  both lie in  $\mathbb{H}$ , so we must have  $\det \gamma = 1$ ; therefore  $\gamma \in \Gamma$  as desired.  $\square$

Lemma 15.9 implies that when studying the  $j$ -function it suffices to study its behavior on  $\Gamma$ -equivalence classes of  $\mathbb{H}$ , that is, the orbits of  $\mathbb{H}$  under the action of  $\Gamma$ . We thus consider the quotient of  $\mathbb{H}$  modulo  $\Gamma$ -equivalence, which we denote by  $\mathbb{H}/\Gamma$ .<sup>2</sup> The actions of  $\gamma$  and  $-\gamma$  are identical, so taking the quotient by  $\text{PSL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})/\{\pm 1\}$  yields the same result, but for the sake of clarity we will stick with  $\Gamma = \text{SL}_2(\mathbb{Z})$ .

We now wish to determine a fundamental domain for  $\mathbb{H}/\Gamma$ , a set of unique representatives in  $\mathbb{H}$  for each  $\Gamma$ -equivalence class. For this purpose we will use the set

$$\mathcal{F} = \{\tau \in \mathbb{H} : \text{re}(\tau) \in [-1/2, 1/2] \text{ and } |\tau| \geq 1, \text{ such that } |\tau| > 1 \text{ if } \text{re}(\tau) > 0\}.$$

**Lemma 15.10.** *The set  $\mathcal{F}$  is a fundamental domain for  $\mathbb{H}/\Gamma$ .*

<sup>2</sup>Some authors write this quotient as  $\Gamma \backslash \mathbb{H}$  to indicate that the action is on the left.

*Proof.* We need to show that for every  $\tau \in \mathbb{H}$ , there is a unique  $\tau' \in \mathcal{F}$  such that  $\tau' = \gamma\tau$ , for some  $\gamma \in \Gamma$ . We first prove existence. Let us fix  $\tau \in \mathbb{H}$ . For any  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$  we have

$$\operatorname{im}(\gamma\tau) = \operatorname{im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\operatorname{im}((a\tau + b)(c\bar{\tau} + d))}{|c\tau + d|^2} = \frac{(ad - bc) \operatorname{im} \tau}{|c\tau + d|^2} = \frac{\operatorname{im} \tau}{|c\tau + d|^2} \quad (6)$$

Let  $c\tau + d$  be a shortest vector in the lattice  $[1, \tau]$ . Then  $c$  and  $d$  must be relatively prime, and we can pick integers  $a$  and  $b$  so that  $ad - bc = 1$ . The matrix  $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  then maximizes the value of  $\operatorname{im}(\gamma\tau)$  over  $\gamma \in \Gamma$ . Let us now choose  $\gamma = T^k \gamma_0$ , where  $k$  is chosen so that  $\operatorname{re}(\gamma\tau) \in [1/2, 1/2)$ , and note that  $\operatorname{im}(\gamma\tau) = \operatorname{im}(\gamma_0\tau)$  remains maximal. We must have  $|\gamma\tau| \geq 1$ , since otherwise  $\operatorname{im}(S\gamma\tau) > \operatorname{im}(\gamma\tau)$ , contradicting the maximality of  $\operatorname{im}(\gamma\tau)$ . Finally, if  $\tau' = \gamma\tau \notin \mathcal{F}$ , then we must have  $|\gamma\tau| = 1$  and  $\operatorname{re}(\gamma\tau) > 0$ , in which case we replace  $\gamma$  by  $S\gamma$  so that  $\tau' = \gamma\tau \in \mathcal{F}$ .

It remains to show that  $\tau'$  is unique. This is equivalent to showing that any two  $\Gamma$ -equivalent points in  $\mathcal{F}$  must coincide. So let  $\tau_1$  and  $\tau_2 = \gamma_1\tau_1$  be two elements of  $\mathcal{F}$ , with  $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , and assume  $\operatorname{im} \tau_1 \leq \operatorname{im} \tau_2$ . By (6), we must have  $|c\tau_1 + d|^2 \leq 1$ , thus

$$1 \geq |c\tau_1 + d|^2 = (c\tau_1 + d)(c\bar{\tau}_1 + d) = c^2|\tau_1|^2 + d^2 + 2cd \operatorname{re} \tau_1 \geq c^2|\tau_1|^2 + d^2 - |cd| \geq 1,$$

where the last inequality follows from  $|\tau_1| \geq 1$  and the fact that  $c$  and  $d$  cannot both be zero (since  $\det \gamma = 1$ ). Thus  $|c\tau_1 + d| = 1$ , which implies  $\operatorname{im} \tau_2 = \operatorname{im} \tau_1$ . We also have  $|c|, |d| \leq 1$ , and by replacing  $\gamma_1$  by  $-\gamma_1$  if necessary, we may assume that  $c \geq 0$ . This leaves 3 cases:

1.  $c = 0$ : then  $|d| = 1$  and  $a = d$ . So  $\tau_2 = \tau_1 \pm b$ , but  $|\operatorname{re} \tau_2 - \operatorname{re} \tau_1| < 1$ , so  $\tau_2 = \tau_1$ .
2.  $c = 1, d = 0$ : then  $b = -1$  and  $|\tau_1| = 1$ . So  $\tau_1$  is on the unit circle and  $\tau_2 = a - 1/\tau_1$ . Either  $a = 0$  and  $\tau_2 = \tau_1 = i$ , or  $a = -1$  and  $\tau_2 = \tau_1 = \rho$ .
3.  $c = 1, |d| = 1$ : then  $|\tau_1 + d| = 1$ , so  $\tau_1 = \rho$ , and  $\operatorname{im} \tau_2 = \operatorname{im} \tau_1 = \sqrt{3}/2$  implies  $\tau_2 = \rho$ .

In every case we have  $\tau_1 = \tau_2$  as desired.  $\square$

**Theorem 15.11.** *The restriction of the  $j$ -function to  $\mathcal{F}$  defines a bijection from  $\mathcal{F}$  to  $\mathbb{C}$ .*

*Proof.* Injectivity follows immediately from Lemmas 15.9 and 15.10. It remains to prove surjectivity. We have

$$g_2(\tau) = 60 \sum_{\substack{n, m \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(m + n\tau)^4} = 60 \left( 2 \sum_{m=1}^{\infty} \frac{1}{m^4} + \sum_{\substack{n, m \in \mathbb{Z} \\ n \neq 0}} \frac{1}{(m + n\tau)^4} \right).$$

The second sum tends to 0 as  $\operatorname{im} \tau \rightarrow \infty$ . Thus we have

$$\lim_{\operatorname{im} \tau \rightarrow \infty} g_2(\tau) = 120 \sum_{m=1}^{\infty} m^{-4} = 120 \zeta(4) = 120 \frac{\pi^4}{90} = \frac{4\pi^4}{3},$$

where  $\zeta(s)$  is the Riemann zeta function. Similarly,

$$\lim_{\operatorname{im} \tau \rightarrow \infty} g_3(\tau) = 280 \zeta(6) = 280 \frac{\pi^6}{945} = \frac{8\pi^6}{27}.$$

Thus

$$\lim_{\operatorname{im} \tau \rightarrow \infty} \Delta(\tau) = \left(\frac{4}{3}\pi^4\right)^3 - 27 \left(\frac{8}{27}\pi^6\right)^2 = 0.$$

(this explains the coefficients 60 and 140 in the definitions of  $g_2$  and  $g_3$ ; they are the smallest pair of integers that ensure this limit is 0). Since  $\Delta(\tau)$  is the denominator of  $j(\tau)$ , the quantity  $j(\tau) = g_2(\tau)^3/\Delta(\tau)$  is unbounded as  $\text{im } \tau \rightarrow \infty$ .

In particular, the  $j$ -function is non-constant, and by Theorem 15.8 it is holomorphic on  $\mathbb{H}$ . The open mapping theorem implies that  $j(\mathbb{H})$  is an open subset of  $\mathbb{C}$ ; see [4, Thm. 3.4.4].

We claim that  $j(\mathbb{H})$  is also a closed subset of  $\mathbb{C}$ . Let  $j(\tau_1), j(\tau_2), \dots$  be an arbitrary convergent sequence in  $j(\mathbb{H})$ , converging to  $w \in \mathbb{C}$ . The  $j$ -function is  $\Gamma$ -invariant, by Lemma 15.9, so we may assume the  $\tau_n$  all lie in  $\mathcal{F}$ . The sequence  $\text{im } \tau_1, \text{im } \tau_2, \dots$  must be bounded, say by  $B$ , since  $j(\tau) \rightarrow \infty$  as  $\text{im } \tau \rightarrow \infty$ , but the sequence  $j(\tau), j(\tau_2), \dots$  converges; it follows that the  $\tau_n$  all lie in the compact set

$$\Omega = \{\tau : \text{re } \tau \in [-1/2, 1/2], \text{im } \tau \in [1/2, B]\}.$$

There is thus a subsequence of the  $\tau_n$  that converges to some  $\tau \in \Omega \subset \mathbb{H}$ . The  $j$ -function is holomorphic, hence continuous, so  $j(\tau) = w$ . It follows that the open set  $j(\mathbb{H})$  contains all its limit points and is therefore closed.

The fact that the non-empty set  $j(\mathbb{H}) \subseteq \mathbb{C}$  is both open and closed implies that  $j(\mathbb{H}) = \mathbb{C}$ , since  $\mathbb{C}$  is connected. It follows that  $j(\mathcal{F}) = \mathbb{C}$ , since every element of  $\mathbb{H}$  is  $\Gamma$ -equivalent to an element of  $\mathcal{F}$  (Lemma 15.10) and the  $j$ -function is  $\Gamma$ -invariant (Lemma 15.9).  $\square$

**Corollary 15.12** (Uniformization Theorem). *For every elliptic curve  $E/\mathbb{C}$  there exists a lattice  $L$  such that  $E = E_L$ .*

*Proof.* Given  $E/\mathbb{C}$ , pick  $\tau \in \mathbb{H}$  so that  $j(\tau) = j(E)$  and let  $L' = [1, \tau]$ . We have

$$j(E) = j(\tau) = j(L') = j(E_{L'}),$$

so  $E$  is isomorphic to  $E_{L'}$ , by Theorem 13.13, where the isomorphism is given by the map  $(x, y) \mapsto (\mu^2 x, \mu^3 y)$  for some  $\mu \in \mathbb{C}^\times$ . If now let  $L = \frac{1}{\mu} L'$ , then  $E = E_L$ .  $\square$

## References

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